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BOUNDARY PROBLEMS FOR THE WAVE EQUATION WITH THE LÉVY LAPLACIAN IN SHILOV'S CLASS

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Abstract. We present solutions to some boundary value and initial-boundary value problems for the "wave" equation with the infinite dimensional Lévy Laplacian Δ_L

$$\frac{\partial^2 U(t,x)}{\partial t^2} = \Delta_L U(t,x)$$

in the Shilov class of functions.

1. Introduction

The theory of linear elliptic and parabolic equations with the Lévy Laplacian is now well developed (see for example [1]).

This paper is devoted to the construction of solution of the boundary value and initial-boundary value problems for the equation

$$\frac{\partial^2 U(t,x)}{\partial t^2} = \Delta_L U(t,x)$$

with the Lévy Laplacian Δ_L ("wave" equation) in fundamental domains of the Shilov functional class.

It should be noted that in the Schilov functional class the Lévy Laplacian is a "derivative" (see (4)). As a result, in this class the equation $\frac{\partial^2 U(t,x)}{\partial t^2} = \Delta_L U(t,x)$ is reduced to the equation

$$\frac{\partial^2 u(t,(a_1,x)_H,\ldots,(a_m,x)_H,\zeta)}{\partial t^2} = \frac{\partial u(t,(a_1,x)_H,\ldots,(a_m,x)_H,\zeta)}{\partial \zeta}\Big|_{\zeta=\frac{\|x\|_H^2}{2}}.$$

2. Preliminaries

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)_H$ and the norm $\|\cdot\|_H$, and let $F(x)$ be a scalar function defined on H.

The infinite dimensional Laplacian was introduced by P. Lévy [2]. If F is twice strongly differentiable at a point x_0 then the Lévy Laplacian of F in this point is defined (if it exists) by the formula

$$\Delta_L F(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} (F''(x_0)f_k, f_k)_H, \tag{1}$$

where $F''(x)$ is the Hessian of the function $F(x)$ and $\{f_k\}_1^\infty$ is a chosen orthonormal basis in H.

Let Ω be a bounded domain in H (that is, a bounded open set in H), $\overline{\Omega} = \Omega \cup \Gamma$, where Γ is the boundary of Ω. We suppose that

$$\Omega = \{x \in H : 0 \le Q(x) < R^2\}, \quad \Gamma = \{x \in H : Q(x) = R^2\}, \tag{2}$$

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with some $R \geq 0$, where $Q(x)$ is a twice differentiable function such that $\Delta_L Q(x) = \gamma$ for a positive nonzero constant γ. Such kind of domains are called fundamental.

For example, the domains

1) $\overline{\Omega} = \{x \in H : \|x\|_H^2 \leq R^2\}$ (a ball)

and

2) $\overline{\Omega} = \{x \in H : (Bx, x)_H \leq R^2\}$, where $B = \gamma E + A$, E is the unit operator, A is a compact linear operator in H (ellipsoid),

are fundamental.

We put

$$T(x) = \frac{R^2 - Q(x)}{\gamma}.$$

Obviously, the real valued function $T(x), x \in H$, possesses the properties

$0 < T(x) \leq \frac{R^2}{\gamma}$ for $x \in \Omega$; $T(x) = 0$ for $x \in \Gamma$; $\Delta_L T(x) = -1$.

3. The Shilov class of functions

Let $\mathfrak{C}$ denote the Shilov class of functions [3], that is a set of functions of the form

$$F(x) = f\Big((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2}\Big), \tag{3}$$

where the elements $a_1, \dots, a_m$ belong to H, $f(\xi_1, \dots, \xi_m, \zeta)$ is a real-valued bounded continuous function of $m+1$ variables, defined and continuous in the domain $G \subseteq R^{m+1}$, and

$$x \in \overline{\Omega} \Longrightarrow \Big((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2}\Big) \in G.$$

Denote by $\mathfrak{C}^*$ the subset of functions from $\mathfrak{C}$ which are continuously differentiable in $\frac{\|x\|_H^2}{2}$. For any $F \in \mathfrak{C}^*$, we have [3]

$$\Delta_L F(x) = \frac{\partial f((a_1, x)_H, \dots, (a_m, x)_H, \zeta)}{\partial \zeta}\Big|_{\zeta = \frac{\|x\|_H^2}{2}}, \quad x \in H. \tag{4}$$

Note that in the Shilov class functions the Lévy Laplacian does not depend on the choice of a basis.

3.1. Initial problem with homogeneous boundary condition.

1. First we consider the problem

$$\frac{\partial^2 V_1(t, x)}{\partial t^2} = \Delta_L V_1(t, x) \quad (t > 0), \tag{5}$$

$$V_1(0, x) = F(x), \tag{6}$$

$$V_1(t, 0) = 0, \tag{7}$$

where $F(x)$ is a given function, and $V_1(t, x) \in C([0, \infty), H) \bigcap C^{2,1}((0, \infty), H)$.

Theorem 1. *Let $F \in \mathfrak{C}^*$,*

$$F(x) = f\Big((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2}\Big),$$

where $f(\xi_1, \dots, \xi_m, \zeta)$ is a bounded, continuous, twice differentiable in ζ function on R^{m+1}, $a_k \in H, k = 1, \dots, m$. Assume also $f(\xi_1, \dots, \xi_m, 0) = 0$.

Then

$$V_1(t, x) = \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f\Big((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big) e^{-z^2} dz \tag{8}$$

is a solution of problem (5)–(7).

Proof. Let us rewrite the function $F(x)$ in the form $F(x) = f\Big(P_a x, \frac{\|x\|_H^2}{2}\Big)$, where P_a is a projection to an m-dimensional subspace spanned on vectors $a_1, \dots, a_m$.

It follows from (8) that

$$\begin{aligned}\frac{\partial^2 V_1(t,x)}{\partial t^2} &= \frac{2f'_\zeta(P_a x, 0)\sqrt{\frac{\|x\|_H^2}{2}}}{\sqrt{\pi}\, t} \exp\left(\frac{-t^2}{2\|x\|_H^2}\right) \\ &- \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f'_\zeta\Big(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big)\frac{1}{2z^2} e^{-z^2} dz \\ &+ \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} t^2 f''_{\zeta\zeta}\Big(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big)\frac{1}{4z^4} e^{-z^2} dz \quad (t>0).\end{aligned} \tag{9}$$

Taking into account (4), we deduce from (8) that

$$\Delta_L V_1(t,x) = \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f'_\zeta\Big(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big) e^{-z^2} dz,$$

because $\Delta_L \frac{\|x\|_H^2}{2} = 1$, $\Delta_L \Big(\frac{\|x\|_H^2}{2}\Big)^{-1/2} = -\frac{1}{2}\Big(\frac{\|x\|_H^2}{2}\Big)^{-3/2}$.

Applying the integration by parts formula, we derive

$$\begin{aligned}\Delta_L V_1(t,x) &= -\frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f'_\zeta\Big(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big)\frac{1}{2z} d\left(e^{-z^2}\right) \\ &= \frac{2f'_\zeta(P_a x, 0)\sqrt{\frac{\|x\|_H^2}{2}}}{\sqrt{\pi}\, t} \exp\left(\frac{-t^2}{2\|x\|_H^2}\right) \\ &- \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f'_\zeta\Big(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big)\frac{1}{2z^2} e^{-z^2} dz \\ &+ \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} t^2 f''_{\zeta\zeta}\Big(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big)\frac{1}{4z^4} e^{-z^2} dz.\end{aligned} \tag{10}$$

Substituting (9) and (10) into (5), we obtain an identity.

Setting $t = 0$ in (8), we obtain $V_1(0, x) = F(x)$, and setting $x = 0$ in (8) we obtain $V_1(t, 0) = 0$. □

2. Now consider the auxiliary problem

$$\frac{\partial^2 V_2(t,x)}{\partial t^2} = \Delta_L V_2(t,x) \quad (t>0,\ x\in\Omega), \tag{11}$$

$$V_2(0,x) = 0, \tag{12}$$

$$V_2(t,x)\Big|_\Gamma = h(t), \tag{13}$$

where $h(t) = V_1(t,x)|_\Gamma$, $V_1(t,x)$ is the solution of problem (5)–(7); $V_2(t,x) \in C\left([0,\infty), \overline{\Omega}\right) \bigcap C^{2,1}\left((0,\infty), \Omega\right)$.

Theorem 2. *Let the conditions of Theorem 1 are satisfied. Let* $f\Big(\xi_1, \dots, \xi_m, \frac{R^2}{2}\Big) = 0$. *Suppose, in addition, that the Fourier sine-transformation* $\hat{h}(\beta) = \sqrt{\frac{2}{\pi}} \int_0^\infty h(\tau) sin\beta\tau\, d\tau$

of the function

$$h(t) = \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{R^2}{2}}}}^{\infty} f\Big((a_1, x)_H, \dots, (a_m, x)_H, \frac{R^2}{2} - \frac{t^2}{4z^2}\Big) e^{-z^2} dz$$

exists, and $\beta^2 \hat{h}(\beta) \exp(\frac{R^2\beta^2}{2}) \in L_1(0, \infty)$.

We put also $\overline{\Omega} = \{x \in H : \|x\|_H^2 \le R^2\}$.

Then

$$V_2(t, x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{h}(\beta) e^{T(x)\beta^2} sint\beta \, d\beta \tag{14}$$

is a solution of problem (11)–(13).

Proof. Since, by the assumptions of the theorem, $f\Big(P_a x, \frac{R^2}{2}\Big) = 0$, we have

$$h(0) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} f\Big(P_a x, \frac{R^2}{2}\Big) e^{-z^2} dz = 0.$$

¿From (14), by direct computation, we deduce

$$\frac{\partial^2 V_2(t, x)}{\partial t^2} = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \beta^2 \hat{h}(\beta) e^{T(x)\beta^2} sint\beta \, d\beta. \tag{15}$$

Taking into account that

$$\Delta_L e^{T(x)\beta^2} = \beta^2 e^{T(x)\beta^2} \Delta_L T(x) = -\beta^2 e^{T(x)\beta^2},$$

we obtain

$$\Delta_L V_2(t, x) = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \beta^2 \hat{h}(\beta) e^{T(x)\beta^2} sint\beta \, d\beta. \tag{16}$$

The substitution of (15) and (16) into (11) gives an identity.

Setting $t = 0$ in (14), we obtain $V_2(0, x) = 0$.

At the boundary Γ we have $T(x) = 0$ that yields

$$V_2(t, x)\Big|_{\Gamma} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{h}(\beta) sint\beta \, d\beta = h(t),$$

where $h(t) = \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{R^2}{2}}}}^{\infty} f\Big(P_a x, \frac{R^2}{2} - \frac{t^2}{4z^2}\Big) e^{-z^2} dz.$ □

3. At the end, we consider the initial-boundary value problem with homogeneous boundary condition for the wave equation with the Lévy Laplacian, namely

$$\frac{\partial^2 V(t, x)}{\partial t^2} = \Delta_L V(t, x) \quad (t > 0, \ x \in \Omega), \tag{1}$$

$$V(0, x) = F(x), \tag{2}$$

$$V(t, x) = 0 \quad \text{on} \quad \Gamma, \tag{3}$$

where $F(x)$ is a given function, and $V(t, x) \in C\left([0, \infty), \overline{\Omega}\right) \bigcap C^{2,1}\left((0, \infty), \Omega\right)$.

Theorem 3. *Suppose, that* $F \in \mathfrak{C}^*$,

$$F(x) = f\Big((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2}\Big)$$

and the conditions of Theorems 1 and 2 are satisfied.

Then

$$\begin{aligned} V(t, x) = {} & \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f\Big((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big) e^{-z^2} dz \\ & - \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{h}(\beta) e^{T(x)\beta^2} sint\beta \, d\beta, \end{aligned} \tag{20}$$

is a solution of problem (17)–(19).

Proof. It follows from (20) and (8), (14) that

$$V(t,x) = V_1(t,x) - V_2(t,x).$$

The function $V(t,x)$ solves (17) $\big($since $\frac{\partial^2 V_1(t,x)}{\partial t^2} = \Delta_L V_1(t,x)$, $\frac{\partial^2 V_2(t,x)}{\partial t^2} = \Delta_L V_2(t,x)\big)$. It satisfies the initial condition (18) as $V(0,x) = F(x) - 0 = F(x)$, and the boundary condition (19), because of $V(t,x)\Big|_\Gamma = h(t) - h(t) = 0$. So the function $V(t,x)$ given by (20) is a solution of problem (17)–(19). □

3.2. **Boundary problem with homogeneous initial condition.** Now we shall deal with the boundary problem

$$\frac{\partial^2 W(t,x)}{\partial t^2} = \Delta_L W(t,x) \quad (t>0,\ x\in\Omega), \tag{4}$$

$$W(0,x) = 0, \tag{5}$$

$$W(t,x) = G(t,x) \quad \text{on} \quad \Gamma, \tag{6}$$

with homogeneous boundary condition, where $G(t,x)$ is a given function, and $W(t,x) \in C\left([0,\infty),\overline{\Omega}\right) \bigcap C^{2,1}\left((0,\infty),\Omega\right)$.

Theorem 4. *Let $\overline{\Omega}$ be a fundamental domain. Suppose that the function $G(t,x)$ is twice differentiable with respect to t and*

$$G(t,x) = g\Big(t, (b_1,x)_H, \dots, (b_n,x)_H, \frac{\|x\|_H^2}{2}\Big),$$

belongs to $\mathfrak{C}^*$ for every $t \in [0,\infty)$. Here $g(t,\xi_1,\dots,\xi_n,\zeta)$ is a function on R^{n+2} such that $g(0,\xi_1,\dots,\xi_n,\zeta) = 0$, $b_k \in H$, $k = 1,\dots,n$. It is assumed also that the functions $t^2 g(t,\xi_1,\dots,\xi_n,\zeta)$ and $t^2 \exp\left(\frac{R^2t^2}{\gamma}\right) \hat{g}(t,\xi_1,\dots,\xi_n,\zeta)$ are absolutely Lebesgue integrable in t over $[0,\infty)$.

Then the formula

$$W(t,x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{g}\Big(\gamma, (b_1,x)_H, \dots, (b_n,x)_H, T(x) + \frac{\|x\|_H^2}{2}\Big) e^{T(x)\gamma^2} sint\gamma \, d\gamma, \tag{24}$$

yields a solution of problem (21)–(23).

Here $\hat{g}(\gamma,\xi_1,\dots,\xi_n,\zeta) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\tau,\xi_1,\dots,\xi_n,\zeta) sin\gamma\tau \, d\tau$.

Proof. Let us rewrite the function $G(t,x)$ in the form $G(t,x) = g\big(t, P_b x, \frac{\|x\|_H^2}{2}\big)$, where P_b is a projection into the n-dimensional space spanned over vectors $b_1,\dots,b_n$. Since the functions $t^k g(t,\xi_1,\dots,\xi_n,\zeta)$, $k = 0,1,2$, are absolutely integrable, their Fourier sine-transformation in t exists.

We deduce from (24)

$$\frac{\partial^2 W(t,x)}{\partial t^2} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \gamma^2 \hat{g}\Big(\gamma, P_b x, T(x) + \frac{\|x\|_H^2}{2}\Big) e^{T(x)\gamma^2} sint\gamma \, d\gamma, \tag{25}$$

$$\begin{aligned} \Delta_L W(t,x) &= -\sqrt{\frac{2}{\pi}} \int_0^\infty \gamma^2 \hat{g}\Big(\gamma, P_b x, T(x) + \frac{\|x\|_H^2}{2}\Big) e^{T(x)\gamma^2} sint\gamma \, d\gamma \\ &+ \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{g}'_\zeta\Big(\gamma, P_b x, T(x) + \frac{\|x\|_H^2}{2}\Big) e^{T(x)\gamma^2} sin\, t\gamma \, d\gamma \Big[\Delta_L T(x) + \Delta_L \frac{\|x\|_H^2}{2}\Big] \\ &= -\sqrt{\frac{2}{\pi}} \int_0^\infty \gamma^2 \hat{g}\Big(\gamma, P_b x, T(x) + \frac{\|x\|_H^2}{2}\Big) e^{T(x)\gamma^2} sint\gamma \, d\gamma, \end{aligned} \tag{26}$$

because

$$\Delta_L e^{T(x)\gamma^2} = \gamma^2 e^{T(x)\gamma^2} \Delta_L T(x) = -\gamma^2 e^{T(x)\gamma^2}, \quad \Delta_L T(x) = -1, \quad \Delta_L \frac{\|x\|_H^2}{2} = 1.$$

Substituting (25) and (26) into (21) we derive an identity.

Setting $t=0$ in (24), we obtain $W(0,x)=0$.

At the boundary Γ we have $T(x)=0$ and $\|x\|_H^2=R^2$, hence (24) yields

$$W(t,x)\Big|_\Gamma = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat g\Big(\gamma, P_b x, \frac{\|x\|_H^2}{2}\Big) sint\gamma\, d\gamma = g\Big(t, P_b x, \frac{\|x\|_H^2}{2}\Big) = G(t,x).$$

□

3.3. **Initial-boundary value problem.** Consider the initial-boundary value problem for the wave equation with the Lévy Laplacian

$$\frac{\partial^2 U(t,x)}{\partial t^2} = \Delta_L U(t,x) \quad (t>0,\ x\in\Omega), \tag{27}$$

$$U(0,x) = F(x), \tag{28}$$

$$U(t,x) = G(t,x) \quad \text{on} \quad \Gamma, \tag{29}$$

where $F(x)$, $G(t,x)$ are given functions, and $U(t,x)\in C\left([0,\infty),\overline{\Omega}\right)\bigcap C^{2,1}\left((0,\infty),\Omega\right)$.

Assume that the conditions of Theorems 3 and 4 are satisfied. The Theorem 3, 4 imply the following assertion

Corollary. *The function*

$$\begin{aligned} U(t,x) = {} & \frac{2}{\sqrt{\pi}}\int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^\infty f\Big((a_1,x)_H,\dots,(a_m,x)_H, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\Big) e^{-z^2} dz \\ & - \sqrt{\frac{2}{\pi}}\int_0^\infty \hat h(\beta) e^{T(x)\beta^2} sint\beta\, d\beta \\ & + \sqrt{\frac{2}{\pi}}\int_0^\infty \hat g\Big(\gamma, (b_1,x)_H,\dots,(b_n,x)_H, \frac{R^2}{2}\Big) e^{T(x)\gamma^2} sint\gamma\, d\gamma, \end{aligned}$$

gives a solution of problem (27)–(29).

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