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HYPERSPACES OF CLOSED LIMIT SETS

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Abstract. We study Michael's lower semifinite topology and Fell's topology on the collection of all closed limit subsets of a topological space. Special attention is given to the subfamily of all maximal limit sets.

1. Introduction

The collection of all closed subsets of a topological space has been for long of interest to topologists and functional analysts. It seems that the modern investigation of the subject began with [10]. It is well known that there is a one-to-one correspondence between the closed two-sided ideals of a C^*-algebra and the closed subsets of its primitive ideal space as detailed in [4, Proposition 3.2.2]. Naturally, this correspondence attracted the interest of operator algebraists in the hyperspace of the closed subsets of a topological space. It led Fell to the definition in [6] of a topology on this hyperspace that is of significance in topology and several branches of analysis. Moreover, according to [2, Proposition 3.2], when one restricts this correspondence to the closed limit subsets of the primitive ideal space, a very interesting class of ideals is obtained. The wealth of information given in [1] on this class of ideals stimulated the present investigation and a significant portion of the results that appear here were proved in [1] for this special family of ideals of a C^*-algebra. However, no knowledge of the theory of C^*-algebras is required for the understanding of the following; we discuss the properties of two topologies on the collection of all the closed limit subsets of a topological space. All the definitions beyond the common knowledge of a topologist or an analyst are given in the next section. Of course, all our results are significant only for non Hausdorff spaces, as the primitive ideal spaces often are.

In section 3 we study the Michael's lower semifinite topology on the family of all closed limit sets. We establish that with this topology this hyperspace is a locally compact Baire space. We restrict the discussion to the collection of all maximal limit sets in section 4. The Fell topology and the lower semifinite topology coincide on this hyperspace. This hyperspace is also a Baire space and if the initial space is second countable and locally compact then the hyperspace of maximal limit sets is a G_δ subspace in the space of all closed limit sets equipped with the Fell topology. Section 5, which is independent of sections 3 and 4, contains a discussion of the space of maximal limit sets of the cartesian product of two topological spaces.

2. Preliminaries

For a topological space X we shall denote by $\mathcal{F}(X)$ the hyperspace of all its closed subsets and $\mathcal{F}'(X)$ will stand for the collection of all the non-void closed subsets of X. A subset L of X is called a limit set if there is a net that converges to all the points of L. By [5, Lemme 9], $L \subset X$ is a limit set if and only if every finite family of open subsets

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that intersect L has a non-void intersection. The collection of all the closed limit sets of X will be denoted by $\mathcal{L}(X)$ and we set $\mathcal{L}'(X) := \mathcal{L}(X) \cap \mathcal{F}'(X)$. It easily follows from the lemma quoted above and Zorn's lemma that each $L \in \mathcal{L}(X)$ is contained in a maximal limit set. Obviously, every maximal limit set is closed and non-void. $\mathcal{ML}(X)$ will denote the collection of all maximal limit sets. There is a natural map $\eta_X : X \to \mathcal{L}'(X)$ defined by $\eta_X(x) := \overline{\{x\}}$. This map is one to one if and only if X is a T_0 space.

Some of the results below are valid under the restriction that the topological space X is locally compact that is, each point in X has a fundamental system of compact neighbourhoods. Such spaces were called locally quasi-compact in [3, I, 9, Ex. 29].

For C be a compact subset and Φ a finite family (possibly empty) of open subsets of X let

$$\mathcal{U}(C, \Phi) := \{A \in \mathcal{F}(X) \mid A \cap C = \emptyset, A \cap O \neq \emptyset, O \in \Phi\}.$$

The collection of all such $\mathcal{U}(C, \Phi)$ forms a base for a topology on $\mathcal{F}(X)$ that was defined by Fell in [6] and which will be denoted here by τ_s. It was shown in [6] that with this topology $\mathcal{F}(X)$ is a compact space that is Hausdorff if X is locally compact. If X is locally compact and has a countable base then $(\mathcal{F}(X), \tau_s)$ is metrizable, see [5, Lemme 2].

The collection of all $\mathcal{U}(\emptyset, \Phi)$ when Φ runs through all the finite families of open subsets of X is the base of a T_0 topology on $\mathcal{F}(X)$, weaker than τ_s, which we shall denote by τ_w. It was called the lower semifinite topology in [10, Definition 9.1] and was further discussed in [8]. It is easily seen that if $\mathcal{B}$ is a base for the topology of X then the collection of all $\mathcal{U}(\emptyset, \Phi)$ when Φ runs through all the finite subfamilies of $\mathcal{B}$ is a base for $(\mathcal{F}(X), \tau_w)$. Thus, if X is second countable then $(\mathcal{F}(X), \tau_w)$ is also second countable. Clearly $\mathcal{F}'(X) = \mathcal{U}(\emptyset, \{X\})$ hence $\mathcal{F}'(X)$ is τ_w-open in $\mathcal{F}(X)$. The only τ_w-open subset of $\mathcal{F}(X)$ to which the empty subset of X belongs is $\mathcal{F}(X)$ itself so $\mathcal{F}'(X)$ is τ_w-dense in $\mathcal{F}(X)$ and $\mathcal{L}'(X)$ is τ_w-dense in $\mathcal{L}(X)$. Obviously, $\mathcal{ML}(X)$ is also τ_w-dense in $\mathcal{L}(X)$. For every $A \in \mathcal{F}(X)$ the τ_w-closure of $\{A\}$ is $\{B \in \mathcal{F}(X) \mid B \subset A\}$ and this entails the T_0 separation property for $(\mathcal{F}(X), \tau_w)$. The map η_X is τ_w continuous; it is a homeomorphism onto its image if X is T_0. Generalizing [1, Proposition 3.1], we claim that always the τ_w-closure of $\eta_X(X)$ is $\mathcal{L}(X)$. Indeed, it is easily seen that $A \in \mathcal{F}(X)$ is in the τ_w-closure of $\eta_X(X)$ if and only if every finite family of open subsets that intersect A has a non-void intersection that is, if and only if $A \in \mathcal{L}(X)$. In particular, $\mathcal{L}(X)$ is τ_w-closed hence also τ_s-closed. Thus $(\mathcal{L}(X), \tau_s)$ is a compact Hausdorff space. From the τ_w-density of $\eta_X(X)$ in $\mathcal{L}(X)$ it follows that $(\mathcal{L}(X), \tau_w)$ is connected when X is connected. However, trivial examples show that $(\mathcal{L}(X), \tau_s)$ need not be connected if X is connected.

Concerning the τ_s-convergence of nets, the following was proved in [11, Lemma H.2]:

Proposition 2.1. *Let $\{A_\iota\}$ be a net of closed subsets of the topological space X and $A \in \mathcal{F}(X)$. The net τ_s-converges to A if (a) given $x_\iota \in A_\iota$ such that the net $\{x_\iota\}$ converges to x, then $x \in A$, and (b) if $x \in A$ then there is a subnet $\{A_{\iota_\kappa}\}$ and there are points $x_{\iota_\kappa} \in A_{\iota_\kappa}$ such that $\{x_{\iota_\kappa}\}$ converges to x. When X is locally compact the converse is true too : the net $\{A_\iota\}$ τ_s-converges to A only if the conditions (a) and (b) hold.*

The characterization of the τ_s-convergence of nets given below is in line with our attempt to investigate the links between the two topologies on the hyperspace of closed subsets noted above. A net in a topological space was called by Fell primitive in [6] if the set of all its limits equals the set of all its cluster points. With this definition we have

Proposition 2.2. *Let X be a topological space. If $\{A_\iota\}$ is a primitive net in $(\mathcal{F}(X), \tau_w)$ and the set of all its τ_w-limits is $\{B \in \mathcal{F}(X) \mid B \subset A\}$ where $A \in \mathcal{F}(X)$ then $\{A_\iota\}$ τ_s-converges to A. If X is locally compact then the converse holds: a net $\{A_\iota\}$ that is τ_s-convergent to A in $\mathcal{F}(X)$ is primitive in $(\mathcal{F}(X), \tau_w)$ and the set of all its τ_w-limits is $\{B \in \mathcal{F}(X) \mid B \subset A\}$.*

Proof. Suppose $\{A_\iota\}$ is a τ_w-primitive net in $\mathcal{F}(X)$ and the set of all its limits is $\{B \mid B \subset A\}$. Let $\mathcal{U}(C,\Phi)$ be a basic τ_s-neighbourhood of A. If we assume that $\{A_\iota\}$ is not eventually in $\mathcal{U}(C,\Phi)$ then, by passing to a subnet and relabelling, we have $A_\iota \cap C \neq \emptyset$ for each ι. We choose points $x_\iota \in A_\iota \cap C$. There is a subnet $\{x_{\iota_\kappa}\}$ that converges to a point x in the compact set C. We claim that $\{A_{\iota_\kappa}\}$ τ_w-converges to $\overline{\{x\}}$. Indeed, let Φ_1 be a finite family of open subsets of X all of which intersect $\overline{\{x\}}$ that is, such that x belongs to the intersection V of all the sets in Φ_1. Then x_{ι_κ} is eventually in V. Thus, for κ large enough, $A_{\iota_\kappa} \cap V \neq \emptyset$ and the claim is established. Accordingly, $\overline{\{x\}}$ is a τ_w-cluster point of the primitive net $\{A_\iota\}$ hence $\overline{\{x\}} \subset A$. We got $A \cap C \neq \emptyset$, a contradiction.

Suppose now that X is locally compact and the net $\{A_\iota\}$ τ_s-converges to A. It follows readily from the definition of the topologies on $\mathcal{F}(X)$ that $\{A_\iota\}$ τ_w-converges to every closed subset B of X which is a subset of A. Assume that there is a subnet $\{A_{\iota_\kappa}\}$ that τ_w-converges to some $B \in \mathcal{F}(X)$ with $B \setminus A \neq \emptyset$ and let $x \in B \setminus A$. There is a compact set $C \subset X$ such that $x \in \operatorname{Int}(C) \subset C \subset X \setminus A$. $\mathcal{U}(C,\{X\})$ is a τ_s-neighbourhood of A hence $A_{\iota_\kappa} \cap C = \emptyset$ eventually. On the other hand, $\mathcal{U}(\emptyset,\{\operatorname{Int}(C)\})$ is a τ_w-neighbourhood of B hence $A_{\iota_\kappa} \cap \operatorname{Int}(C) \neq \emptyset$ and we got a contradiction. We have proved that each τ_w-cluster point of $\{A_\iota\}$ is a subset of A and we are done. □

3. The topology τ_w

First we want to establish the local compactness of $\mathcal{F}(X)$, $\mathcal{F}'(X)$, $\mathcal{L}(X)$, and $\mathcal{L}'(X)$ with their τ_w-topology when the space X is locally compact. The result for the first two spaces is likely to be known but we have no reference for it. The local compactness of $\mathcal{L}(X)$ and $\mathcal{L}'(X)$ was established when X is the primitive ideal space of a C^*-algebra in [1, Theorem 3.7] by using special properties of such spaces.

Lemma 3.1. *Let $C_1, \dots C_n$ be compact subsets of the topological space X. Then $\mathcal{S} := \{A \in \mathcal{F}(X) \mid A \cap C_i \neq \emptyset, 1 \leq i \leq n\}$ is τ_w-compact.*

Proof. Let $\{M_\alpha \mid \alpha \in \mathcal{A}\}$ be a net in $\mathcal{S}$ and $x_\alpha^i \in M_\alpha \cap C_i$. By passing to successive subnets we may suppose that each of the nets $\{x_\alpha^i \mid \alpha \in \mathcal{A}\}$, $1 \leq i \leq n$, converges to a point $x_i \in C_i$. Denote by M the closure of $\{x_1, \dots x_n\}$ and suppose $\mathcal{E} := \{U_1, \dots U_p\}$ is a finite family of open subsets of X such that $M \in \mathcal{U}(\emptyset, \mathcal{E})$. Then for each k, $1 \leq k \leq p$, there is $1 \leq i_k \leq n$ such that $x_{i_k} \in U_k$. Hence there is $\alpha_0 \in \mathcal{A}$ such that for all $1 \leq k \leq p$ and $\alpha > \alpha_0$ we have $x_\alpha^{i_k} \in U_k$. Thus, if $\alpha > \alpha_0$ then $M_\alpha \cap U_k \neq \emptyset$, $1 \leq k \leq p$. We have established that $\{M_\alpha\}$ converges weakly to M and clearly $M \in \mathcal{S}$. □

Theorem 3.2. *If X is a locally compact space then $\mathcal{F}(X)$, $\mathcal{F}'(X)$, $\mathcal{L}(X)$, and $\mathcal{L}'(X)$ are locally compact spaces with their τ_w topology.*

Proof. Suppose X is a locally compact space and let A be a closed subset of X. For a basic τ_w-neighbourhood $\mathcal{U}(\emptyset, \{U_i\}_{i=1}^n)$ of A we choose $x_i \in A \cap U_i$, $1 \leq i \leq n$. Let V_i be a compact neighbourhood of x_i contained in U_i and $W_i := \operatorname{Int}(V_i)$. Then

$$A \in \mathcal{U}(\emptyset, \{W_i\}_{i=1}^n) \subset \mathcal{V} := \{B \in \mathcal{F}(X) \mid B \cap V_i \neq \emptyset, 1 \leq i \leq n\} \subset \mathcal{U}(\emptyset, \{U_i\}_{i=1}^n).$$

Thus $\mathcal{V}$ is a neighbourhood of A that is compact by the preceding lemma. We have proved that $(\mathcal{F}(X), \tau_w)$ is locally compact.

As remarked above, $\mathcal{F}'(X)$ is τ_s-open in $F(X)$, $\mathcal{L}(X)$ is τ_w-closed, $\mathcal{L}'(X) = \mathcal{L}(X) \cap \mathcal{F}'(X)$ is relatively open in $\mathcal{L}(X)$ and the conclusion follows. □

The next result was stated in [1, Proposition 3.4] for the primitive ideal space of a C^*-algebra. However, the proof given there is valid for any topological space and we reproduce it here.

Proposition 3.3. *If X is a Baire topological space then $(\mathcal{L}(X), \tau_w)$ and $(\mathcal{L}'(X), \tau_w)$ are Baire spaces.*

Proof. For each natural number n let $\mathcal{U}_n$ be a τ_w-dense open subset of $\mathcal{L}(X)$. Since $\eta_X(X)$ is τ_w-dense in $\mathcal{L}(X)$ and η_X is τ_w-continuous, $\eta_X^{-1}(\mathcal{U}_n)$ is an open dense subset of X. From the hypothesis it follows that $\cap_{n\geq 1}\eta_X^{-1}(\mathcal{U}_n)$ is dense in X. But then

$$\eta_X(\cap_{n\geq 1}\eta_X^{-1}(\mathcal{U}_n)) = \eta_X(X)\bigcap(\cap_{n\geq 1}\mathcal{U}_n)$$

is τ_w-dense in $Li(X)$. In particular, $\cap_{n\geq 1}\mathcal{U}_n$ is τ_w-dense in $\mathcal{L}(X)$.

$\mathcal{L}'(X)$ is an open dense subset of $(\mathcal{L}(X), \tau_w)$ so it is a Baire space too. □

Proposition 3.4. *If X has a base consisting of open and compact sets then the same is true for $(\mathcal{F}(X), \tau_w)$ and its subspaces $\mathcal{F}'(X)$, $\mathcal{L}(X)$, and $\mathcal{L}'(X)$.*

Proof. Suppose $\mathcal{B}$ is a base for the topology of X consisting of open and compact sets. Then the collection of all the families $\mathcal{U}(\emptyset, \Phi)$ where Φ runs through all the finite subfamilies of $\mathcal{B}$ is a base for $(\mathcal{F}(X), \tau_w)$. Each $\mathcal{U}(\emptyset, \Phi)$ is τ_w-compact by Lemma 3.1. We get a base for $\mathcal{F}'(X)$ by requiring Φ to run through the nonempty finite subfamilies of $\mathcal{B}$. Intersecting each of the elements of the bases we got for $\mathcal{F}(X)$ and $\mathcal{F}'(X)$ with the τ_w-closed set $\mathcal{L}(X)$ we get bases as needed for $\mathcal{L}(X)$ and $\mathcal{L}'(X)$, respectively. □

4. The hyperspace $\mathcal{ML}(X)$

The next result generalizes [1, Theorem 4.2] where the framework is that of a certain family of ideals of a C^*-algebra and the proof uses C^*-algebraic methods. The "if" part of the statement is also a consequence of [5, Lemme 15].

Theorem 4.1. *The identity map $(\mathcal{L}(X), \tau_w) \to (\mathcal{L}(X), \tau_s)$ is continuous at $A \in \mathcal{L}(X)$ if and only if $A \in \mathcal{ML}(X)$.*

Proof. Suppose A is a maximal limit set. Let C be a compact subset of X and Φ a finite family of open subsets of X such that $A \in \mathcal{U}(C, \Phi)$. We claim that there is a finite family $\Psi \supset \Phi$ of open subsets of X each of which has a nonempty intersection with A and such that $\mathcal{U}(\emptyset, \Psi) \cap \mathcal{L}(X) \subset \mathcal{U}(C, \Phi) \cap \mathcal{L}(X)$. This, of course, will establish the continuity of the identity map at A.

Assume there is no such Ψ. Then for each finite family $\Psi \supset \Phi$ of open subsets of X such that every set in Ψ has a nonempty intersection with A there is $B_\Psi \in (\mathcal{U}(\emptyset, \Psi) \setminus \mathcal{U}(C, \Phi)) \cap \mathcal{L}(X)$. Denote the collection of all such families Ψ by $\mathbf{\Lambda}$ and order it by inclusion. Clearly $\Psi \in \mathbf{\Lambda}$ implies $B_\Psi \cap C \neq \emptyset$. Choose $x_\Psi \in B_\Psi \cap C$. The net $\{x_\Psi\}$ has a converging subnet to some point $x \in C$. We have $x \notin A$ hence $A \cup \{x\} \supsetneqq A$. We shall show that $A \cup \{x\}$ is a limit set hence $A \cup \overline{\{x\}} \in \mathcal{L}'(X)$, and this will yield a contradiction to the maximality of A.

Let $\mathcal{N}$ be the family of all the open neighbourhoods of x. We order $\mathcal{N} \times \mathbf{\Lambda}$ by defining $(V_1, \Psi_1) \prec (V_2, \Psi_2)$ if $V_1 \supset V_2$ and $\Psi_1 \subset \Psi_2$). Denote by $\mathbf{\Gamma}$ the collection of all the pairs $(V, \Psi) \in \mathcal{N} \times \mathbf{\Lambda}$ such that the finite family of open sets $\{V\} \cup \Psi$ has a nonempty intersection. For $(V_1, \Psi_1), (V_2, \Psi_2) \in \mathcal{N} \times \mathbf{\Lambda}$ there is $(V, \Psi) \in \mathbf{\Gamma}$ such that $(V_1, \Psi_1) \prec (V, \Psi)$ and $(V_1, \Psi_2) \prec (V, \Psi)$. Indeed, $V := V_1 \cap V_2$ is an open neighbourhood of x and $\Psi_1 \cup \Psi_2 \in \mathbf{\Lambda}$ hence there is $\Psi \in \mathbf{\Lambda}$ that satisfies $\Psi \supset \Psi_1 \cup \Psi_2$ and $x_\Psi \in V$. Thus $x_\Psi \in V \cap B_\Psi$ and since B_Ψ is a limit set that belongs to $\mathcal{U}(\emptyset, \Psi)$, the family of open sets $\{V\} \cup \Psi$ has a nonempty intersection by the previously quoted Lemme 9 of [5]. We got $(V, \Psi) \in \mathbf{\Gamma}$ as needed. In particular, $\mathbf{\Gamma}$ is a directed set with this order restricted to it. For each $(V, \Psi) \in \mathbf{\Gamma}$ we choose $y_{(V,\Psi)}$ in the intersection of the family $\{V\} \cup \Psi$. The net $\{y_{(V,\Psi)}\}$ converges to every point of $\{x\} \cup A$. It is clear that the net converges to x. Let now y be a point of A and W an open neighbourhood of y. With $\Psi_0 := \{W\} \cup \Phi$ we have $(X, \Psi_0) \in \mathcal{N} \times \mathbf{\Lambda}$. By the order property of $\mathbf{\Gamma}$ proved above there is $(V_1, \Psi_1) \in \mathbf{\Gamma}$ such that $(X, \Psi_0) \prec (V_1, \Psi_1)$. Clearly if $(V, \Psi) \in \mathbf{\Gamma}$ and $(V_1, \Psi_1) \prec (V, \Psi)$ then $W \in \Psi$ hence $y_{(V,\Psi)} \in \cap\{O \mid O \in \Psi\} \subset W$. We have proved that the net $\{y_{(V,\Psi)} \mid (V, \Psi) \in \mathbf{\Gamma}\}$ converges to y as claimed.

Let now L be a non-maximal closed limit set of X. There are $z \in X \setminus L$ and a net that converges to all the points of $L \cup \overline{\{z\}}$. The set L belongs to the τ_s-open set $\mathcal{U}(\{z\}, \{X\})$ but no τ_w-neighbourhood of L in $\mathcal{L}(X)$ is contained in $\mathcal{U}(\{z\}, \{X\})$ thus the identity map from $(\mathcal{L}(X), \tau_w)$ to $(\mathcal{L}(X), \tau_s)$ is not continuous at L. Indeed, if $\mathcal{U}(\emptyset, \Phi)$ is any basic τ_w-neighbourhood of L then $L \cup \overline{\{z\}} \in \mathcal{U}(\emptyset, \Phi) \cap \mathcal{L}(X)$ but $L \cup \overline{\{z\}} \notin \mathcal{U}(\{z\}, \{X\})$. $\square$

Corollary 4.2. *The restrictions of τ_w and τ_s to $\mathcal{ML}(X)$ coincide.*

A point y of a topological space Y is called separated in Y if, for every $z \in Y \setminus \overline{\{y\}}$, y and z have disjoint neighbourhoods; equivalently, $\overline{\{y\}}$ is a maximal limit set(see [5, Dèfinition 16]). It is proved in [5, Thèoréme 19] that if Y is a second countable locally compact Baire space then the subset of all separated points in Y is a dense G_δ. For any topological space X the density of the set of all separated points in $(\mathcal{L}(X), \tau_w)$ is an immediate corollary of the next result.

Theorem 4.3. *Let X be a topological space. An element A of $\mathcal{L}(X)$ is separated in $(\mathcal{L}(X), \tau_w)$ if and only if A is a maximal limit set.*

Proof. If $A \in \mathcal{L}(X)$ is not maximal then there is $A_1 \in \mathcal{L}(X)$ such that $A_1 \supsetneqq A$. Then A_1 does not belong to the τ_w-closure of $\{A\}$ in $\mathcal{L}(X)$. However, A is in the τ_w-closure of $\{A_1\}$ in $\mathcal{L}(X)$ hence A and A_1 cannot be separated by disjoint τ_w-open sets.

Suppose now that A is a maximal limit set and $A_1 \in \mathcal{L}(X)$ does not belong to the τ_w-closure of $\{A\}$ that is, A_1 is not included in A. Then $A \cup A_1 \in \mathcal{F}(X) \setminus \mathcal{L}(X)$. By [5, Lemme 9] there is a finite family Φ of open subsets of X such that each of them has a nonempty intersection with $A \cup A_1$ but the intersection of all the sets in Φ is void. Let Ψ be the subfamily of Φ consisting of those sets that have a nonempty intersection with A. Since $A_1 \in \mathcal{L}(X)$ we must have, by the above quoted lemma of Dixmier, $\Psi \neq \emptyset$. Similarly, $\Psi_1 := \Phi \setminus \Psi$ is not empty since $A \in \mathcal{L}(X)$. Now, $\mathcal{U}(\emptyset, \Psi) \cap \mathcal{L}(X)$ is a τ_w-neighbourhood of A in $\mathcal{L}(X)$ and $\mathcal{U}(\emptyset, \Psi_1) \cap \mathcal{L}(X)$ is a τ_w-neighbourhood of A_1 in $\mathcal{L}(X)$. We have

$$\mathcal{U}(\emptyset, \Psi) \cap \mathcal{U}(\emptyset, \Psi_1) \cap \mathcal{L}(X) = \emptyset$$

hence A and A_1 can be separated by disjoint τ_w-open sets. Indeed, if the above equality does not hold and $B \in \mathcal{U}(\emptyset, \Psi) \cap \mathcal{U}(\emptyset, \Psi_1) \cap \mathcal{L}(X)$ then

$$\cap\{V \mid V \in \Phi\} = (\cap\{V \mid V \in \Psi\}) \bigcap (\cap\{V \mid V \in \Psi_1\}) \neq \emptyset$$

since $B \in \mathcal{L}(X)$, a contradiction. $\square$

The following two propositions were stated and proved in [1] in the language of C^*-algebras. We only had to rewrite the proofs to be fit in a more general situation.

Proposition 4.4 ([1, Proposition 4.9])**.** *$\mathcal{ML}(X)$ is a Baire space if X is a Baire space.*

Proof. Let $\{\mathcal{V}_n\}$ be a sequence of τ_w-open subsets of $\mathcal{L}(X)$ such that every $\mathcal{U}_n := \mathcal{V}_n \cap \mathcal{ML}(X)$ is dense in $\mathcal{ML}(X)$. Since $\mathcal{ML}(X)$ is τ_w-dense in $\mathcal{L}(X)$ we get that each $\mathcal{V}_n$ is τ_w-dense in $\mathcal{L}(X)$. By Proposition 3.3, $\cap\mathcal{V}_n$ is τ_w-dense in $\mathcal{L}(X)$. Let now $\mathcal{U}$ be an open set in $\mathcal{ML}(X)$. Then $\mathcal{U} = \mathcal{V} \cap \mathcal{ML}(X)$, $\mathcal{V}$ being a τ_w-open set in $\mathcal{L}(X)$. There exist $B \in (\cap\mathcal{V}_n) \cap \mathcal{V}$ and $B_1 \in \mathcal{ML}(X)$ with $B \subset B_1$. Since B_1 belongs to any τ_w-open set of $\mathcal{L}(X)$ to which B belongs, we have

$$B_1 \in (\cap\mathcal{V}_n) \cap \mathcal{V} \cap \mathcal{ML}(X) = (\cap\mathcal{U}_n) \cap \mathcal{U}.$$

Hence $\cap\mathcal{U}_n$ is dense in $\mathcal{ML}(X)$. $\square$

Proposition 4.5 ([1, Corollary 4.6]). *If X is a second countable locally compact Baire space then the family $\{\overline{\{x\}} \mid x \text{ is separated in } X\}$ is dense in $\mathcal{ML}(X)$.*

Proof. As mentioned before, [5, Thèoréme 19] asserts that the set

$$T := \{x \in X \mid x \text{ is separated in } X\}$$

is dense in X. Since $\eta_X(X)$ is τ_w-dense in $\mathcal{L}(X)$ and η_X is τ_w-continuous we can infer that $\eta_X(T)$ is τ_w-dense in $\mathcal{L}(X)$. In particular, $\eta_X(T) = \eta_X(X) \cap \mathcal{ML}(X)$ is dense in $\mathcal{ML}(X)$. □

We shall have more to say about the set considered in the statement of Proposition 4.5 in Corollary 4.10.

Theorems 4.1 and 4.3 give us some information about the way $\mathcal{ML}(X)$ is imbedded in $\mathcal{L}(X)$. Theorem 4.8 will show us another aspect of this imbedding when the space is second countable. First we need two lemmas.

Lemma 4.6. *Let Y be a compact space, $M \subset Y \times Y$ and*

$$S(M) := \{y \in Y \mid \{y\} \times Y \subset M\}.$$

If M is open then $S(M)$ is open and if M is a G_δ set then $S(M)$ is a G_δ set too.

Proof. Suppose M is an open set. If $y \in S(M)$ then, by using the compactness of Y, we can infer that there are open subsets $\{U_i\}_{i=1}^n$ and $\{V_i\}_{i=1}^n$ of Y such that

$$\{y\} \times Y \subset \cup_{i=1}^n (U_i \times V_i) \subset M.$$

Then

$$y \in \cap_{i=1}^n U_i \subset S(M)$$

and $\cap_{i=1}^n U_i$ is open.

Suppose now M is a G_δ set, $M = \cap_1^\infty M_n$ with each M_n open in $Y \times Y$. Since $S(M) = \cap_1^\infty S(M_n)$ and $S(M_n)$ is open by the first part of the proof, the conclusion obtains. □

Lemma 4.7. *Let X be a locally compact space. Then*

$$\mathcal{E} := \{(A, B) \in \mathcal{L}(X) \times \mathcal{L}(X) \mid A \subset B\}$$

is $(\tau_s \times \tau_s)$-closed in $\mathcal{L}(X) \times \mathcal{L}(X)$.

Proof. Let $\{(A_\iota, B_\iota)\}$ be a net in $\mathcal{E}$ that $(\tau_s \times \tau_s)$-converges to (A, B). Given $x \in A$ there exists, by Proposition 2.1, a subnet $\{A_{\iota_\kappa}\}$ of $\{A_\iota\}$ and points $x_{\iota_\kappa} \in A_{\iota_\kappa} \subset B_{\iota_\kappa}$ such that $\{x_{\iota_\kappa}\}$ converges to x. Again by Proposition 2.1, $x \in B$ and we have shown $A \subset B$ that is $(A, B) \in \mathcal{E}$. □

Theorem 4.8. *If X is a second countable locally compact space then $\mathcal{ML}(X)$ is a G_δ subset of $(\mathcal{L}(X), \tau_s)$.*

Proof. Set

$$\mathcal{D} := \{(A, A) \mid A \in \mathcal{L}(X)\}, \quad \mathcal{E} := \{(A, B) \in \mathcal{L}(X) \times \mathcal{L}(X) \mid A \subset B\},$$

and

$$\mathcal{T} := \mathcal{L}(X) \times \mathcal{L}(X) \setminus (\mathcal{E} \setminus \mathcal{D}).$$

Then for $A \in \mathcal{L}(X)$ we have $A \in \mathcal{ML}(X)$ if and only if $\{A\} \times \mathcal{L}(X) \subset \mathcal{T}$. $\mathcal{T}$ is a G_δ set since $\mathcal{E} \setminus \mathcal{D}$ is an F_σ set. The conclusion follows now from Lemma 4.6. □

Remark 4.9. If X is a second countable locally compact space then $\mathcal{ML}(X)$ is a Baire space since it is a G_δ subset of the compact metrizable space $(\mathcal{L}(X), \tau_s)$.

Corollary 4.10. *If X is a second countable locally compact space in which every closed subset is a Baire space then $\{\overline{\{x\}} \mid x \quad \text{is separated in} \quad X\}$ is a G_δ subset of $\mathcal{L}(X)$; it is also a dense subset of $\mathcal{ML}(X)$.*

Proof. By [5, Thèoréme 7], $\eta_X(X)$ is a G_δ subset of $\mathcal{F}(X)$ hence it is a G_δ subset of $\mathcal{L}(X)$. Then $\{\overline{\{x\}} \mid x \quad \textit{is separated in} \quad X\} = \eta_X(X) \cap \mathcal{ML}(X)$ is a dense G_δ subset of $\mathcal{ML}(X)$ by Proposition 4.5 and a G_δ subset of $\mathcal{L}(X)$ by Theorem 4.8. □

Remark 4.11. The primitive ideal space of a separable C^*-algebra with its hull-kernel topology satisfies the hypothesis of Corollary 4.10.

5. Maximal limit sets of a cartesian product

Let now X_1 and X_2 be topological spaces. We are going to discuss $\mathcal{ML}(X_1 \times X_2)$ but first we shall establish two results for $\mathcal{F}(X_1) \times \mathcal{F}(X_2)$. Everywhere in this section $\phi : \mathcal{F}(X_1) \times \mathcal{F}(X_2) \to \mathcal{F}(X_1 \times X_2)$ will be the map given by $\phi(A_1, A_2) := A_1 \times A_2$. The continuity of ϕ was proved in [9, Lemma 1] but be reproduce here the simple proof for the sake of completeness.

Proposition 5.1. *Let $\mathcal{F}(X_1)$, $\mathcal{F}(X_2)$, and $\mathcal{F}(X_1 \times X_2)$ have their τ_w topologies. Then ϕ is continuous and its restriction to $\mathcal{F}'(X_1) \times \mathcal{F}'(X_2)$ is a homeomorphism onto its image (which is contained in $\mathcal{F}'(X_1 \times X_2)$).*

Proof. Let U be an open subset of $X_1 \times X_2$, say $U = \cup(U_\iota^1 \times U_\iota^2)$ where U_ι^j are open subsets of X_j. We have

$$\begin{aligned}\{(A_1, A_2) \in \mathcal{F}(X_1) \times \mathcal{F}(X_2) \mid \phi(A_1, A_2) \cap U \neq \emptyset\}\\ = \cup(\{A_1 \in \mathcal{F}(X_1) \mid A_1 \cap U_\iota^1 \neq \emptyset\} \times \{A_2 \in \mathcal{F}(X_2) \mid A_2 \cap U_\iota^2 \neq \emptyset\}).\end{aligned}$$

The latter is an open subset of $\mathcal{F}(X_1) \times \mathcal{F}(X_2)$ so the continuity of ϕ is established. Let now $V_1 \subset X_1$, $V_2 \subset X_2$ be open sets. Then

$$\begin{aligned}\phi(\{(A_1, A_2) \in \mathcal{F}'(X_1) \times \mathcal{F}'(X_2) \mid A_1 \cap V_1 \neq \emptyset, \quad A_2 \cap V_2 \neq \emptyset\})\\ = \phi(\mathcal{F}'(X_1) \times \mathcal{F}'(X_2)) \cap \{A \in \mathcal{F}'(X_1) \times X_2) \mid A \cap (V_1 \times V_2) \neq \emptyset\}.\end{aligned}$$

Since the restriction of ϕ to $\mathcal{F}'(X_1) \times \mathcal{F}'(X_2)$ is one to one this shows that it is also an open map with respect to the relative topology of its image and the proof is complete. □

A similar result is valid for the Fell topology but with an additional hypothesis on the spaces.

Proposition 5.2. *Let X_1 and X_2 be locally compact spaces. We suppose that $\mathcal{F}(X_1)$, $\mathcal{F}(X_2)$, and $\mathcal{F}(X_1 \times X_2)$ are endowed with their τ_s-topologies. Then ϕ is continuous and its restriction to $\mathcal{F}'(X_1) \times \mathcal{F}'(X_2)$ is a homeomorphism onto its image.*

Proof. As in the proof of Proposition 5.1 we let $U := \cup(U_\iota^1 \times U_\iota^2)$ where U_ι^j are open subsets of X_j. For the first step of the continuity proof we let C_k^j, $1 \leq k \leq n$, be compact subsets of X_j. Then

$$\begin{aligned}
&\big\{(A_1,A_2)\in\mathcal{F}(X_1)\times\mathcal{F}(X_2)\mid (A_1\times A_2)\cap\cup(C_k^1\times C_k^2)=\emptyset,\quad (A_1\times A_2)\cap U\neq\emptyset\big\}\\
&\quad=\cap_{k=1}^n\big\{(A_1,A_2)\in\mathcal{F}(X_1)\times\mathcal{F}(X_2)\mid (A_1\times A_2)\cap(C_k^1\times C_k^2)=\emptyset,\\
&\qquad (A_1\times A_2)\cap U\neq\emptyset\}\\
&\quad=\cap_{k=1}^n\big(\cup_\iota\big\{(A_1,A_2)\in\mathcal{F}(X_1)\times\mathcal{F}(X_2)\mid (A_1\times A_2)\cap(C_k^1\times C_k^2)=\emptyset,\\
&\qquad (A_1\times A_2)\cap(U_\iota^1\times U_\iota^2)\neq\emptyset\big\}\big)\\
&\quad=\cap_{k=1}^n\big\{\cup_\iota\big(\big\{A_1\in\mathcal{F}(X_1)\mid A_1\cap C_k^1=\emptyset,\quad A_1\cap U_\iota^1\neq\emptyset\big\}\\
&\qquad\times\big\{A_2\in\mathcal{F}(X_2)\mid A_2\cap U_\iota^2\neq\emptyset\big\}\big)\\
&\qquad\cup\cup_\iota\big(\big\{A_1\in\mathcal{F}(X_1)\mid A_1\cap U_\iota^1\neq\emptyset\big\}\times\big\{A_2\in\mathcal{F}(X_2)\mid A_2\cap C_k^2=\emptyset,\\
&\qquad A_2\cap U_\iota^2\neq\emptyset\big\}\big)\big\}.
\end{aligned}$$

Clearly the latter is an open subset of $\mathcal{F}(X_1)\times\mathcal{F}(X_2)$.

Let now C be an arbitrary compact subset of $X_1\times X_2$. To finish the proof of the continuity of ϕ we have to show that

$$\mathcal{U}:=\{(A_1,A_2)\in\mathcal{F}(X_1)\times\mathcal{F}(X_2)\mid (A_1\times A_2)\cap C=\emptyset,\quad (A_1\times A_2)\cap U\neq\emptyset\}$$

is open in $\mathcal{F}(X_1)\times\mathcal{F}(X_2)$. Suppose $(A_1^0,A_2^0)\in\mathcal{U}$. Then $C\subset(X_1\times X_2)\setminus(A_1^0\times A_2^0)$ hence there are compact subsets $\{C_k^j\}_{k=1}^n\}$ of X_j, $j=1,2$, such that

$$C\subset\cup_{k=1}^n\operatorname{int}(C_k^1\times C_k^2)\subset\cup_{k=1}^n(C_k^1\times C_k^2)\subset(X_1\times X_2)\setminus(A_1^0\times A_2^0).$$

Thus

$$\begin{aligned}
(A_1^0,A_2^0)\in\{&(A_1,A_2)\in\mathcal{F}(X_1)\times\mathcal{F}(X_2)\mid (A_1\times A_2)\cap\cup_{k=1}^n(C_k^1\times C_k^2)=\emptyset,\\
&(A_1\times A_2)\cap U\neq\emptyset\}
\end{aligned}$$

which is an open subset of $\mathcal{F}(X_1)\times\mathcal{F}(X_2)$ by the first part of the proof. Clearly

$$\begin{aligned}
\{&(A_1,A_2)\in\mathcal{F}(X_1)\times\mathcal{F}(X_2)\mid (A_1\times A_2)\cap\cup_{k=1}^n(C_k^1\times C_k^2)=\emptyset,\\
&(A_1\times A_2)\cap U\neq\emptyset\}\subset\mathcal{U}
\end{aligned}$$

and this shows that $\mathcal{U}$ is indeed open.

The remainder of the proof parallels that of the corresponding statement in Proposition 5.1. Let $C_j\subset X_j$ be compact and $V_j\subset X_j$ be open, $j=1,2$. We have

$$\begin{aligned}
&\phi\left(\{A_1\in\mathcal{F}'(X_1)\mid A_1\cap C_1=\emptyset,\quad A_1\cap V_1\neq\emptyset\}\times\{A_2\in\mathcal{F}'(X_2)\mid A_2\cap C_2=\emptyset,\right.\\
&\qquad\left.A_2\cap V_2\neq\emptyset\}\right)\\
&\quad=\phi(\mathcal{F}'(X_1)\times\mathcal{F}'(X_2))\cap\{A\in\mathcal{F}'(X_1\times X_2)\mid A\cap(C_1\times C_2)=\emptyset,\\
&\qquad A\cap(V_1\times V_2)\neq\emptyset\}
\end{aligned}$$

hence the restriction of ϕ to $\mathcal{F}'(X_1)\times\mathcal{F}'(X_2)$ is an open map onto its image. $\square$

We have $\phi(\mathcal{L}'(X_1)\times\mathcal{L}'(X_2))\subset\mathcal{L}'(X_1\times X_2)$; indeed, if $L_j\in\mathcal{L}'(X_j)$ then every collection of open subsets of $X_1\times X_2$ that intersect $L_1\times L_2$ has a non-void intersection hence $\phi(L_1,L_2)\in\mathcal{L}'(X_1\times X_2)$. Very simple examples show that the inclusion can be strict. For instance, if X_1 and X_2 are T_1 spaces but not Hausdorff then one sees immediately that the inclusion is strict. However the situation with the spaces of maximal limit sets is different. We shall use below the restriction of the τ_w topology to the spaces of maximal limit sets that we discuss.

Theorem 5.3. *Let X_1 and X_2 be topological spaces and ϕ the map defined above. Then the restriction of ϕ to $\mathcal{ML}(X_1)\times\mathcal{ML}(X_2)$ is a homeomorphism onto $\mathcal{ML}(X_1\times X_2)$.*

Proof. We have only to show that $\phi(\mathcal{ML}(X_1) \times \mathcal{ML}(X_2)) = \mathcal{ML}(X_1 \times X_2)$, the rest being a consequence of Proposition 5.1. Denote by p_j the projection of $X_1 \times X_2$ onto X_j, $j = 1, 2$, and let L be a closed limit subset of $X_1 \times X_2$. Then $p_j(L)$ is a limit subset of X_j, $L \subset p_1(L) \times p_2(L)$ and if L is a maximal limit set then one must have $L = p_1(L) \times p_2(L)$. Clearly in this case $p_j(L)$ must also be a maximal limit subset of X_j. Thus we have proved $\phi(\mathcal{ML}(X_1) \times \mathcal{ML}(X_2)) \supset \mathcal{ML}(X_1 \times X_2)$. Let now $M_j \in \mathcal{ML}(X_j)$, $j = 1, 2$. We have $(M_1 \times M_2) \in \mathcal{L}'(X_1 \times X_2)$. If M is a limit subset of $X_1 \times X_2$ with $(M_1 \times M_2) \subset M$ then $p_j(M)$ is a limit subset of X_j and

$$(M_1 \times M_2) \subset M \subset p_1(M) \times p_2(M).$$

The maximality of M_j forces $M_j = p_j(M)$ and we got $M_1 \times M_2 = M$. Thus $M_1 \times M_2$ is a maximal limit subset of $X_1 \times X_2$ and we have shown $\phi(\mathcal{ML}(X_1) \times \mathcal{ML}(X_2)) \subset \mathcal{ML}(X_1 \times X_2)$. $\square$

Remark 5.4. Theorem 5.3 allows us to give an alternative proof for Theorem 1.1 of [7] on the minimal primal ideal space of the spatial tensor product of two C^*-algebras. The details, in a more general setting, will appear elsewhere.

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