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A SPECTRAL DECOMPOSITION IN ONE CLASS OF NON-SELFADJOINT OPERATORS

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Abstract. In this paper, a class of special finite dimensional perturbations of Volterra operators in Hilbert spaces is investigated. The main result of the article is finding necessary and sufficient conditions for an operator in a chosen class to be similar to the orthogonal sum of a dissipative and an anti-dissipative operators with finite dimensional imaginary parts.

1. Integral estimates of the resolvent norms

1.1. Let B be an arbitrary Volterra dissipative operator with trivial kernel, acting on a separable Hilbert space $\mathfrak{H}$. We note that in this article the operator is called dissipative, if it satisfies $\operatorname{Im} B := \frac{1}{2i}(B - B^*) \geqslant 0$. In what follows we assume that $\operatorname{Im} B$ is an operator of rank n, i.e., the dimenion of the non-selfadjoint subspace $\mathcal{L} := (\operatorname{Im} B)\mathfrak{H}$ is equal to n. The main objects of the investigation make operators of the type

$$Kh = B^*h + \sum_{k=1}^{n}(h, f_k)g_k, \quad h \in \mathfrak{H}, \tag{1.1}$$

where $\{g_k\}_1^n$ is a some basis of the subspace $\mathcal{L}$, $f_k (1 \leqslant k \leqslant n)$ are arbitrary vectors of space $\mathfrak{H}$.

We briefly discuss only two reasons that make a study of operators of type (1.1) of some interest. Firstly, for a lot of concrete examples of operators B the problem of the corresponding operator K roots vectors unconditional basis being property is of the independent interest. For example, the eigen vectors of the operator K can be the vectors exponents, the property of being an unconditional basis indication of which find the important applications in problems of control theory for systems with the distributed parameters [1].

Secondly, the investigation of the spectral problems of type

$$\frac{dx(t)}{dt} = i\lambda\mathcal{H}(t)x(t), \quad x(0) = Ax(a), \quad a > 0,$$

where $\mathcal{H}(t)$ is non-negative almost everywhere on $[0, a]$ matrix-valued function, amounts to studying the operators K in case all the vectors f_k belongs to $\mathcal{L}$ also.

The main result of this paper (theorem 2.1) is finding the conditions, under which the operator K is similar to the orthogonal sum of dissipative and anti-dissipative operators with the finite-dimensional imaginary parts. Now these operators are investigated enough complete [2], [3].

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As result of simple calculations we'll get

$$K(I-zK)^{-1}h = B^*(I-zB^*)^{-1}h + \sum_{k=1}^{n} f_k(h,z)(I-zB^*)^{-1}g_k, \quad h \in \mathfrak{H}, \tag{1.2}$$

where the functionals $f_k(h,z)$ are determined by formulae

$$f_k(h,z) = \sum_{j=1}^{n} \Psi_{kj}(z)\left((I-zB^*)^{-1}h, f_j\right), \quad \Psi(z) := \Phi^{-1}(z), \quad 1 \leqslant k \leqslant n, \tag{1.2$'$}$$

where, in turn, the elements of $\Phi(z)$ are calculating in the way

$$\Phi_{kj}(z) = \delta_{jk} - z\left((I-zB^*)^{-1}g_j, f_k\right), \quad 1 \leqslant k, \quad j \leqslant n. \tag{1.3}$$

For the formulation this section main result of we'll need the next concepts. Firstly, the (A_2)-Muckenhoupt condition for almost everywhere non-negative on the real axis $(n \times n)$-matrix weight W is in the [4]

$$\sup_{\Delta}\left\{\left\|\left(\frac{1}{|\Delta|}\int_{\Delta} W(x)dx\right)^{1/2}\left(\frac{1}{|\Delta|}\int_{\Delta} W^{-1}(x)dx\right)^{1/2}\right\|\right\} < \infty, \tag{A_2}$$

where Δ is an arbitrary interval of real axis and $|\Delta|$ is its length.

The second concept is connected with the theory of non-selfadjoint operators. Let B be Volterra dissipative operator with n-dimensional imaginary part, i.e.

$$\frac{1}{i}(B-B^*)h = \sum_{k=1}^{n}(h,\varphi_k)\varphi_k, \quad h \in \mathfrak{H}. \tag{1.4}$$

The entire matrix-valued function Θ, which elements are determined by equalities

$$\Theta_{jk}(z) = \delta_{kj} + iz\left((I-zB)^{-1}\varphi_k, \varphi_j\right), \quad 1 \leqslant k, \quad j \leqslant n \tag{1.5}$$

is called the characteristic matrix-valued function of operator B. If in these formulae we'll turn to the another system of vectors $\{\varphi_k\}$, then the according characteristic matrix-valued function is got from $\Theta(z)$ by multiplication from the left and from the right on the constant unitary matrix. We note, that matrix-valued function $\Theta(z)$ is inner in the $\mathbb{C}_+$, i.e.

$$\Theta(z)\Theta^*(z) - E_n \leqslant 0, \quad z \in \mathbb{C}_+, \quad \Theta(x)\Theta^*(x) - E_n = 0, \quad x \in \mathbb{R},$$

where as E_n the identity matrix is denoted.

With every operator K of type (1.1) we'll connect the matrix weight

$$W(x) := \Phi(x)\Phi^*(x), \quad x \in \mathbb{R}, \tag{1.6}$$

where Φ is determined by formulae (1.3). Further, the entire function $\Delta(z) = \det \Phi(z)$ roots set we denote as Λ. It follows from the formula (1.2), that $\sigma(K) = \left\{\lambda_k^{-1} : \lambda_k \in \Lambda\right\} \cup \{0\}$, moreover, the numbers λ_k^{-1} belong to the discrete spectrum of operator K.

The next result plays the important role in this paper constructions.

Theorem 1.1. *We assume, the operator K of type (1.1) doesn't have the real eigenvalues. Then, if the matrix weight $W(x)$ is determined by equality (1.6) and satisfies the condition (A_2), then the integral estimation*

$$\int_{\mathbb{R}} \left\|K(I-xK)^{-1}h\right\|^2 dx \leqslant M\|h\|^2, \quad h \in \mathfrak{H} \tag{1.7}$$

holds, and here M is a some constant. Conversely, let for all $h \in \mathfrak{H}$ the unequality (1.7) holds and let the characteristic matrix-valued function $\Theta(z)$ of operator B is such that elements of matrix $e^{-i\delta z}\Theta(z)$ are bounded in $\mathbb{C}_+$ under some $\delta > 0$. Then the matrix weight $W(x)$ satisfies (A_2)-Muckenhoupt condition.

We'll presuppose some subsidiary statements to theorem 1.1 proof. We'll turn to the functional model of operator B for it. As $\operatorname{Ker} B = \{0\}$, then operator B is unitary equalent to the operator

$$(Bh)(z) := z^{-1}\left(h(z) - h(0)\right), \tag{1.8}$$

acting in the model space $\mathcal{K}_\Theta := H^2_+(\mathbb{C}^n) \ominus \Theta H^2_+(\mathbb{C}^n)$, where $H^2_+(\mathbb{C}^n)$ is Hardy vector class in $\mathbb{C}_+$, and Θ is a characteristic matrix-valued function of operator B [5]. It isn't difficult to verify the operator B^* acts by formula

$$(B^*h)(z) = z^{-1}\left(h(z) - \Theta(z)h(0)\right), \quad h \in \mathcal{K}_\Theta. \tag{1.9}$$

We note, that every operator of type (1.1) is unitary equal to the operator

$$Kh = z^{-1}\left(h(z) - \Theta(z)h(0)\right) + \sum_{k=1}^{n}(h, f_k)g_k \tag{1.10}$$

in space $\mathcal{K}_\Theta$, where f_k are the arbitrary vectors from $\mathcal{K}_\Theta$ $(1 \leqslant k \leqslant n)$, and vectors g_k are defined by equalities

$$g_k = z^{-1}(E_n - \Theta(z))c_k, \quad 1 \leqslant k \leqslant n. \tag{1.11}$$

In these formulae the vectors system $\{c_k\}_1^n$ runs through the set of all the basises of space $\mathbb{C}^n$. Really, it follows from formulae (1.8), (1.9), that

$$\frac{B - B^*}{i}h = i\frac{E_n - \Theta(z)}{z}h(0) = i\frac{E_n - \Theta(z)}{z}\sum_{k=1}^{n}(h(0), e_k)e_k,$$

where e_k $(1 \leqslant k \leqslant n)$ are the standard orths of space $\mathbb{C}^n$. If we assume $\varphi_k = z^{-1}(E_n - \Theta(z))e_k$, we'll get

$$\frac{B - B^*}{i}h = i\sum_{k=1}^{n}(h(0), e_k)_{\mathbb{C}^n}\varphi_k = \frac{1}{2\pi}\sum_{k=1}^{n}(h, \varphi_k)_{\mathcal{K}_\Theta}\varphi_k,$$

i.e. the subspace of model operator non-selfadjointion $\mathcal{L}$ is stretched on the system of vectors $\{\varphi_k\}_1^n$. In such a way, an arbitrary basis of subspace $\mathcal{L}$ consists of vectors (1.11). It is known [6], that under the unitary equivalence of operator to its functional model, the subspace of non-selfadjointion is transferring into the subspace of model operator non-selfadjointion. So, every operator of considered class is unitary equivalent to some operator of type (1.10).

1.2. *In what follows we'll denote as Q_n the set of operators K in separable Hilbert space $\mathfrak{H}$, which are defined by formulae (1.1).* With the every such operator we'll connect the mapping

$$\mathcal{D}(z, h) := -\frac{1}{2\pi i}\operatorname{row}\left\{\left((I - zB^*)^{-1}g_k, h\right)_{\mathfrak{H}}\right\}_1^n. \tag{1.12}$$

We note, in this formula vectors $\{g_k\}_1^n$ make a basis of subspace $\mathcal{L} := (\operatorname{Im} B)\mathfrak{H}$. Moreover, in the next formulation the norm in $\mathbb{C}^n$ is Euclidian one, i.e. if $\alpha = \operatorname{row}\{\alpha\}_1^n$, then $\|\alpha\|^2 = \sum_{k=1}^n |\alpha_k|^2$.

Lemma 1.1. *There exist such constants $m, M > 0$, that for all $h \in \mathfrak{H}$ the two-sided estimation*

$$m\|h\|^2_{\mathfrak{H}} \leqslant \int_{\mathbb{R}} \|\mathcal{D}(x, h)\|^2_{\mathbb{C}^n}\, dx \leqslant M\|h\|^2_{\mathfrak{H}}$$

holds.

Proof. In power of the theorem about the unitary equivalence of functional model it is enough to proove lemma for the operator B^* acting by formula (1.9) in the space $\mathcal{K}_\Theta$. The vectors g_k are given by equalities (1.11).

Step 1. At first we'll proove the correctness of equalities

$$g_k = (I + iB^*)\mathbb{P}_\Theta \frac{c_k}{x+i}, \quad 1 \leqslant k \leqslant n, \tag{1.13}$$

where $\mathbb{P}_\Theta$ is the orthoprojector from $H^2_+(\mathbb{C}^n)$ onto $\mathcal{K}_\Theta$. Let $\mathbb{P}_-$ be the orthoprojector from $L^n_2(\mathbb{R})$[1] onto Hardy class $H^2_-(\mathbb{C}^n)$. Taking into consideration the formula $\mathbb{P}_\Theta = \Theta\mathbb{P}_-\Theta^*$, we'll get

$$\mathbb{P}_\Theta \frac{c_k}{x+i} = \Theta\mathbb{P}_- \frac{\Theta^*(x)c_k}{x+i} = \Theta(x)\frac{\Theta^*(x) - \Theta^*(i)}{x+i}c_k = \frac{(E_n - \Theta(x)\Theta^*(i))}{x+i}c_k, \quad 1 \leqslant k \leqslant n.$$

From (1.9) the formula

$$(I + iB^*)h = z^{-1}\left((z+i)h(z) - i\Theta(z)h(0)\right)$$

is following. Therefore

$$\begin{aligned}(I + iB^*)\mathbb{P}_\Theta \frac{c_k}{x+i} &= z^{-1}\left((E_n - \Theta(z)\Theta^*(i))\,c_k - \Theta(z)\left(E_n - \Theta^*(i)\right)c_k\right) \\ &= z^{-1}\left(E_n - \Theta(z)\right)c_k = g_k,\end{aligned}$$

i.e. the equalities (1.13) are proved.

Step 2. Let us proove, that for all $z \in \mathbb{C}_-$ the formulae

$$(I + zB^*)^{-1}g_k = \mathbb{P}_\Theta \frac{c_k}{x+z}, \quad 1 \leqslant k \leqslant n \tag{1.14}$$

hold.

Really, it is easily concluding from (1.9), that

$$(I + zB^*)^{-1}g_k = \frac{\lambda g_k(\lambda) + z\Theta(\lambda)\Theta^{-1}(-z)g_k(-z)}{z+\lambda}, \quad 1 \leqslant k \leqslant n.$$

Taking into consideration (1.13) we get

$$\begin{aligned}(I + zB^*)^{-1}g_k &= (I + zB^*)^{-1}(I + zB^*)\mathbb{P}_\Theta \frac{c_k}{x+i} \\ &= \frac{\lambda+i}{\lambda+z}\mathbb{P}_\Theta \frac{c_k}{x+i} + \frac{z-i}{\lambda+z}\Theta(\lambda)\Theta^{-1}(-z)\mathbb{P}_\Theta \frac{c_k}{x+i}, \quad \lambda \in \mathbb{C}_+.\end{aligned}$$

We remark, the representations

$$\mathbb{P}_\Theta \frac{c_k}{x+i} - \frac{c_k}{\lambda+i} = \Theta(\lambda)h_k(\lambda), \quad h_k \in H^2_+(\mathbb{C}^n), \quad 1 \leqslant k \leqslant n$$

hold.

Therefore, it follows from the previous equality, that

$$(I + zB^*)^{-1}g_k = \frac{\lambda+i}{\lambda+z}\frac{c_k}{\lambda+i} + \Theta(\lambda)\left(\frac{\lambda+i}{\lambda+z}h_k(\lambda) + \frac{z-i}{\lambda+z}\Theta^{-1}(-z)\mathbb{P}_\Theta \frac{c_k}{x+i}\right).$$

As $\mathcal{K}_\Theta = H^2_+(\mathbb{C}^n) \ominus \Theta H^2_+(\mathbb{C}^n)$, the equalities (1.14)

$$(I + zB^*)^{-1}g_k = \mathbb{P}_\Theta(I + zB^*)^{-1}g_k = \mathbb{P}_\Theta \frac{c_k}{x+z}, \quad 1 \leqslant k \leqslant n$$

follow from it.

Step 3. For each vector $h(\lambda) = \operatorname{col}(h_k(\lambda))_1^n$ from space $\mathcal{K}_\Theta$ and for each $z \in \mathbb{C}_-$, taking account of (1.14) we'll calculate the inner products

$$-\frac{1}{2\pi i}\left((I - zB^*)^{-1}g_k, h\right)_{\mathcal{K}_\Theta} = -\frac{1}{2\pi i}\left(\mathbb{P}_\Theta \frac{c_k}{x-z}, h\right)_{\mathcal{K}_\Theta} = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{(c_k, h(x))}{x-z}dx.$$

[1]As $L^n_2(\mathbb{R})$ the standard space L_2 of $\mathbb{C}^n$-valued functions on real axis is denoted.

If we input the notations

$$h^*(z) = \text{row}\,\left(\overline{h_j(\bar{z})}\right)_1^n, \quad c_k = \text{col}\,(c_{kj})_{j=1}^n, \quad 1 \leqslant k \leqslant n,$$

then we can transform

$$(c_k, h(x))_{\mathbb{C}^n} = \sum_{j=1}^n c_{kj} h_j^*(x) = \left\{h^*(x)\,{}^tC\right\}_k,$$

where C is the matrix composed of columns $c_1, c_2, \dots, c_n$, ${}^tC$ is a transposed matrix C, and as $\{h^*(x)C\}_k$ the k component of line $h^*(x)C$ is denoted.

So, from (1.12) the formula

$$\mathcal{D}(z,h)({}^tC)^{-1} = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{h^*(x)}{x-z}dx = h^*(z), \quad z \in \mathbb{C}_- \tag{1.15}$$

follows.

Therefore, the equality

$$\int_{\mathbb{R}} \left\|\mathcal{D}(x,h)({}^tC)^{-1}\right\|_{\mathbb{C}^n}^2 dx = \int_{\mathbb{R}} \|h^*(x)\|_{\mathbb{C}^n}^2\, dx = \|h\|_{\mathcal{K}_\Theta}^2$$

holds, and the statement of lemma follows from it. □

Let us remind that the entire matrix-valued function $\Phi(z)$ is defined by formulae (1.3), and the functionals $f_k(h,z)$ are computed by formulae (1.2').

Lemma 1.2. *Let K be an arbitrary operator of class Q_n without real eigenvalues. The matrix weight $W(x) := \Phi(x)\Phi^*(x)$, $x \in \mathbb{R}$ satisfies the (A_2) condition if and only if the constant $M > 0$ such that for all $h \in \mathfrak{H}$*

$$\int_{\mathbb{R}} \sum_{k=1}^n |f_k(h,x)|^2\, dx \leqslant M\,\|h\|_{\mathfrak{H}}^2 \tag{1.16}$$

exists.

Proof. Step 1. If the vector-valued function $l(x) = \text{col}\,(l_k(x))_1^n$ is continious and finite on $\mathbb{R}$, then vector φ of type

$$\varphi = \int_{\mathbb{R}} \sum_{k=1}^n (I - zB^*)^{-1} g_k l_k(x) dx \tag{1.17}$$

belongs to the space $\mathfrak{H}$. For each $h \in \mathfrak{H}$ taking account of lemma 1.1 we have

$$\begin{aligned}|(\varphi,h)| &\leqslant \int_{\mathbb{R}} \Big|\sum_{k=1}^n \left((I - xB^*)^{-1} g_k, h\right) l_k(x)\Big| dx \leqslant 4\pi^2 \int_{\mathbb{R}} \|\mathcal{D}(x,h)\|_{\mathbb{C}^n} \|l(x)\|_{\mathbb{C}^n}\, dx \\ &\leqslant 4\pi^2 \Big(\int_{\mathbb{R}} \|\mathcal{D}(x,h)\|^2\, dx\Big)^{1/2} \Big(\int_{\mathbb{R}} \|l(x)\|^2 dx\Big)^{1/2} \leqslant M_1 \Big(\int_{\mathbb{R}} \|l(x)\|^2 dx\Big)^{1/2} \|h\|.\end{aligned}$$

Therefore integral (1.17) can be given a sense for each vector-valued function $l \in L_2^n(\mathbb{R})$, moreover,

$$\|\varphi\|^2 \leqslant M_1^2 \|l\|^2. \tag{1.18}$$

Step 2. We'll input under consideration the vector-valued function $f(h,z) := \text{col}\,\{f_k(h,z)\}_1^n$ and calculate it's value for vector $h = \varphi$ (see formula (1.7)). At first we notice, that if

$$h = \sum c_k (I - \lambda B^*)^{-1} g_k, \quad c_k \in \mathbb{C}, \quad c = \text{col}\,(c_k)_1^n,$$

then, taking account of formulae (1.2'), (1.3), we get

$$f_k(h,z) = \sum_{j=1}^{n} \Psi_{kj}(z) \sum_{m=1}^{n} c_m \left((I - zB^*)^{-1}(I - \lambda B^*)^{-1} g_m, f_j\right)$$
$$= \sum_{m=1}^{n} \sum_{j=1}^{n} \Psi_{kj}(z)(\lambda - z)^{-1} \left(\Phi_{jm}(z) - \Phi_{jm}(\lambda)\right) c_m.$$

We note, that here the next variant of Hilbert identity

$$(I - zB^*)^{-1}(I - \lambda B^*)^{-1} = (z - \lambda)^{-1} \left(z(I - zB^*)^{-1} - \lambda(I - \lambda B^*)^{-1}\right)$$

was used.

The received formulae can be rewritten in the vector form

$$f(h,z) = (\lambda - z)^{-1}\Phi^{-1}(z)\left(\Phi(z) - \Phi(\lambda)\right)c.$$

Now let φ be defined by formula (1.17), in which l run through the space $L_2^n(\mathbb{R})$. It follows from the previous equality, that

$$f(\varphi, z) = \int_{\mathbb{R}} f\Big(\sum_{k=1}^{n} l_k(y)(I - yB^*)^{-1} g_k, z\Big) dy = \Phi^{-1}(z) \int_{\mathbb{R}} \frac{\Phi(z) - \Phi(y)}{y - z} l(y) dy.$$

Step 3. Now let the estimation (1.16) holds. We'll consider it on the vectors φ of type (1.7). Using the calculated value $f(\varphi, z)$ and inequality (1.18) we'll find

$$\int_{\mathbb{R}} \|f(\varphi, x)\|_{\mathbb{C}^n}^2 \, dx = \int_{\mathbb{R}} \Big\|\Phi^{-1}(x) \int_{\mathbb{R}} \frac{\Phi(x) - \Phi(y)}{y - x} l(y) dy\Big\|_{\mathbb{C}^n}^2 dx$$
$$\leqslant M\|h\|_{\mathfrak{H}}^2 \leqslant M M_1^2 \int_{\mathbb{R}} \|l(x)\|_{\mathbb{C}^n}^2 dx. \tag{1.19}$$

Taking account of boundness of Hilbert transform $\mathcal{H}$, from (1.19) we conclude the estimate

$$\int_{\mathbb{R}} \left\|\Phi^{-1}(x)\mathcal{H}\Phi(y)l(y)\right\|_{\mathbb{C}^n}^2 dx \leqslant M_2 \int_{\mathbb{R}} \|l(x)\|_{\mathbb{C}^n}^2 \, dx$$

for all $l \in L_2^n(\mathbb{R})$. It follows from here [4], that weight $\left(\Phi^{-1}(x)\right)^* \Phi^{-1}(x)$ and weight $\Phi(x)\Phi^*(x)$ satisfy the (A_2) condition on $\mathbb{R}$ both.

Step 4. Conversely, let weight $W(x)$ satisfies the (A_2) condition. Then operator $\Phi^{-1}\mathcal{H}\Phi$ is bounded in the space $L_2^n(\mathbb{R})$ [4] and, therefore, the estimate (1.19) holds, i.e.

$$\int_{\mathbb{R}} \|f(\varphi, x)\|_{\mathbb{C}^n}^2 \, dx \leqslant M M_1^2 \int_{\mathbb{R}} \|l(x)\|_{\mathbb{C}^n}^2 \, dx, \tag{1.20}$$

where φ and l are connected by equality (1.17). As we remarked yet, we may consider that in (1.17) B^* acts in $\mathcal{K}_\Theta$ by formula (1.9), and vectors g_k are defined by equalities (1.11).

Let C be the matrix composed of columns $c_1, c_2, \ldots, c_n$, which input in formula (1.11). We assume

$$l(x) = -\frac{1}{2\pi i} C^{-1} \varphi_0(x), \tag{1.21}$$

where φ_0 is an arbitrary function from $\mathcal{K}_\Theta$, and we'll calculate the vector-valued function φ according in force of (1.17). Taking account of formulae (1.12), (1.15) we'll get for each

$h \in \mathcal{K}_\Theta$

$$(\varphi, h)_{\mathcal{K}_\Theta} = -\frac{1}{2\pi i}\int_{\mathbb{R}} \sum_{k=1}^{n} \left((I - zB^*)^{-1} g_k, h\right) \left\{C^{-1}\varphi_0(x)\right\}_k dx$$
$$= \int_{\mathbb{R}} \sum_{j=1}^{n} \left\{\mathcal{D}(x,h)C^{-1}\right\}_j \left\{\varphi_0(x)\right\}_j dx$$
$$= \int_{\mathbb{R}} \sum_{j=1}^{n} \left\{\varphi_0(x)\right\}_j \left\{h^*(x)\right\}_j dx = (\varphi_0, h)_{\mathcal{K}_\Theta},$$

i.e. $\varphi = \varphi_0$. In such way, if in (1.20) $l \in \mathcal{K}_\Theta$, then φ run through all the space $\mathcal{K}_\Theta$ and in force of (1.21)

$$\int_{\mathbb{R}} \|l(x)\|^2 dx \leqslant M_2 \int_{\mathbb{R}} \|\varphi_0(x)\|^2 dx = M_2 \int_{\mathbb{R}} \|\varphi(x)\|^2 dx,$$

i.e. the inequality (1.16) holds. □

Let us remind, that the characteristic matrix-valued function $\Theta(z)$ of operator B is determined by formulae (1.4), (1.5).

Lemma 1.3. *If under some $\delta > 0$ the elements of matrix $e^{-i\delta z}\Theta(z)$ are bounded in $\mathbb{C}_+$, then for each basis $\{g_k\}_1^n$ of non-selfadjointion space $\mathcal{L}$ the constant $\alpha > 0$, such that*

$$\Big\| \sum_{k=1} c_k (I - xB^*)^{-1} g_k \Big\|_{\mathfrak{H}}^2 \geqslant \alpha \|C\|_{\mathbb{C}^n}^2, \quad x \in \mathbb{R}$$

for each vector $C := \operatorname{col}(c_k)_1^n$, exists.

Proof. Without loss of generality, we can consider $g_k = \varphi_k$ $(1 \leqslant k \leqslant n)$, where the basis $\{\varphi_k\}_1^n$ is contained in formulae (1.4), (1.5). Then the equality [6]

$$E_n - \Theta(z)\Theta^*(z) = \operatorname{Im} z R(z), \quad z \in \mathbb{C} \tag{1.22}$$

holds. Here the elements of matrix $R(z)$ are determined by formulae

$$R_{kj}(z) = \left((I - \bar{z}B^*)^{-1} g_k, (I - \bar{z}B^*)^{-1} g_j\right), \quad 1 \leqslant k, \quad j \leqslant n.$$

Therefore it follows, from lemma condition and (1.22), that under some $\eta > 0$

$$\Big\| \sum_{k=1}^{n} c_k \left(I - (x - i\eta)B^*\right)^{-1} g_k \Big\|^2 = \sum_{k,j=1}^{n} c_k R_{kj}(x + i\eta)\bar{c}_j \geqslant \alpha_0 \sum_{k=1}^{n} |c_k|^2, \tag{1.23}$$

where $\alpha_0 > 0$. As $\operatorname{Ker} B = \{0\}$ and the operator B is dissipative one, then there exist non-bounded densely given operator $(B^*)^{-1}$ which is also dissipative one. Therefore, the semigroup $U(t) := \exp\left\{i(B^*)^{-1}t\right\}$ is contractive and nilpotent [7], and, consequently,

$$(2\pi)^{-1} \int_{\mathbb{R}} \left\|B^*(I - xB^*)^{-1} h\right\|^2 dx = \int_0^\infty \|U(t)h\|^2 dt \leqslant M\|h\|^2, \quad h \in \mathfrak{H}. \tag{1.24}$$

It is easily follows from here, that for each vector $g \in \mathfrak{H}$ the two-sided estimation

$$\Big\|\left(I - (x - i\eta)B^*\right)^{-1} g\Big\| \asymp \left\|(I - xB)^{-1} g\right\|, \quad x \in \mathbb{R} \tag{1.25}$$

holds. At last we assume $g = \sum_{k=1}^n c_k g_k$ here and take account of inequality (1.23). □

The proof of theorem 1.1. Let weight $W(x)$ satisfies the matrix Muckenhoupt condition. As $\|\Theta(z)c\| \leqslant \|c\|$, $z \in \mathbb{C}_+$, $c \in \mathbb{C}^n$ [6], then it follows from (1.22), that $\|(I + (x - i\eta)B^*)^{-1} g_k\|$, $1 \leqslant k \leqslant n$ is bounded on $\mathbb{R}$ and, therefore in force of (1.25),

$$\left\|(I - xB^*)^{-1} g_k\right\| \leqslant M, \quad 1 \leqslant k \leqslant n.$$

Now the estimate (1.7) is easily following from formulae (1.2), (1.16), (1.24).

Conversely, let for all $h \in \mathfrak{H}$ the inequality (1.7) holds. It follows from formula (1.2) taking account of (1.24), that

$$\int_{\mathbb{R}} \Big\| \sum_{k=1}^{n} f_k(h,x)(I - xB^*)^{-1} g_k \Big\|_{\mathfrak{H}}^2 dx \leqslant M\|h\|^2, \quad h \in \mathfrak{H}. \tag{1.26}$$

Applying lemma 1.3, we'll get

$$\alpha \sum_{k=1}^{n} |f_k(h,x)|^2 \leqslant \| \sum_{k=1}^{n} f_k(h,x)(I - xB^*)^{-1} g_k\|^2, \quad h \in \mathfrak{H}, \quad x \in \mathbb{R}.$$

Now the estimate (1.16) follows from (1.26) and we take account of lemma 1.2. □

The proved theorem will be used in the next paragraph in theorem 2.1 proof.

2. The spectral decomposition of class Q_n operators

2.1. In this paragraph we'll continue the class Q_n operators investigation. The further progress is connected with studying of vector-valued functions

$$K(h,z) := \mathrm{col}\, \left\{ \left((I - zK)^{-1} h, g_j \right) \right\}_1^n, \quad h \in \mathfrak{H} \tag{2.1}$$

properties, where K is an arbitrary operator of type (1.1). As a result of elementary calculations we get

$$\left((I - zK)^{-1} h, g_j \right) = \left((I - zB^*)^{-1} h, g_j \right) + z \sum_{k=1}^{n} f_k(h,z) \left((I - zB^*)^{-1} g_k, g_j \right). \tag{2.2}$$

We'll put in consideration the column

$$K_0(h,z) = \mathrm{col}\, \left\{ \left((I - zB^*)^{-1} h, g_j \right) \right\}_1^n, \quad z \in \mathbb{C}, \quad h \in \mathfrak{H}.$$

We'll remind, that the vectors system $\{g_k\}_1^n$ forms a basis of subspace $(\operatorname{Im} B)\mathfrak{H}$ in this formula.

Lemma 2.1. *The next statements are correct:*

1) *the constants $m, M > 0$ such that*

$$m\|h\|_{\mathfrak{H}}^2 \leqslant \int_{\mathbb{R}} \|K_0(h,x)\|_{\mathbb{C}^n}^2 \, dx \leqslant M\|h\|_{\mathfrak{H}}^2, \quad h \in \mathfrak{H}$$

exist;

2) *for each $h \in \mathfrak{H}$*

$$K_0(h,x) \in H_-^2(\mathbb{C}^n), \quad \Theta(x) K_0(h,x) \in H_+^2(\mathbb{C}^n), \quad x \in \mathbb{R},$$

where Θ is the operator B characteristic matrix-valued function.

Proof. We'll consider the line

$$\begin{aligned} \overline{{}^t K_0(h,-\bar z)} &= \mathrm{row}\, \left\{ \left(g_j, (I + \bar z B^*)^{-1} h \right) \right\}_1^n \\ &= \mathrm{row}\, \left\{ \left(I - z(-B) \right)^{-1} g_j, h \right\}_1^n = -2\pi i \mathcal{D}_1(z,h), \end{aligned} \tag{2.3}$$

where $\mathcal{D}_1(z,h)$ is defined by formula (1.12) for operator $B_1 := (-B)^*$. We'll remark, that $\{g_j\}_1^n$ is also a basis of subspace $(\operatorname{Im} B_1)\mathfrak{H}$ and the easily verified equality

$$\Theta_1(z) = \Theta^*(-\bar z), \quad z \in \mathbb{C},$$

where Θ_1 is a characteristic matrix-valued function of operator B_1 holds.

Now it is clear, that from lemma 1.1 and from equality (2.3) the first statement of lemma follows. Further, we conclude from formula (1.15), that components of line

$\overline{{}^tK_0(h,-\bar z)}$, $h\in\mathfrak{H}$, belong to Hardy class H^2_- and, therefore, $K_0(h,x)\in H^2_-(\mathbb{C}^n)$. Further, it follows again from (1.15) for the transformation $\mathcal{D}_1(z,h)$, that components of line

$$\mathcal{D}_1(z,h)\Theta_1(z) = -\frac{1}{2\pi i}\cdot\overline{{}^tK_0(h,-\bar z)}\Theta^*(-\bar z)$$

belong to H^2_+. It is equaivalent to fact $\Theta(x)K_0(h,x)\in H^2_+(\mathbb{C}^n)$ for all $h\in\mathfrak{H}$. □

Lemma 2.2. *If the matrix weight $W(x)=\Phi(x)\Phi^*(x)$, $x\in\mathbb{R}$ satisfies the (A_2) condition, then the constant $M>0$, such that*

$$\int_{\mathbb{R}}\|K(h,x)\|^2_{\mathbb{C}^n}\,dx\leqslant M\|h\|^2_{\mathfrak{H}},\quad h\in\mathfrak{H},$$

exists.

Proof. Without loss of generality we can consider that in (2.1) $g_k=\varphi_k$ $(1\leqslant k\leqslant n)$, where basis $\{\varphi_k\}_1^n$ is contained in formulae (1.4), (1.5). Therefore from (2.2) taking account of (1.5) we conclude the equalities

$$\big((I-zK)^{-1}h,g_j\big)=\big((I-zB^*)^{-1}h,g_j\big)-i\sum_{k=1}^n\big(\delta_{jk}-\Theta^*_{jk}(\bar z)\big)\,f_k(h,z),\quad 1\leqslant j\leqslant n,$$

which can be rewritten in a vector form

$$K(h,z)=K_0(h,z)-i\,(E_n-\Theta^*(\bar z))\,f(h,z),\quad h\in\mathfrak{H},\tag{2.4}$$

where $f(h,z)=\operatorname{col}\{f_k(h,z)\}_1^n$, $\Theta(z)$ is the operator B characteristic matrix-valued function. As $\Theta(z)$ is an inner one in $\mathbb{C}_+$, then

$$\|K(h,x)\|_{\mathbb{C}^n}\leqslant\|K_0(h,x)\|_{\mathbb{C}^n}+2\|f(h,x)\|_{\mathbb{C}^n}.$$

Now the statement of lemma follows from the lemma 2.1 and lemma 1.2. □

2.2. In this article we consider the problem of conditions under which the lower bound

$$m\|h\|^2_{\mathfrak{H}}\leqslant\int_{\mathbb{R}}\|K(h,x)\|^2_{\mathbb{C}^n}\,dx,\quad h\in\mathfrak{H}\tag{2.5}$$

holds, where m is a some positive constant. We'll start from the factorizations of the entire matrix-valued function Φ, which elements are defined by equalities (1.3).

Lemma 2.3. *Let the entire matrix-valued function Φ corresponds to operator $K\in Q_n$. Then*

1) *in the domain $\mathbb{C}_+$ the factorization*

$$\Phi(z)\Theta(z)=w_+(z)Q_+(z)$$

holds. Here w_+ is the an outer matrix-valued function and Q_+ is the inner one [2] in $\mathbb{C}_+$.

2) *in the domain $\mathbb{C}_-$ the factorization*

$$\Phi(z)=w_-(z)Q_-(z)$$

holds. Here w_- is the outer matrix-valued function and Q_- is the inner one in $\mathbb{C}_-$.

Proof. It follows from equalities (1.3) and (1.12), that the columns of matrix $z^{-1}(E_n-\Phi(z))$ lie in the image of transform $\mathcal{D}(z,h)$. It follows from (1.15), that parameters of vector-valuated function $\mathcal{D}(z,h)\Theta(z)$ belong to H^2_+. Therefore the elements of matrix $z^{-1}\left(E_n-\Phi(z)\right)\Theta(z)$ belong to H^2_+ and, consequently, the elements of matrix $(z+i)^{-1}\Phi(z)\Theta(z)$ have this property also. In such way, the factorization [8].

$$(z+i)^{-1}\Phi(z)\Theta(z)=\mathring{w}_+(z)Q_+(z),\quad z\in\mathbb{C}_+$$

is correct. Here $\overset{\circ}{w}_+$ is an outer matrix-valued function and Q_+ is an inner one in a domain $\mathbb{C}_+$. As $w_+(z) := (z+i)\overset{\circ}{w}_+(z)$ is outer one also, then the first statement of lemma is proved. The second statement is prooving analogously. □

Let v be some inner in $\mathbb{C}_+$ matrix-valued function of order n. In the model space $\mathcal{K}_v = H^2_+(\mathbb{C}^n) \ominus vH^2_+(\mathbb{C}^n)$ we'll consider the operator

$$(T_a\varphi)(x) = \mathbb{P}_v e^{-iax}\varphi(x), \quad \varphi \in \mathcal{K}_v,$$

where $\mathbb{P}_v$ is the orthoprojector from $H_2^+(\mathbb{C}^n)$ onto $\mathcal{K}_v$. As for each $h_+ \in H^2_+(\mathbb{C}^n)$ the equality $(\varphi, \Theta(x)h_+) = 0$ holds, then

$$\big(e^{-iax}\varphi(x), \Theta(x)h_+\big) = \big(\varphi(x), \Theta(x)e^{iax}h_+(x)\big) = 0.$$

It follows from here, that the formula

$$(T_a\varphi)(x) = \mathbb{P}_+ e^{-iax}\varphi(x), \quad \varphi \in \mathcal{K}_v \tag{2.6}$$

is correct. Here $\mathbb{P}_+$ is the orthoprojector from $L_2^n(\mathbb{R})$ onto $H^2_+(\mathbb{C}^n)$.

It is assumed, that $v(z)$ is analytical in some neighbourhood $z = 0$ and the condition $v(0) = E_n$ holds.

Lemma 2.4. *If the condition*

$$\inf_{\operatorname{Im}\lambda>0} \big\{|\det v(\lambda)| + \big|e^{ia\lambda} - 1\big|\big\} > 0$$

holds, then 1 does not belong to operator T_a spectrum.

Proof. In space $\mathcal{K}_v$ we'll consider the semigroup of contraction operators [3]

$$T_t = \mathbb{P}_+ e^{-itx}\varphi(x), \quad \varphi \in \mathcal{K}_v$$

and we'll compute the next integral, assuming $\operatorname{Im}\lambda > 0$

$$\int_0^\infty e^{i\lambda t} T_t\varphi dt = \mathbb{P}_+ \int_0^\infty e^{i(\lambda - xt)}\varphi(x)dt = -i\mathbb{P}_+ \frac{\varphi(x)}{x-\lambda} = -i\frac{\varphi(x) - \varphi(\lambda)}{x-\lambda}, \quad \varphi \in \mathcal{K}_v.$$

On the other hand we'll consider the operator

$$(Af)(z) = zf(z) - \lim_{y\to\infty} iyf(iy) \tag{2.7}$$

on the maximal by inclusion domain of definition in space $\mathcal{K}_v$. The simple calculations show, that

$$(A - \lambda I)^{-1}\varphi = \frac{\varphi(x) - \varphi(\lambda)}{x-\lambda}.$$

From here it follows [7], that $T_t = \exp\{-iAt\}$ and it is necessary to formulate the conditions, under which $1 \notin \sigma(T_a^*)$. It is made with the help of theorem about the mapping of spectrum in functional calculus Sz.-Nagy-Foias [9]. For the contraction operator

$$V = (A^* - iI)(A^* + iI)^{-1}$$

we have

$$T_a^* = \exp\{iA^*a\} = u(V), \quad u(z) = \exp\left\{-\frac{1+z}{1-z}a\right\} \tag{2.8}$$

with the help of the standard transform [9] we'll turn from space $\mathcal{K}_v$ to space

$$\mathcal{K}_w(\mathcal{D}) := H^2_+(\mathcal{D}) \ominus wH^2_+(\mathcal{D}), \quad w(z) := v\left(i(1+z)(1-z)^{-1}\right), \quad z \in \mathcal{D},$$

where $H^2_+(\mathcal{D})$ is a Hardy class in the unit disk $\mathcal{D}$. Coming from the formula (2.7) it isn't difficult to verify that operator V is defined by equality

$$(Vf)(z) = \mathbb{P}_w zf(z), \quad f \in \mathcal{K}_w(\mathcal{D}),$$

where $\mathbb{P}_w$ is the orthoprojector from $H^2_+(\mathcal{D})$ onto $\mathcal{K}_w(\mathcal{D})$. We'll denote the minimal function of contraction V as $m_V(z)$, $z \in \mathcal{D}$. Then from (2.8) $1 \notin \sigma(T_a^*)$ follows if and only if [9] the condition

$$\inf_{z\in\mathcal{D}} \{|m_V(z)| + |u(z) - 1|\} > 0 \tag{2.9}$$

holds. As $\det w(z)$ is divided into $m_V(z)$ in algebra H^∞ [2], then from the condition

$$\inf_{z\in\mathcal{D}} \{|\det w(z)| + |u(z) - 1|\} > 0$$

the (2.9) follows. Now it is left to make a change $z = (\lambda - i)(\lambda + i)^{-1}$, $\lambda \in \mathbb{C}_+$ in the last inequality. □

In the constructions what follow we'll use formula (2.4) for the vector-valued function $K(h, z)$, i.e.

$$K(h, z) = K_0(h, z) - i\,(E_n - \Theta^*(\bar z))\, f(h, z),$$

where $K_0(h, z) = \operatorname{col}\left\{\left((I - zB^*)^{-1}h, g_j\right)\right\}$, the parameters $f_k(h, z)$ of column $f(h, z)$ are defined by formulae (1.2'), $\Theta(z)$ is a characteristic matrix-valued function of operator B. We'll input the column of entire functions $F(h, z) := \operatorname{col}\left\{\left((I - zB^*)^{-1}h, f_j\right)\right\}_1^n$ and notice, that

$$f(h, z) = \Phi^{-1}(z)F(h, z), \quad z \notin \Lambda. \tag{2.10}$$

We'll remind, that as Λ we denote the sequence of roots of equation $\det \Phi(z) = 0$. We'll input the notations

$$\Lambda_\pm := \Lambda \cap \mathbb{C}_\pm; \quad \mu_k^+ := \overline{\lambda_k^-}, \quad \lambda_k^- \in \Lambda_-; \quad \mu_k^- := \overline{\lambda_k^+}, \quad \lambda_k^+ \in \Lambda_+. \tag{2.10'}$$

Further, as $b_+(z)$ we'll denote the Blaschke product in $\mathbb{C}_+$ with zeroes on sequence $\{\mu_k^+\}$. Analogously, let $b_-(z)$ be Blaschke product in $\mathbb{C}_-$ with zeroes $\{\mu_k^-\}$. We note, that both products are built taking account of $\det \Phi(z)$ zeroes multiplicity.

The next lemma will be proved in that special case when $\Theta(z) = e^{iaz}E_n$. Moreover, we assume $\Lambda \cap \mathbb{R} = \varnothing$.

Lemma 2.5. *Let the operator $K \in Q$ is such that $\Theta(z) = e^{iaz}E_n$ and let the weight $W(x) = \Phi(x)\Phi^*(x)$, $x \in \mathbb{R}$ satisfies the (A_2) matrix condition. Then if the conditions*

$$\inf_{\operatorname{Im}\lambda>0} \left\{|b_+(\lambda)| + |e^{ia\lambda} - 1|\right\} > 0, \qquad \inf_{\operatorname{Im}\lambda<0} \left\{|b_-(\lambda)| + |e^{-ia\lambda} - 1|\right\} > 0$$

hold, then the estimate (2.5) holds.

Proof. Step 1. As $\Theta(z) = e^{iaz}E_n$, then it follows from (2.4), (2.10), that

$$K(h, z) = K_0(h, z) - i(1 - e^{-iaz})\Phi^{-1}(z)F(h, z), \quad h \in \mathfrak{H}. \tag{2.11}$$

The existance of factorization $\Phi(z) = w_-(z)Q_-(z)$, $z \in \mathbb{C}_-$ follows from lemma 2.3. Therefore

$$\Phi^{-1}(x)F(h, x) = Q_-^{-1}(x)w_-^{-1}(x - i0)F(h, x), \quad x \in \mathbb{R}$$

and it follows from (1.16), that

$$\int_{\mathbb{R}} \left\|w_-^{-1}(x - i0)F(h, x)\right\|^2 dx \leqslant M\|h\|^2, \quad h \in \mathfrak{H}. \tag{2.11'}$$

Also we note, that $W(x) = \Phi(x)\Phi^*(x) = w_-(x - i0)w_-^*(x - i0)$, where w_- is the outer matrix-valued function. Further, the parameters of $F(h, z)$ are the entire functions of exponential type, not overestimated a. Therefore it follows from (2.11′) [10], that

$$w_-^{-1}(x - i0)F(h, x) \in H^2_-(\mathbb{C}^n), \quad h \in \mathfrak{H}.$$

In domain $\mathbb{C}_+$ we'll consider the inner matrix-valued function

$$\Theta_+(z) := Q_-^*(\bar z), \quad z \in \mathbb{C}_+. \tag{2.12}$$

Then the functions of type

$$\mathbb{P}_+\Phi^{-1}(x)F(h,x) = \mathbb{P}_+\Theta_+(x+i0)w_-^{-1}(x-i0)F(h,x)$$

are orthogonal to subspace $\Theta_+(x)H_+^2(\mathbb{C}^n)$, i.e. they belong to the model space $\mathcal{K}_{\Theta_+}$. Taking account of lemma 2.1 and equality (2.11) we come to the lower bound

$$\int_{\mathbb{R}} \|K(h,x)\|^2_{\mathbb{C}^n}\,dx \geqslant \int_{\mathbb{R}} \|\mathbb{P}_+K(h,x)\|^2_{\mathbb{C}^n}\,dx \geqslant \int_{\mathbb{R}} \left\|(1-\mathbb{P}_+e^{-iax})\mathbb{P}_+\Phi^{-1}(x)F(h,x)\right\|^2 dx.$$

If we assume the condition of lemma 2.4 holds, i.e.

$$\inf_{\operatorname{Im}\lambda>0}\left\{|\det\Theta_+(\lambda)| + |e^{ia\lambda}-1|\right\} > 0,$$

then previous estimate can be continued

$$\int_{\mathbb{R}} \|K(h,x)\|^2_{\mathbb{C}^n}dx \geqslant m\int_{\mathbb{R}} \|\mathbb{P}_+\Phi^{-1}(x)F(h,x)\|^2dx. \tag{2.13}$$

It follows from lemma 2.3 and equality (2.12), that

$$\overline{\det\Phi(\bar z)} = f(z)\det\Theta_+(z), \quad z\in\mathbb{C}_+,$$

where $f(z)$ is some outer function. From this equality $\det\Theta_+(z) = e^{i\alpha z}b_+(z)$, $\alpha\geqslant 0$ follows [11]. Therefore from the lemma 2.5 condition we conclude, that the conditions of lemma 2.4 hold, i.e. estimate (2.13) holds.

Step 2. We remind that the factorization

$$\Phi(z)e^{iaz} = w_+(z)Q_+(z)$$

holds in domain $\mathbb{C}_+$. So, from (2.11) under $z\in\mathbb{C}_+$ we find

$$e^{iaz}K(h,z) = e^{iaz}K_0(h,z) - i(e^{iaz}-1)Q_+^{-1}(z)w_+^{-1}(z)e^{iaz}F(h,x).$$

We'll input the inner matrix-valued function in domain $\mathbb{C}_-$

$$\Theta_-(z) := Q_+^*(\bar z), \quad z\in\mathbb{C}_-. \tag{2.13$'$}$$

Then we'll get the representation

$$e^{iax}K(h,x) = e^{iax}K_0(h,x) - i(e^{iax}-1)\Theta_-(x-i0)w_+^{-1}(x+i0)F(h,x)e^{iax}. \tag{2.14}$$

As from (1.16) we conclude the estimate

$$\int_{\mathbb{R}} \left\|w_+^{-1}(x+i0)F(h,x)e^{iax}\right\|^2 dx \leqslant M\|h\|^2, \quad h\in\mathfrak{H},$$

then it follows from the paper [10] results again, that

$$w_+^{-1}(x+i0)F(h,x)e^{iax}\in H_-^2(\mathbb{C}^n), \quad h\in\mathfrak{H}.$$

Therefore, functions of type

$$\varphi_-(x) := \mathbb{P}_-\Theta_-(x-i0)w_+^{-1}(x+i0)F(h,x)e^{iax}, \quad h\in\mathfrak{H}$$

belongs to the space $H_-^2(\mathbb{C}^n)\ominus\Theta_-H_-^2(\mathbb{C}^n)$.

Now from (2.14) and lemma 2.1 we conclude the lower bound

$$\int_{\mathbb{R}} \|K(h,x)\|^2\,dx \geqslant \int_{\mathbb{R}} \|P_-K(h,x)\|^2\,dx \geqslant \int_{\mathbb{R}} \left\|(1-\mathbb{P}_-e^{iax})\varphi_-(x)\right\|^2 dx.$$

From lemma 2.4 analog for operator

$$(V_a\varphi_-)(x) := \mathbb{P}_-e^{iax}\varphi_-(x), \quad \varphi_-\in H_-^2(\mathbb{C}^n)\ominus\Theta_-H_-^2(\mathbb{C}^n)$$

it follows, that in case

$$\inf_{\operatorname{Im}\lambda<0}\left\{|\det\Theta_-(\lambda)| + |e^{-ia\lambda}-1|\right\} > 0 \tag{2.15}$$

the previous estimate can be continued

$$\begin{aligned}\int_{\mathbb{R}} \|K(h,x)\|^2 dx &\geqslant m \int_{\mathbb{R}} \|\varphi_-(x)\|^2 dx \\ &= m \int_{\mathbb{R}} \|\mathbb{P}_- \Theta_-(x) w_+^{-1}(x+i0) F(h,x)\|^2 dx = m \int_{\mathbb{R}} \|\mathbb{P}_- \Phi^{-1}(x) F(h,x)\|^2 dx.\end{aligned} \tag{2.16}$$

Now we'll prove that condition (2.15) holds. Really, it follows from lemma 2.3 and equality (2.13), that

$$e^{-inaz}\overline{\det \Phi(\bar{z})} = g(z) \det \Theta_-(z), \quad z \in \mathbb{C}_-,$$

where $g(z)$ is some outer function in $\mathbb{C}_-$. Therefore, $\det \Theta_-(z) = e^{-i\beta z} b_-(z)$, $\beta \geqslant 0$, $z \in \mathbb{C}_-$ and, so, inequality (2.15) is a corollary of proved lemma conditions.

Step 3. We'll assume, that the estimate (2.5) doesn't hold, i.e. the sequence h_n, such that $\|h_n\| = 1$ and

$$\int_{\mathbb{R}} \|K(h_n, x)\|^2 dx \to 0, \quad n \to \infty$$

exists. Then it follows from (2.12) and (2.16), that

$$\Phi^{-1}(x) F(h_n, x) \to 0, \quad n \to \infty$$

in metric of space $L_2^n(\mathbb{R})$. Now from (2.11) we conclude, that $K_0(h_n, x) \to 0$ in $L_2^n(\mathbb{R})$ and in force of lemma 2.1 $h_n \to 0$, which is impossible. □

Remark. If the part of Fredholm spectrum Λ_- is an empty or finite set, then the condition

$$\inf_{\operatorname{Im}\lambda > 0} \left\{ |\det \Theta_+(\lambda)| + |e^{ia\lambda} - 1| \right\} > 0$$

concluding the estimate (2.13) is certainly realized. Therefore in this case the first inequality in the lemma 2.5 formulation can be excepted. Analogously, if the set Λ_+ is empty or finite, the second inequality in the lemma 2.5 formulationn can be excepted.

2.3. In the reasoning what follow, we'll denote the operator K of class Q_n with $\Theta(z) = e^{iaz} E_n$ as K_a, and a corresponding vector-valued function of type (2.1) as $K_a(h,z)$. So, in conditions of lemma 2.5 the double inequality

$$m\|h\|_{\mathfrak{H}}^2 \leqslant \int_{\mathbb{R}} \|K_a(h,x)\|_{\mathbb{C}^n}^2 dx \leqslant M \|h\|_{\mathfrak{H}}^2, \quad h \in \mathfrak{H} \tag{2.17}$$

is correct.

We'll consider the integral

$$\mathcal{P}h := \frac{1}{2\pi i} \int_{\mathbb{R}} \sum_{k=1}^{n} f_k(x,h)(I - xB^*)^{-1} g_k dx \tag{2.18}$$

for each $h \in \mathfrak{H}$, where functionals $f_k(x,h)$ have previous sence and the integration is carried on in the direction of parameter x increase. It is integral of type (1.17) and so, in force of lemma 1.2 and (1.18), it gives the bounded operator

$$\|\mathcal{P}h\|^2 \leqslant C \int_{\mathbb{R}} \sum_{k=1}^{n} |f_k(x,h)|^2 dx \leqslant C_1 \|h\|^2, \quad h \in \mathfrak{H}.$$

Lemma 2.6. *Let K be an arbitrary operator of class Q_n without real eigenvalues and let weight $W(x) = \Phi(x)\Phi^*(x)$, $x \in \mathbb{R}$ satisfies the (A_2) matrix condition. Then the equality*

$$K(\mathcal{P}h, x) = \mathbb{P}_+ K(h,x), \quad h \in \mathfrak{H}, \quad x \in \mathbb{R} \tag{2.19}$$

where $\mathbb{P}_+$ is the orthoprojector from $L_2^n(\mathbb{R})$ onto $H_+^2(\mathbb{C}^n)$, is correct.

Proof. It follows from formulae (2.1) and (2.18), that

$$K(\mathcal{P}h,z)=\frac{1}{2\pi i}\int_{\mathbb{R}}\sum_{k=1}^{n} f_k(x,h)K((I-xB^*)^{-1}g_k,z)dx.$$

In force of (2.4) taking account of formula $\Theta^*(\bar{z})=\Theta^{-1}(z)$ [6], we'll get

$$\begin{aligned}K\left((I-xB^*)^{-1}g_k,z\right)&=K_0\left((I-xB^*)^{-1}g_k,z\right)\\&\quad-i\left(E_n-\Theta^{-1}(z)\right)f\left((I-xB^*)^{-1}g_k,z\right).\end{aligned}\tag{2.20}$$

We'll remind that vectors g_k, $1\leqslant k\leqslant n$ are contained in formulae (1.4), (1.5) by definition of matrix-valued function $\Theta(z)$. So

$$\begin{aligned}K_0\left((I-xB^*)^{-1}g_k,z\right)&=\operatorname{col}\left\{\left((I-zB^*)^{-1}(I-xB^*)^{-1}g_k,g_j\right)\right\}_{j=1}^{n}\\&=\operatorname{col}\left\{\frac{z\left((I-zB^*)^{-1}g_k,g_j\right)}{z-x}-\frac{x\left((I-xB^*)^{-1}g_k,g_j\right)}{z-x}\right\}_{j=1}^{n}\\&=\operatorname{col}\left\{i\frac{\Theta_{jk}^{-1}(z)-\Theta_{jk}^{-1}(x)}{z-x}\right\}_{j=1}^{n}=-i(x-z)^{-1}\left(\Theta^{-1}(z)-\Theta^{-1}(x)\right)e_k,\end{aligned}$$

where e_k $(1\leqslant k\leqslant n)$ are the standard orths of space $\mathbb{C}^n$. Also, we'll take account of the fact, that in lemma 1.2 proof (step 2) the formula

$$f\left((I-xB^*)^{-1}g_k,z\right)=(x-z)^{-1}\Phi^{-1}(z)\left(\Phi(z)-\Phi(\lambda)\right)e_k$$

was got.

Therefore, if we return to (2.20), then we find

$$\begin{aligned}K\left((I-xB^*)^{-1}g_k,z\right)&=-i(x-z)^{-1}\left(\Theta^{-1}(z)-\Theta^{-1}(x)\right)e_k\\&\quad-i(x-z)^{-1}\left(E_n-\Theta^{-1}(z)\right)\Phi^{-1}(z)\left(\Phi(z)-\Phi(x)\right)e_k\\&=i(x-z)^{-1}\left(\Theta^{-1}(x)-E_n\right)e_k+i(x-z)^{-1}\left(E_n-\Theta^{-1}(z)\right)\Phi^{-1}(z)\Phi(x)e_k.\end{aligned}$$

Therefore formula for $K(Ph,z)$ can be written in a form

$$\begin{aligned}K(\mathcal{P}h,z)&=\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{i(\Theta^{-1}(x)-E_n)f(h,x)dx}{x-z}\\&\quad+\frac{1}{2\pi}\left(E_n-\Theta^{-1}(z)\right)\Phi^{-1}(z)\int_{\mathbb{R}}\frac{\Phi(x)f(h,x)}{x-z}dx.\end{aligned}\tag{2.21}$$

The second summand of this equality is equal to 0 under $z\in\mathbb{C}_+$. Really, it follows from (2.10) that the entire vector-valued function $F(h,z)=\operatorname{col}\left\{\left((I-zB^*)^{-1}h,f_j\right)\right\}_1^n$ is bounded in $\mathbb{C}_-$ (dissipativity of operator B) and such, that in force of lemma 1.2

$$\begin{aligned}\int_{\mathbb{R}}\left(W^{-1}(x)F(h,x),F(h,x)\right)dx&=\int_{\mathbb{R}}\left\|\Phi^{-1}(x)F(h,x)\right\|^2dx\\&=\int_{\mathbb{R}}\|f(h,x)\|^2dx\leqslant\mathcal{M}\|h\|^2,\quad h\in\mathfrak{H}.\end{aligned}$$

From here it follows that $F(h,z)$ belongs to Hardy weight's class in $\mathbb{C}_-$ [10], i.e. under $z\in\mathbb{C}_+$ the second summand of (2.21) is equal to 0. Directing $z\to x$ non-tangently and taking account of (2.4) and lemma 2.1 we get

$$\begin{aligned}K(\mathcal{P}h,x)&=\mathbb{P}_+\left(-i(E_n-\Theta^{-1}(x))f(h,x)\right)\\&=\mathbb{P}_+\left(K_0(h,x)-i(E_0-\Theta^{-1}(x))f(h,x)\right)=\mathbb{P}_+K(h,x),\end{aligned}$$

and it proves the lemma. □

Now we'll use the proved lemma to the operators K_a. We input the notations for images

$$\mathfrak{H}_1 = \mathcal{P}\mathfrak{H}, \quad \mathfrak{H}_2 = (I - \mathcal{P})\mathfrak{H}.$$

Then $h = h_1 + h_2$, $h_1 = \mathcal{P}h$, $h_2 = (I - \mathcal{P})h$ for each $h \in \mathfrak{H}$ and, moreover, from (2.17) we conclude the two-sided estimates

$$\begin{aligned} \|h\|^2_{\mathfrak{H}} \asymp \int_{\mathbb{R}} \|K_a(h,x)\|^2 dx &= \int_{\mathbb{R}} \|\mathbb{P}_+ K_a(h,x)\|^2 dx + \int_{\mathbb{R}} \|\mathbb{P}_- K_a(h,x)\|^2 dx \\ &= \int_{\mathbb{R}} \|K_a(h_1,x)\|^2 dx + \int_{\mathbb{R}} \|K_a(h_2,x)\|^2 dx \asymp \|h_1\|^2_{\mathfrak{H}} + \|h_2\|^2_{\mathfrak{H}}. \end{aligned}$$

From here it is following that $\mathcal{P}$ and $(I - \mathcal{P})$ are the bounded projectors onto subspaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ and, also, $\mathfrak{H} = \mathfrak{H}_1 \dot{+} \mathfrak{H}_2$.

Now we'll consider the linear operator

$$(Sh)(x) = K_a(h,x), \quad h \in \mathfrak{H}$$

from $\mathfrak{H}$ into space $L^n_2(\mathbb{R})$. It follows from (2.17), that S is the isomorphism of $\mathfrak{H}$ onto its image. Moreover, the equalities

$$\begin{aligned} (SKh)(x) &= \operatorname{col}\left\{\left((I - xK)^{-1}Kh, g_k\right)\right\} \\ &= x^{-1}\left(\operatorname{col}\left\{((I - xK)^{-1}h, g_k)\right\} - \operatorname{col}\left\{(h, g_k)\right\}\right) \\ &= x^{-1}\left((Sh)(x) - (Sh)(0)\right) = k(Sh)(x), \quad h \in \mathfrak{H}, \end{aligned}$$

where the operator k is defined by formula

$$(kf)(x) = x^{-1}(f(x) - f(0)), \quad f \in S\mathfrak{H},$$

are correct.

We note, that $h_1 \in \mathfrak{H}_1$ if and only if then $K_a(h_1, x) \in H^2_+(\mathbb{C}^n)$. From here it is easily concluding, that the subspace $\mathfrak{H}_1$ is invariant under the operator K and, consequently, subspace $S\mathfrak{H}_1$ is invariant under the operator k. Therefore [12], the inner in $\mathbb{C}_+$ matrix-valued function U_+, such that $S\mathfrak{H}_1 = H^2_+(\mathbb{C}^n) \ominus U_+ H^2_+(\mathbb{C}^n)$, exists. Analogously, the image $S\mathfrak{H}_2 = H^2_-(\mathbb{C}^n) \ominus U_- H^2_+(\mathbb{C}^n)$, where U_- is some inner matrix-valued function in domain $\mathbb{C}_-$. So we come to the equalities

$$S\mathfrak{H} = \mathcal{K}_{U_+} \oplus \mathcal{K}_{U_-}, \quad SK_a = (k_+ \oplus k_-)S, \tag{2.22}$$

where operators k_+, k_- are defined by formulae

$$\begin{aligned} (k_+ f)(z) &= z^{-1}(f(z) - f(0)), \quad f \in \mathcal{K}_{U_+} \\ (k_- g)(z) &= z^{-1}(g(z) - g(0)), \quad g \in \mathcal{K}_{U_-}. \end{aligned} \tag{2.23}$$

Lemma 2.7. *Let K be an arbitrary operator of class Q_n without the real eigenvalues. If the conditions of lemma 2.5 hold, then there exist inner matrix-valued functions V_+, V_- in domains $\mathbb{C}_+, \mathbb{C}_-$ correspondingly such that operator K is similar to the operator $k_+ \oplus k_-$ in space $\mathcal{K}_{V_+} \oplus \mathcal{K}_{V_-}$.*[2]

Proof. Step 1. Let K be an arbitrary operator of class Q_n. We can consider, that K is acting in space $\mathcal{K}_\Theta$ by formulae (1.10), (1.11), i.e.

$$Kh = B^*h + \sum_{k=1}^{n} (h, f_k) g_k, \quad g_k = z^{-1}(E_n - \Theta(z))c_k,$$

[2]In this formulation operators k_+, k_- act by formulae (2.23) in spaces $\mathcal{K}_{V_+}, \mathcal{K}_{V_-}$ correspondingly.

where vectors $\{c_k\}_1^n$ forms some basis in $\mathbb{C}^n$. We denote as a the exponential type of characteristic matrix-valued function $\Theta(z)$. In space $\mathcal{K}_a := H_+^2(\mathbb{C}^n) \ominus e^{iaz} H_+^2(\mathbb{C}^n)$ we'll consider the operator

$$(B_a^* h)(z) = z^{-1}\left(h(z) - e^{iaz} h(0)\right), \quad h \in \mathcal{K}_a.$$

It is known, that Θ is a divisor of $e^{iaz} E_n$ and so subspace $\mathcal{K}_\Theta \subseteq \mathcal{K}_a$. It is invariant under operator B_a and, moreover, $B = B_a | \mathcal{K}_\Theta$ [6]. Now we consider the operator

$$K_a h = B_a^* h + \sum_{k=1}^{n} (h, f_k) \tilde{g}_k, \quad \tilde{g}_k = z^{-1}(1 - e^{iaz}) c_k,$$

which belongs to class Q_n and $\Theta(z) = e^{iaz} E_n$ in space $\mathcal{K}_a$. It isn't difficult to verify the correctness of equalities

$$\mathbb{P}_\Theta \tilde{g}_k = g_k, \quad 1 \leqslant k \leqslant n,$$

where $\mathbb{P}_\Theta$ is the orthoprojector from $\mathcal{K}_a$ onto $\mathcal{K}_\Theta$. So $\mathbb{P}_\Theta \mathcal{K}_a h = Kh$, $h \in \mathcal{K}_\Theta$ and then

$$K_a^* h = K^* h, \quad h \in \mathcal{K}_\Theta. \tag{2.24}$$

Now we'll compute the matrix-valued functions corresponding to operators K and K_a in force of formula (1.3). We have

$$\begin{aligned}\Phi_{kj}^a(z) :=& \delta_{jk} - z\left((I - zB_a^*)^{-1} \tilde{g}_k, f_j\right) = \delta_{kj} - z\left(\mathbb{P}_\Theta (I - zB_a^*)^{-1} \tilde{g}_k, f_j\right) \\ =& \delta_{kj} - z\left((I - zB^*)^{-1} g_k, f_j\right) = \Phi_{kj}(z), \quad 1 \leqslant k, \quad j \leqslant n,\end{aligned}$$

i.e. $\Phi^a(z) \equiv \Phi(z)$, $z \in \mathbb{C}$.

Step 2. As $\Phi^a = \Phi$, then the conditions of lemma 2.5 hold for operator K_a, i.e. the two-sided inequality (2.17) holds. Therefore the equality (2.22) is correct, and, so

$$K_a^* S^* h = S^*(k_+^* \oplus k_-^*) h, \quad h \in \mathcal{K} := \mathcal{K}_{U_+} \oplus \mathcal{K}_{U_-},$$

where $U_+ (U_-)$ is inner $(n \times n)$-matrix-valued function in $\mathbb{C}_+$ $(\mathbb{C}_-)$. We'll define the subspace $\mathcal{L}$ of space $\mathcal{K}$ by equality

$$\mathcal{L} = \{l \in \mathcal{K} : S^* l \in \mathcal{K}_\Theta\}.$$

If we take account of (2.24) then it is following from the last equality, that

$$K^* S^* l = S^*(k_+^* \oplus k_-^*) l, \quad l \in \mathcal{L}. \tag{2.25}$$

So, the subspace $\mathcal{L}$ is invariant under $k_+^* \oplus k_-^*$, where both operators have only discrete spectrum. Moreover, the dimensions of their imaginary parts do not overestimate n. Further, k_+^* is anti-dissipative, and k_-^* is dissipative operator. Therefore [13], the space $\mathcal{L}$ can be represented as $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, where subspace $\mathcal{L}_1$ is invariant under k_+^*, and subspace $\mathcal{L}_2$ is invariant under k_-^*. Consequently, it follows from (2.25), that operator K^* is similar to the orthogonal sum of dissipative and anti-dissipative operators. Moreover, both summands have only discrete spectrum and the dimensions of their imaginary parts don't overestimate n. So operator K is similar to the orthogonal sum with the analogous properties of summands. At last we use the theorem about functional models of dissipative operators with the discrete spectrum and n-dimensional imaginary parts [5]. □

2.4. In this article the main result of paper about spectral structure of special finite-dimensional perturbations of Volterra operators will be proved. We'll remind, that the question is about operators of type (class Q_n)

$$Kh = B^* h + \sum_{k=1}^{n} (h, f_k) g_k, \quad h \in \mathfrak{H},$$

where B is Volterra dissipative operator with imaginary part $\operatorname{Im} B$ of rank n, f_k $(1 \leqslant k \leqslant n)$ are the arbitrary vectors of space $\mathfrak{H}, g_k$ $(1 \leqslant k \leqslant n)$ is a some basis of subspace $(\operatorname{Im} B)\mathfrak{H}$. The entire matrix-valued function $\Phi(z)$ with elements

$$\Phi_{kj}(z) = \delta_{jk} - z\left((I - zB^*)^{-1} g_j, f_k\right), \quad 1 \leqslant k, \quad j \leqslant n$$

correspond to each operator $K \in Q_n$. We denote the equation $\det \Phi(z) = 0$ roots set as Λ (taking account of multiplicities), and, moreover, we assume

$$\Lambda_{\pm} := \Lambda \cap \mathbb{C}_{\pm}; \quad \Lambda_+ = \{\lambda_k^+\}; \quad \Lambda_- = \{\lambda_k^-\}.$$

Further, as $\mathcal{B}_+(\lambda)$ the Blaschke product in $\mathbb{C}_+$ with zeroes on sequence Λ_+ is denoted. As $\mathcal{B}_-(\lambda)$ the Blaschke product in $\mathbb{C}_-$ with zeroes on Λ_- is denoted (taking account of multiplicities).

Let k be an arbitrary completely continious dissipative operator with the trivial kernel and imaginary part $\operatorname{Im} k$ of rank not more then n. The set of such operators acting in the separable Hilbert space we denote as $\mathcal{D}_n$. If $k \in \mathcal{D}_n$ then the non-bounded operator k^{-1} exists, moreover the semigroup $\exp\{-ik^{-1}t\}$, $t \geqslant 0$ is contractive. The set of operators $k \in \mathcal{D}_n$ such that $\exp\{-ik^{-1}t\}$ has negative exponential type we denote as $\mathcal{D}_n^-$.

At last we remind that characteristic matrix-valued function $\Theta(z)$ of operator $\mathcal{B}$ is defined by formulae (1.4), (1.5), a is the exponential type of $\Theta(z)$.

Theorem 2.1. *Let $K \in Q_n$ and doesn't have the real eigenvalues. If the matrix weight $\Phi(x)\Phi^*(x)$, $x \in \mathbb{R}$ satisfies the (A_2) condition and the inequalities*

$$\inf_{\operatorname{Im} z > 0} \left\{|\mathcal{B}_+(z)| + |e^{iaz} - 1|\right\} > 0, \quad \inf_{\operatorname{Im} z < 0} \left\{|\mathcal{B}_-(z)| + |e^{-iaz} - 1|\right\} > 0 \tag{2.25}$$

are correct, then the operator K is similar to the orthogonal sum $k_1 \oplus (-k_2)$, where k_1, k_2 are the some operators of class $\mathcal{D}_n^-$.

Conversely, let the operator $K \in Q_n$ be similar to the orthogonal sum $k_1 \oplus (-k_2)$, $k_1, k_2 \in \mathcal{D}_n^-$. If under some $\delta > 0$ the matrix-valued function $e^{-iz\delta}\Theta(z)$ is bounded in $\mathbb{C}_+$, then the weight $\Phi(x)\Phi^*(x)$ satisfies the (A_2) condition and the inequalities (2.25) hold.

Remark. If the set Λ_+ (Λ_-) is finite or empty then in theorem 2.1 formulation the first (second) inequality (2.25) must be excepted.

Proof. It isn't difficult to see that the inequalities (2.25) are equivalent to the inequalities which are contained in the lemma 2.5 formulation. Therefore, it follows from lemma 2.7, that K is similar to $k_1 \oplus (-k_2)$, where k_1, k_2 are the operators of class $\mathcal{D}_n$. Now we prove that both operators $k_1, k_2 \in \mathcal{D}_n^-$. Really, the correctness of estimate (1.7) follows from the theorem 1.1. Therefore

$$\int_{\mathbb{R}} \left\|(k_1^{-1} - xI)^{-1} f\right\|^2 dx \leqslant M_1 \|f\|^2, \quad \int_{\mathbb{R}} \left\|(k_2^{-1} - xI)^{-1} g\right\|^2 dx \leqslant M_1 \|g\|^2, \tag{2.26}$$

for all f, g from the spaces where k_1, k_2 act. From resolvent generator representation by Laplace transform of semigroup [7] it follows that

$$\int_{\mathbb{R}} \left\|\exp\{-ik_1^{-1}t\} f\right\|^2 dx \leqslant M_1 \|f\|^2, \quad \int_{\mathbb{R}} \left\|\exp\{-ik_2^{-1}t\} g\right\|^2 dx \leqslant M_2 \|g\|^2. \tag{2.27}$$

It is known [14], that from here the negativity of both semigroups exponential types follows, i.e. $k_1, k_2 \in \mathcal{D}_n^-$.

Conversely, let $K \in Q_n$ is similar to the orthogonal sum $k_1 \oplus (-k_2)$. As exponential types $\exp\{-ik_1^{-1}t\}$, $\exp\{-ik_2^{-1}t\}$ are negative, then the estimates (2.27) hold and so the estimate (1.7) is correct for the operator K. It follows from the theorem 1.1, that the weight $\Phi(x)\Phi^*(x)$ satisfies the matrix Muckenhoupt condition. Further, the inequalities (2.25) are equivalent to fact that 1 does not contained in the operators spectrums

$\exp\{-ik_1^{-1}a\}$, $\exp\{-ik_2^{-1}a\}$. As $k_1, k_2 \in \mathcal{D}_n^-$, the spectral radiuses of these operators are less then 1, i.e. (2.25) hold. □

The further consideration of class Q_n operators spectral properties is connected with the more detailed investigation of operators k_1, k_2. We hope to dedicate the separate publication to it.

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