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SYSTEMS OF ONE-DIMENSIONAL SUBSPACES OF A HILBERT SPACE

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ABSTRACT. We study systems of one-dimensional subspaces of a Hilbert space. For such systems, symmetric and orthoscalar systems, as well as graph related configurations of one-dimensional subspaces have been studied.

INTRODUCTION

Studies of systems $\mathcal{L} = (V; V_1, \dots, V_n)$ of subspaces $\{V_k\}$, $k = 1, \dots, n$, of a linear space V, in particular, a description of indecomposable systems up to similarity, a description of indecomposable representations of finite partially ordered sets on a space V, and related problems, have by now became classical, see, e.g., [24, 2, 6, 5].

Let $S = (H; H_1, H_2, \dots, H_n)$ be a system of subspaces of a finite dimensional or a countably dimensional complex Hilbert space H. Denote by P_j the orthogonal projections that map H onto H_j, $j = 1, \dots, n$, correspondingly. Since there is a one-to-one correspondence between subspaces and the orthogonal projections, a description of a collection of subspaces or involution representations of the $*$-algebras $\mathcal{P}_n = \langle p_1, \dots, p_n | p_j^2 = p_j^* = p_j, j = 1, \dots, n\rangle$ generated by a collection of the orthogonal projections is the same problem.

We say that two systems $S = (H; H_1, \dots, H_n)$ and $\tilde{S} = (\tilde{H}; \tilde{H}_1, \dots, \tilde{H}_n)$ are unitarily equivalent if there exists a unitary operator $U : H \to \tilde{H}$ such that $UH_k = \tilde{H}_k, k = 1, \dots, n$. Using the projections, the condition for the unitary equivalence becomes $UP_k = \tilde{P}_k U$. A system S is called irreducible if an arbitrary linear operator $C \in B(H)$ that commutes with all orthogonal projections, $CP_k = P_k C$ for all $k = 1, \dots, n$, is necessarily a multiple of the identity operator, $C = \lambda I$.

There are many works dealing with a description of systems of subspaces of a Hilbert space up to the unitary equivalence, e.g., [4, 10, 25, 9] and others. For two subspaces, the problem has been solved, see [4, 10] and others. This result has numerous applications. For three subspaces, even with the condition that two of them are orthogonal, the problem of describing all irreducible systems is $*$-wild, see [17, 18].

A number of works deal with a description of systems of subspaces of a Hilbert space with additional conditions imposed on the subspaces. Among the conditions there is a condition that S is a configuration of subspaces, see [28, 21, 22] and others, the condition that the system is orthoscalar, see, e.g., [15, 16, 14, 20, 1, 13, 23] and others.

In Sections 2–4, we consider various classes of systems of *one-dimensional* subspaces, $\dim(H_k) = 1$, for all $k = 1, \dots, n$. Let us remark that irreducible pairs of subspaces, and irreducible configurations of subspaces (see Section 3) that correspond to trees and unicycle graphs are always given with systems of one-dimensional subspaces [28]. In

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Section 2, we give a description of symmetric systems of one-dimensional subspaces. In Section 3, we study graph related configurations of one-dimensional subspaces for various graphs. Section 4 deals with a study of orthoscalar systems of one-dimensional subspaces such that the sum of orthogonal projections onto these subspaces is a scalar operator. All these classes of one-dimensional subspaces were studied using the corresponding Gram matrices. In Section 1, we give conditions for a unitary equivalence and an irreducibility of such systems in terms of the corresponding Gram matrices.

1. On systems of one-dimensional subspaces of a Hilbert space

In this section, we discuss a simple relation between irreducible systems of one-dimensional subspaces and Gram matrices.

1.1. Let $S = (H; H_1, H_2, \dots, H_n)$ be an irreducible system of one-dimensional subspaces of a Hilbert space H, and let $v_k \in H_k$, $k = 1, 2, \dots, n$, be a set of unit vectors. Then v_k generate H_k, $k = 1, 2, \dots, n$, since each space is one-dimensional, and the set of vectors $\{v_k : k = 1, 2, \dots, n\}$ generates the entire space H, since the system is irreducible. Thus, these vectors define the system S uniquely. The vectors give rise to the Gram matrix $G = (< v_j, v_k >)_{j,k=1}^n$, where $< \cdot, \cdot >$ denotes the inner product in H. On the other hand, the system S is also defined by the vectors $\tilde{v}_k = e^{i\psi_k} v_k$ for arbitrary $\psi \in (0, 2\pi)$, and $\tilde{v}_k$ having length one. It is clear that the Gram matrix $\tilde{G}$ for the vectors $\tilde{v}_k$ is related to G via the identity $VG = \tilde{G}V$, where $V = \operatorname{diag}(e^{i\psi_1}, \dots, e^{i\psi_n})$. Thus, every system of one-dimensional subspaces defines a class of Gram matrices up to a diagonal unitary operator.

Conversely, a nonnegative definite matrix G is a Gram matrix of a system of vectors of a Hilbert space H (its dimension coincides with the rank of the matrix), and these vectors are determined up to the action of a unitary operator. Hence, a nonnegative definite matrix defines a certain system S of one-dimensional subspaces of the space H up to the unitary equivalence.

Proposition 1. *Systems of one-dimensional subspaces $S = (H; H_1, \dots, H_n)$ and $\tilde{S} = (\tilde{H}, \tilde{H}_1, \dots, \tilde{H}_n)$ are unitarily equivalent if and only if there is a unitary operator $V = \operatorname{diag}(e^{i\psi_1}, \dots, e^{i\psi_n})$ such that $VG = \tilde{G}V$, where G and $\tilde{G}$ are Gram matrices of the systems of unit vectors $\{v_k \in H_k : k = 1 \dots, n\}$ and $\{\tilde{v}_k \in \tilde{H}_k : k = 1 \dots, n\}$, correspondingly.*

Proof. The systems S and $\tilde{S}$ are unitarily equivalent if and only if there is a unitary operator $U : H \to \tilde{H}$ such that $UH_k = \tilde{H}_k$, $k = 1, \dots, n$, and then we have that $\hat{v}_k = Uv_k$ is a unit vector in $\tilde{H}_k$ for an arbitrary unit vector $v_k \in H_k$. Since U is unitary, we have that $< \hat{v}_j, \hat{v}_k >_{\tilde{H}} = < Uv_j, Uv_k >_{\tilde{H}} = < v_j, v_k >_H$. Hence, the Gram matrix G for the vectors $\{v_k : k = 1, \dots, n\}$ and the Gram matrix for the vectors $\{\hat{v}_k : k = 1, \dots, n\}$ coincide. As follows from the above, the Gram matrix $\tilde{G}$ for any system of unit vectors $\{\tilde{v}_k \in \tilde{H}_k, k = 1 \dots, n\}$ and the matrix G satisfy $VG = \tilde{G}V$, where $V = \operatorname{diag}(e^{i\psi_1}, \dots, e^{i\psi_n})$ is a unitary operator.

Conversely, if $VG = \tilde{G}V$, then G is the Gram matrix for the vectors $\{v_1, \dots, v_n\}$, and $\tilde{G}$ is the same for the vectors $\{\tilde{e}_k = e^{i\psi_k} e_k : k = 1, \dots, n\}$. These vectors define the same system S of one-dimensional subspaces. □

1.2. Let us find conditions on the Gram matrix for the corresponding system of vectors to be reducible. Recall (see, e.g., [11]) that an $n \times n$-matrix A ($n \geq 2$) is called *decomposable* if there is a permutation matrix $P \in M_n$ and a number $1 \leq r \leq n-1$ such that $P^T A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where $B \in M_r$, $D \in M_{n-r}$, $C \in M_{r,n-r}$, $0 \in M_{n-r,r}$. Otherwise, the matrix is called *indecomposable*.

If a matrix is symmetric, then it is indecomposable if and only if it can not be reduced to a block-diagonal form by simultaneously permuting rows and columns. Note that multiplying the matrix by a diagonal matrix does not change its property of being decomposable, hence, for a system S of one-dimensional subspaces, the Gram matrix G for unit vectors $\{v_k \in H_k : k = 1 \ldots, n\}$ will be decomposable or indecomposable regardless of a particular choice of the vectors v_k.

Proposition 2. *A system $S = (H; H_1, \ldots, H_n)$ of one-dimensional subspaces is irreducible if and only if the Gram matrix G is indecomposable.*

Proof. If the Gram matrix is decomposable, then we can assume that $G = G_1 \oplus G_2$, where G_1 is the Gram matrix for the vectors $v_1, \ldots, v_m$, and G_2 is the Gram matrix for the vectors $v_{m+1}, \ldots, v_n$. Then the orthogonal projection P onto the subspace $\langle v_1, \ldots, v_m \rangle$ spanned by the first m vectors commutes with all orthogonal projections P_k, $k = 1, \ldots, n$. Since $\langle v_1, \ldots, v_m \rangle$ does not coincide with the whole space H, the system S will be reducible.

On the other hand, if the matrix G is indecomposable, then for arbitrary $1 \le k, j \le n$ there is the unitary operator,

$$U_{kj} = \prod_{\substack{i_l:\ g_{i_l, i_{l+1}} \neq 0, \\ i_1 = k,\ i_m = j}} \frac{P_{i_{l+1}} P_{i_l}}{g_{i_l, i_{l+1}}},$$

such that $U_{kj} v_k = v_j$. Then, if the operator C commutes with all the orthogonal projections P_k, then it will also commute with all the operators U_{kj}. Hence, $CP_k = P_k C$, thus $Cv_k = c_k v_k$, and, since $U_{kj} C = C U_{kj}$, we see that $U_{kj} C v_k = U_{kj} c_k v_k = c_k v_j$. On the other hand, $U_{kj} C v_k = C U_{kj} v_k = C v_j = c_j v_j$, whence $c_k = c_j$ for all k, j. This shows that C is a scalar operator. $\square$

Thus studying systems of irreducible systems of one-dimensional subspaces up to the unitary equivalence is equivalent to studying all indecomposable Hermitian nonnegative definite matrices G that have 1 on the main diagonal up to the equivalence indicated in Proposition 1, finding ranks of these matrices, etc. In Sections 2–4, we will study various classes of systems of one-dimensional subspaces.

2. Symmetric systems of one-dimensional subspaces

Among systems of subspaces of a Hilbert space H, we single out symmetric systems.

Definition 1. A system S is called *symmetric* if the collections of orthogonal projections $\{P_j : j = 1, \ldots, n\}$ and $\{P_{\sigma(j)} : j = 1, \ldots, n\}$ are unitarily equivalent for all $\sigma \in S_n$.

It directly follows from the definition that $\dim H_j = \dim H_k$ for all $j, k = 1, \ldots, n$, and the operators $P_{i_1} P_{i_2} \ldots P_{i_k}$ and $P_{\sigma(i_1)} P_{\sigma(i_2)} \ldots P_{\sigma(i_k)}$ are unitarily equivalent for all $k \ge 1$ and $\sigma \in S_n$; as a consequence, $P_j P_k P_j$ and $P_l P_m P_l$ are unitarily equivalent for all mutually distinct $j, k, l, m = 1, \ldots, n$.

Requiring that the spaces be *one-dimensional* one can get a complete description of such systems. We give such a description in terms of the Gram matrices, fixing the parameter $\tau =< e_1, e_2 >$ ($|e_k| = 1$, $\langle e_k \rangle = H_k$, $k = 1, 2$).

Theorem 1. *Symmetric systems of one-dimensional subspaces, up to the unitarily equivalence, are the following.*

- *If $\tau = 0$, then there are no irreducible nonzero symmetric systems.*
- *If $0 < \tau < \frac{1}{n-1}$, then all irreducible nonequivalent symmetric systems are $S_+ = (H; H_1^+, \ldots, H_n^+)$ and $S_- = (H; H_1^-, \ldots, H_n^-)$, where H is the n-dimensional*

Hilbert space $H_i^{\pm} = \langle e_i^{\pm} \rangle$, $i = 1, \dots, n$, *and the collections of vectors* $(e_1^+, \dots, e_n^+)$ *and* $(e_1^-, \dots, e_n^-)$ *are defined by the Gram matrices*

$$G_+ = \begin{pmatrix} 1 & \tau & \dots & \tau \\ \tau & 1 & \dots & \tau \\ \vdots & \vdots & \ddots & \vdots \\ \tau & \tau & \dots & 1 \end{pmatrix} \quad \text{and} \quad G_- = \begin{pmatrix} 1 & -\tau & \dots & -\tau \\ -\tau & 1 & \dots & -\tau \\ \vdots & \vdots & \ddots & \vdots \\ -\tau & -\tau & \dots & 1 \end{pmatrix},$$

correspondingly.

- *If* $\tau = \frac{1}{n-1}$, *there is a unique irreducible system of subspaces of a space of dimension* n, $S = (H; H_1, \dots, H_n)$, $H_i = \langle e_i \rangle$, $i = 1, \dots, n$, *and the collection of vectors* $(e_1, \dots, e_n)$ *is defined by the Gram matrix* G_+, *there is also one irreducible system in a space of dimension* $n-1$, $S = (H; H_1, \dots, H_n)$, $H_i = \langle e_i \rangle, i = 1, \dots, n$, *and the collection of vectors* $(e_1, \dots, e_n)$ *is defined by the Gram matrix* G_-.
- *If* $\frac{1}{n-1} < \tau < 1$, *then there is a unique system* $S = (H; H_1, \dots, H_n)$, *where* H *is a Hilbert space of dimension* n, $H_i = \langle e_i \rangle$, $i = 1, \dots, n$, *and the collection of vectors* $(e_1, \dots, e_n)$ *is defined by the Gram matrix* G_+.
- *If* $\tau = 1$, *a unique irreducible symmetric system, up to the unitary equivalence, is* $S = (\mathbb{C}; \mathbb{C}, \dots, \mathbb{C})$.

Proof. If $\tau = 0$, then all subspaces are orthogonal, the Gram matrix is the identity matrix, and the system is reducible. If $\tau = 1$, the subspaces coincide, and thus the system is $S = (\mathbb{C}; \mathbb{C}, \dots, \mathbb{C})$.

Now consider the case $0 < \tau < 1$. Since the operators $P_1 P_2 P_1$ and $P_j P_k P_j$ are unitarily equivalent, it follows that $| < e_j, e_k > |^2 = \tau^2$. That is, the system S is defined by the Gram matrix $G_\varphi = \left(e^{i\varphi_{jk}}\tau\right)_{j,k=1}^n$, where $\varphi_{jk} = 2\pi - \varphi_{kj} \in [0, 2\pi)$. Up to the unitary equivalence, we can assume that $\varphi_{1k} = \varphi_{k1} = 0$, $k = 2, \dots, n$, and the two systems are unitarily equivalent if and only if $G_\varphi = G_{\widetilde{\varphi}}$.

Let now an irreducible system $S = (H; H_1, \dots, H_n)$ be defined by the Gram matrix G_φ. Then it is clear that the system $\sigma(S) = (H; H_{\sigma(1)}, \dots, H_{\sigma(n)})$ is given by the Gram matrix $G_{\widetilde{\varphi}} = P_\sigma G_\varphi P_\sigma$, where P_σ is a permutation matrix corresponding to $\sigma \in S_n$.

So, if the system S is symmetric, the matrix G_φ is invariant with respect to simultaneous permutations of the j-th and the k-th columns and the j-th and the k-th rows, in particular for $j, k \neq 1$. This immediately implies that the matrix G_φ is real and, moreover, has the form

$$G_{\varphi_1} = G_+ = \begin{pmatrix} 1 & \tau & \dots & \tau \\ \tau & 1 & \dots & \tau \\ \vdots & \vdots & \ddots & \vdots \\ \tau & \tau & \dots & 1 \end{pmatrix} \quad \text{or} \quad G_{\varphi_2} = \begin{pmatrix} 1 & \tau & \tau & \dots & \tau \\ \tau & 1 & -\tau & \dots & -\tau \\ \tau & -\tau & 1 & \dots & -\tau \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tau & -\tau & -\tau & \dots & 1 \end{pmatrix}.$$

It is clear that, in the first case, the matrix will be positive definite for arbitrary $\tau \in (0, 1)$ and symmetric with respect to simultaneous permutations of the j-th and the k-th columns and the j-th and k-th rows for $j, k = 1, \dots, n$, and, hence, the corresponding system of subspaces is symmetric, and the dimension of the space H is n.

The matrices G_{φ_2} and G_- satisfy $G_- = V G_{\varphi_2} V^*$, where $V = \operatorname{diag}(-1, 1, \dots, 1)$. Thus, these two matrices, if they are positive definite, define the same system of one-dimensional subspaces. It is clear that G_- is invariant with respect to simultaneous permutations of the j-th and the k-th columns and the j-th and the k-th rows, $j, k = 1, \dots, n$. Hence, the corresponding system of subspaces is symmetric.

The matrix G_- is positive definite for $\tau < \frac{1}{n-1}$, nonnegative definite and has rank $n-1$ for $\tau = \frac{1}{n-1}$, and negative definite for $\tau > \frac{1}{n-1}$. This finishes the proof. $\square$

3. On configurations of one-dimensional subspaces

Fix a simple (with no multiple edges or loops) connected nonoriented graph Γ and a mapping $\tau(\cdot) : R\Gamma \to (0,1)$ that, to each edge γ_{kj}, assigns a number $\tau(\gamma_{kj}) = \tau_{kj}^2$ ($\tau_{kj} = \tau_{jk}$) from the set $(0,1]$. A collection of subspaces such that

$$\left\{ \begin{array}{ll} P_kP_jP_k = \tau_{kj}^2 P_k, P_jP_kP_j = \tau_{kj}^2 P_j, & \text{if } \gamma_{k,j} \in R\Gamma, \\ P_kP_j = P_jP_k = 0, & \text{if } \gamma_{k,j} \notin R\Gamma, \end{array} \right. \tag{1}$$

is called (see [22]) a *simple configuration*, connected with the graph Γ and the collection of angles τ, of subspaces of a Hilbert space H. Such systems are related to representations of Temperley-Lieb algebras and generalized Temperley-Lieb algebras, see [26, 8, 12] and others. For a description of simple configurations, see [21, 22, 28] and others.

Each system of *one-dimensional* subspaces is a simple configuration with the collection of angles $\tau_{kj} = | < e_k, e_j > | = |g_{kj}|$ and a corresponding graph $\Gamma = (V\Gamma, R\Gamma)$ such that $V\Gamma = 1, \ldots, n$, and $\gamma_{k,j} \in R\Gamma$ if and only if $\tau_{kj} \neq 0$.

3.1. Fix a graph Γ and a number $\tau \in (0,1]$. Set $\tau_{kj} = \tau$ for all $\gamma_{k,j} \in R\Gamma$. In this paragraph, we will study the question of what are the values of τ such that there exist corresponding configurations of one-dimensional subspaces.

Recall (see, e.g., [3]) that the *adjacency matrix* of a graph Γ is an $n \times n$-matrix A_Γ, where $n = |V\Gamma|$, consisting of zeros and ones and such that $a_{k,j} = 1$ if and only if $\gamma_{k,j} \in R\Gamma$. The *index*, $\mathrm{ind}(\Gamma)$, of the graph Γ is the greatest eigen value of the adjacency matrix A_Γ of this graph, $\mathrm{ind}(A_\Gamma)$.

Introduce a matrix $A_{\Gamma,\Phi} = A_{\Gamma,\varphi_1,\ldots,\varphi_{\nu(\Gamma)}}$, which differs from the matrix A_Γ by that $a_{k,j} = e^{i\varphi_l}$ and $a_{j,k} = e^{-i\varphi_l}$ for edges γ_{kj} which, being removed from the graph Γ, make it a tree. The number of such edges coincides with the *cyclotomic number* of the graph, $\nu(\Gamma) = R\Gamma - V\Gamma + 1$. In the case of *one-dimensional* subspaces, irreducible configurations connected with the graph Γ, for $\tau \leq \frac{1}{\mathrm{ind}(\Gamma)}$, are parametrized with $\nu(\Gamma)$ parameters running over $[0, 2\pi)$ up to the unitary equivalence, see [22].

If the matrix $I - \tau A_{\Gamma,\Phi}$ is nonnegative definite for fixed τ and $\Phi = \{\varphi_1, \ldots, \varphi_{\nu(\Gamma)}\}$, then it is the Gram matrix of an irreducible system of vectors, hence it defines a configuration of one-dimensional subspaces. For different Φ, such configurations will not be unitarily equivalent.

Denoting $\mathrm{ind}_{\mathbb{C}}(\Gamma) = \min\{\mathrm{ind}(A_{\Gamma,\Phi}) | \varphi_l \in [0, 2\pi), l = 1, \ldots, \nu(\Gamma)\}$, the previous reasoning gives the following.

Proposition 3. *For a pair Γ, τ there exists a corresponding irreducible configuration of subspaces if and only if $\tau \in (0, \frac{1}{\mathrm{ind}_{\mathbb{C}}(\Gamma)}]$.*

Example 1. ([21]). If Γ is a tree, then $\nu(\Gamma) = 0$ and, hence, $\mathrm{ind}_{\mathbb{C}}(\Gamma) = \mathrm{ind}(A_\Gamma) = \mathrm{ind}(\Gamma)$. So, there exist corresponding configurations of subspaces for $\tau \in (0, \frac{1}{\mathrm{ind}(\Gamma)}]$. Thus, if Γ is a Euclidean graph, i.e., an extended Dynkin diagram, then the corresponding configuration exists for $\tau \in (0, \frac{1}{2}]$.

Example 2. ([7]). If Γ is a unicyclic graph, that is, a graph that has one cycle, the cyclotomic number of such a graph is $\nu(\Gamma) = 1$. These graphs satisfy $\mathrm{ind}_{\mathbb{C}}(\Gamma) = \min\{\mathrm{ind}(A_\Gamma), \mathrm{ind}(A_{\Gamma,\pi})\} = \mathrm{ind}(A_{\Gamma,\pi})$, see [7]. Denote $\mathrm{ind}(A_{\Gamma,\pi})$ by $\mathrm{ind}_\pi(\Gamma)$. Then the corresponding configurations of subspaces exist only for $\tau \in (0, \frac{1}{\mathrm{ind}_\pi(\Gamma)}]$, e.g., for a cycle C_n, the corresponding configurations exist for $\tau \in (0, \frac{1}{2\cos\frac{\pi}{n}}]$.

Example 3. For complete graphs, $\Gamma = K_n$, $\mathrm{ind}_{\mathbb{C}}(\Gamma) = \mathrm{ind}(A_{\Gamma,\pi,\ldots,\pi}) = \mathrm{ind}(-A_\Gamma) = 1$. Hence, configurations of n one-dimensional subspaces such that cosine of the angle between a pair of subspaces equals τ exist for all $\tau \in (0,1]$.

3.2. Let us give a description of all irreducible configurations of subspaces for some classes of Γ and $\tau_{kj} = \tau \in (0, \frac{1}{\mathrm{ind}_{\mathbb{C}}(\Gamma)}]$.

If Γ is a tree, then all irreducible configurations of the corresponding subspaces are configurations of one-dimensional subspaces. A description of such configurations is the following.

Proposition 4. (see [21]). *Let Γ be a tree and $\tau_{kj} = \tau$ for all pairs k, j.*

- *If $0 < \tau < \frac{1}{\mathrm{ind}(\Gamma)}$, then there exists a unique, up to the unitary equivalence, irreducible configuration S that corresponds to this graph, and its dimension equals n.*
- *If $\tau = \frac{1}{\mathrm{ind}(\Gamma)}$, there exists a unique, up to the unitary equivalence, irreducible configuration S corresponding to this graph, and its dimension equals $n-1$.*
- *If $\tau > \frac{1}{\mathrm{ind}(\Gamma)}$, no corresponding configurations exist.*

If Γ is a unicyclic graph, then all irreducible configurations are also necessarily systems of one-dimensional subspaces. Irreducible configurations that are connected to a unicyclic graph are parametrized, up to the unitary equivalence, with one parameter $\varphi \in [\alpha, 2\pi - \alpha]$, where $\alpha \in [0, \pi]$ depends on τ, see [28]. Let us give a description of these configurations.

Proposition 5. ([7]). *Let Γ be a unicyclic graph with n vertices, and let $\tau_{kj} = \tau$ for all pairs k, j.*

- *If $\tau < \frac{1}{\mathrm{ind}(\Gamma)}$, then there is a corresponding configuration $S_{\tau,\varphi}$ for arbitrary $\varphi \in [0, 2\pi)$, and $\dim H = n$.*
- *If $\tau = \frac{1}{\mathrm{ind}(\Gamma)}$, there is an infinite family of configurations $S_{\tau,\varphi}$ parametrized with a parameter in $[0, 2\pi)$, and $\dim H = n$ for $\varphi \neq 0$, and $\dim H = n-1$ for $\varphi = 0$.*
- *If $\frac{1}{\mathrm{ind}(\Gamma)} < \tau < \frac{1}{\mathrm{ind}_\pi(\Gamma)}$, there exists an infinite family of configurations $S_{\tau,\varphi}$, parametrized with a parameter in a segment $[a, b] \subset [0, 2\pi)$ that depends on τ. Here, $\dim H = n$ for $\varphi \in (a, b)$, and $\dim H = n-1$ for $\varphi = a$ or $\varphi = b$.*
- *If $\tau = \frac{1}{\mathrm{ind}_\pi(\Gamma)}$, then there is a unique configuration S corresponding to Γ, τ for $\varphi = \pi$, and the dimension of the space is $n-2$.*
- *If $\tau > \frac{1}{\mathrm{ind}_\pi(\Gamma)}$, no corresponding configurations exist.*

If Γ has more than one cycle, the problem of describing irreducible configurations related to the graph Γ becomes $*$-wild for some collections of angles, see [22].

In the case of *one-dimensional* subspaces, irreducible configurations connected with the graph Γ for $\tau < \frac{1}{\mathrm{ind}(\Gamma)}$ are parametrized up to the unitary equivalence with $\nu(\Gamma)$ parameters in $[0, 2\pi)$.

We will give a description of irreducible configurations of one-dimensional subspaces in the case where $\frac{1}{\mathrm{ind}(\Gamma)} < \tau \leq \frac{1}{\mathrm{ind}_{\mathbb{C}}(\Gamma)}$ only for $\Gamma = K_4$.

Example 4. Let $\Gamma = K_4$ be a complete graph with four vertices. Then $\nu(\Gamma) = 3$ and, generally speaking, all configurations are parametrized with three parameters $\varphi_1, \varphi_2, \varphi_3$ in $[0, 2\pi)$. Consider the case $\tau_{kj} = \tau$ for all pairs k, j.

- Configurations connected with K_4 of subspaces exist for all $\tau \in (0, 1]$, and the dimension of H can take the values $4, 3$, or 1 depending on τ and $\varphi_1, \varphi_2, \varphi_3$.
- If $0 < \tau < \frac{1}{3}$, then $\varphi_1, \varphi_2, \varphi_3$ are arbitrary in $[0, 2\pi)$, and the dimension of the space equals 4 for all values of the free parameters.
- If $\tau = \frac{1}{3}$, then for arbitrary $\varphi_1, \varphi_2, \varphi_3$ in $[0, 2\pi)$ there is a corresponding system of subspaces. The dimension of the space is 4, if not all φ_j are 0, and $\dim H = 3$ if $\varphi_j = 0$ for all j.

- If $\frac{1}{3} < \tau < 1$, then the family of irreducible nonequivalent systems are parametrized with $\varphi_1, \varphi_2, \varphi_3$ in the set
$$M_\tau = \{\varphi_1, \varphi_2, \varphi_3 \in [0, 2\pi) | \varphi_1, \varphi_2, \varphi_3, (\varphi_1 + \varphi_2 - \varphi_3) \in [n\alpha, 2\pi - n\alpha]\},$$
where $\alpha \in (0, 2\pi)$ is such that $\tau = \frac{1}{2\cos\alpha}$, and the dimension of the space is 4 if the parameters lie in the interior of the region M_τ, and the dimension of the space is 3 if the values belong to the boundary.
- If $\tau = 1$, then there is a unique, up to the unitary equivalence, irreducible collection of subspaces that corresponds to the complete graph, it is the one-dimensional collection $S = (\mathbb{C}; \mathbb{C}, \ldots, \mathbb{C})$.

4. Orthoscalarity

Let us consider systems of subspaces, $S = (H; H_1, \ldots, H_n)$, such that the orthogonal projections $P_1, \ldots, P_n$ on $H_1, \ldots, H_n$ satisfy the relation

$$P_1 + P_2 + \cdots + P_n = \gamma I_H \tag{2}$$

for some $\gamma > 0$. We will call such systems orthoscalar.

Systems satisfying such conditions were studied in [15, 14, 16, 1, 20, 13] and other works.

Let S be an irreducible orthoscalar system of *one-dimensional* subspaces. Spectrum of the Gram matrix G for the collection of unit vectors $\{v_k \in H_k : k = 1 \ldots, n\}$ does not depend on the choice of the vectors but only on the system S, since the Gram matrices are unitarily equivalent for different sets of vectors. Spectrum of the Gram matrix G is connected to the spectrum of the sum of corresponding orthogonal projections as follows.

Proposition 6. *If the system S is irreducible, then*
$$\sigma(P_1 + P_2 + \cdots + P_n) = \sigma(G_S) \setminus \{0\}.$$

Proof. Denote $A = P_1 + \cdots + P_n$ and let $\lambda \in \sigma(A)$, Then there exists a vector $v \in H$ such that $Av = \lambda v$. Since the system is irreducible, v is a linear combination of vectors $v_1, \ldots, v_n$, that is, $v = c_1 v_1 + \cdots + c_n v_n$, where not all c_k are equal to zero. Thus, we have the following:
$$\lambda(\sum_{j=1}^{n} c_j v_j) = \lambda v = Av = \left(\sum_{j=1}^{n} P_j\right) v = \sum_{j=1}^{n}\sum_{k=1}^{n} c_k < v_j, v_k > v_j = \sum_{j=1}^{n}\sum_{k=1}^{n} c_k g_{j,k} v_j.$$
By equating the coefficients at v_j, $j = 1, \ldots, n$, we get the identity in a matrix form
$$Gc = \lambda c, \quad c = (c_1, \ldots, c_n) \neq 0.$$
That is, $\lambda \in \sigma(G)$. □

As a corollary, we get a criterion for an irreducible system of one-dimensional subspaces to be orthoscalar.

Corollary 1. *A sum of orthogonal projections on subspaces of the system is a scalar operator, γI, if and only if the spectrum of Gram matrix G is $\{\gamma, 0\}$ with some multiplicities.*

Example 5. Irreducible symmetric orthoscalar systems of one-dimensional subspaces are the only systems that correspond to $\tau = 1$ ($\gamma = n$) and $\tau = -\frac{1}{n-1}$ ($\gamma = \frac{n}{n-1}$). The corresponding Gram matrices are the following:
$$G_1 = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}, \quad G_{-\frac{1}{n-1}} = \begin{pmatrix} 1 & \ldots & -\frac{1}{n-1} \\ \vdots & \ddots & \vdots \\ -\frac{1}{n-1} & \ldots & 1 \end{pmatrix}.$$

Remark. Orthoscalar systems of subspaces are closely connected with involution representations of the $*$-algebras

$$A_{abo} = \mathbb{C}\langle q_1, \dots, q_m, q | q_k^2 = q_k^* = q_k, q^2 = q^* = q, k, j = 1, \dots, m; q_1 + \cdots + q_n = e\rangle$$

that are generated by a system of "all but one" projections; these algebras were studied, e.g., in [27]. In particular, if G is the Gram matrix that corresponds to an orthoscalar system S of one-dimensional subspaces, then $Q = \frac{1}{\gamma}G$ is an orthogonal projection on the space $\mathbb{C}^m$, which, together with the orthogonal projections $Q_1, \dots, Q_m$ onto basis vectors $e_1, \dots, e_m$, gives a $*$-representation of the quotient $*$-algebra

$$\mathbb{C}\langle q_1, \dots, q_m, q | q_k^2 = q_k^* = q_k, q^2 = q^* = q, k, j = 1, \dots, m; q_1 + \cdots + q_n = e; q_k q q_k = q_j q q_j\rangle.$$

And all $*$-representations of this $*$-algebra with the condition that $\dim H_k = 1$, $H_k = Q_k H$ are unitarily equivalent to the above.

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