ALGEBRAICALLY ADMISSIBLE CONES IN FREE PRODUCTS OF ∗-ALGEBRAS

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ABSTRACT. It was proved in [7] that a ∗-algebra is C^*-representable, i.e., ∗-isomorphic to a self-adjoint subalgebra of bounded operators acting on a Hilbert space if and only if there is an algebraically admissible cone in the real space of Hermitian elements of the algebra such that the algebra unit is an Archimedean order unit. In the present paper we construct such cones in free products of C^*-representable ∗-algebras generated by unitaries. We also express the reducing ideal of any algebraically bounded ∗-algebra with corepresentation $\mathcal{F}/\mathcal{J}$ where $\mathcal{F}$ is a free algebra as a closure of the ideal $\mathcal{J}$ in some universal enveloping C^*-algebra.

1. INTRODUCTION

Let $\mathbf{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. Subalgebras of $\mathbf{B}(\mathcal{H})$ are called *operator algebras*. A well-know result of Varopoulos [9] gives a necessary and sufficient conditions for a Banach algebra to be bi-continuously isomorphic to an operator algebra. In modern operator space language this characterization can be restated in the following way (see [2]). A Banach algebra $\mathcal{A}$ is bi-continuously isomorphic to an operator algebra if and only if there exists a constant K such that for each $r \geq 1$ the r-fold multiplication regarded as a map from the r-fold Haagerup tensor product of $MAX(\mathcal{A})$ into $MIN(\mathcal{A})$ has norm less than K^r. The reader should consult [2] for the definitions used in the above characterization.

Another characterization of subalgebras of $\mathbf{B}(\mathcal{H})$ was given by P. G. Dixon [3]. He proved that a Banach algebra $\mathcal{A}$ is an operator algebra if and only if, for every n and every polynomial in non-commuting indeterminates $p(X_1, \ldots, X_n)$ without constant term $\|p\|_{\mathcal{A}} \leq \|p\|_{\mathbf{B}(\mathcal{H})}$, where $\|p\|_{\mathcal{B}}$ denotes the supremum of $\|p(x_1, \ldots, x_n)\|$ over all $x_1, \ldots, x_n$ in the unit ball of the Banach algebra $\mathcal{B}$. The most often used characterization of operator algebras given by Blecher, Ruan and Sinclair. It is given in terms of hierarchy of matrix norms (see [8]).

In [5] the author obtained an abstract characterization of subalgebras of $\mathbf{B}(\mathcal{H})$ in algebraic terms without using a norm structure. In particular, an associative algebra $\mathcal{A}$ is isomorphic to an operator algebra if and only if the ∗-double $\mathcal{A} * \mathcal{A}$ is ∗-isomorphic to a self-adjoint operator algebra. Thus the characterization of operator algebras (among the associative algebras) is reduced to a characterization of self-adjoint operator algebras (among the ∗-algebras) which was previously done in [6, 7]. ∗-Algebra that has a faithful representation on a Hilbert space will be called C^*-*representable* in the sequel.

There is an example (see [7]) of an associative algebra which is not isomorphic to a subalgebra in any Banach algebra. In particular, such an algebra is not an operator algebra. Below we present a more delicate example of an associative algebra which

possesses a norm and which is a Banach algebra with respect to this norm but such that it is not isomorphic (as an associative algebra) to an operator algebra.

Let $\mathbf{B}(l^1(\mathbb{Z}))$ denote the Banach algebra of all bounded linear operators acting on a Banach space $l^1(\mathbb{Z})$. Let $\mathbf{B}_F(l^1(\mathbb{Z}))$ denote the ideal of finite-rank operators in $\mathbf{B}(l^1(\mathbb{Z}))$. The closure of this ideal is called the ideal of approximable operators and denoted by $\mathbf{B}_A(l^1(\mathbb{Z}))$ (for more details see [4]). Let $\mathcal{A}$ be $\mathbf{B}(l^1(\mathbb{Z}))$ considered as an abstract associative algebra.

Proposition 1. *There is no an algebraic isomorphism of $\mathcal{A}$ with a (not necessarily closed) subalgebra of bounded operators acting on a Hilbert space.*

Proof. Assume that $\pi\colon \mathcal{A} \to \mathbf{B}(\mathcal{H})$ is an injective homomorphism then by setting $\|a\|_\pi = \|\pi(a)\|$ for all $a \in \mathcal{A}$ one defines an algebra norm on $\mathcal{A}$. Since $l^1(\mathbb{Z})$ is isometrically isomorphic to the direct sum $l^1(\mathbb{Z}) \oplus l^1(\mathbb{Z})$, $\|\cdot\|_\pi$ is equivalent to the operator norm on $\mathbf{B}(l^1(\mathbb{Z}))$ by [4, Cor. 6.2.8]. Thus the restriction of π to $\mathbf{B}_F(l^1(\mathbb{Z}))$ is a continuous injective homomorphism of $\mathbf{B}_F(l^1(\mathbb{Z}))$ into $\mathbf{B}(\mathcal{H})$. Since $l^1(\mathbb{Z})$ is non-reflexive π must be a zero homomorphism by [4, Corollary on p. 81]. This contradiction proves that such π does not exist. $\square$

Remark 2. Note that the non-existence of a faithful representation of $A = \mathbf{B}(l^1(\mathbb{Z}))$ can also be proved along the same lines using the fact that $\mathbf{B}_A(\mathcal{X})$ is Arens regular if and only if the Banach space $\mathcal{X}$ is reflexive together with the well know fact that all subalgebras of $\mathbf{B}(\mathcal{H})$ are Arens regular.

A characterization of self-adjoint operator algebras was obtained in [7]. It is given in terms of algebraically admissible cones (see definition 3 below). The aim of the paper is to construct such cones for a particular class of $*$-algebras that the free products of $*$-algebras generated by unitaries. In Section 3 we also present a description of $*$-radicals in terms of the closure in some universal C^*-algebras.

2. C^*-REPRESENTABILITY OF FREE PRODUCTS

Firstly we give necessary definitions and fix notations. Let $\mathcal{A}_{sa}$ denote the set of self-adjoint elements in $\mathcal{A}$. A subset $C \subset \mathcal{A}_{sa}$ containing the unit e of $\mathcal{A}$ is an *algebraically admissible* cone provided that

Definition 3.

    (i) C is a cone in $\mathcal{A}_{sa}$, i.e., $\lambda x + \beta y \in C$ for all x, $y \in C$ and $\lambda \geq 0$, $\beta \geq 0$, $\lambda, \beta \in \mathbb{R}$;
    (ii) $C \cap (-C) = \{0\}$;
    (iii) $xCx^* \subseteq C$ for every $x \in \mathcal{A}$;

We call $e \in \mathcal{A}_{sa}$ an *order unit* if for every $x \in \mathcal{A}_{sa}$ there exists $r > 0$ such that $re + x \in C$. An order unit e is *Archimedean* if $re + x \in C$ for all $r > 0$ implies that $x \in C$
The following theorem was proved in [6].

Theorem 4. *Let $\mathcal{A}$ be a $*$-algebra with unit e and $C \subseteq \mathcal{A}_{sa}$ be a cone containing e. If $xCx^* \subseteq C$ for every $x \in \mathcal{A}$ and e is an Archimedean order unit then there is a unital $*$-representation $\pi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ such that $\pi(C) = \pi(\mathcal{A}_{sa}) \cap \mathbf{B}(\mathcal{H})^+$ where $\mathbf{B}(\mathcal{H})^+$ is the set of positive operators. Moreover,*

    (1) $\|\pi(x)\| = \inf\{r > 0 : r^2 \pm x^*x \in C\}$.
    (2) $\ker \pi = \{x : x^*x \in C \cap (-C)\}$.
    (3) *If $C \cap (-C) = \{0\}$ then $\ker \pi = \{0\}$, $\|\pi(a)\| = \inf\{r > 0 : r \pm a \in C\}$ for all $a = a^* \in \mathcal{A}$ and $\pi(C) = \pi(\mathcal{A}) \cap \mathbf{B}(\mathcal{H})^+$.*

In particular a unital ∗-algebra $\mathcal{A}$ which possesses an algebraically admissible cone such that the algebra unit is an Archimedean order unit has a faithful ∗-representation by bounded operators on a Hilbert space.

We will apply Theorem 4 to prove that a free product of two C^*-representable ∗-algebras is C^*-representable. This result is folklore and can be deduced from [1] but the author was unable to trace an explicit proof in the literature. Moreover, it is important to have examples of algebraically admissible cones satisfying conditions of Theorem 4.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two unital ∗-algebras and ϕ_1, ϕ_2 be linear unital functionals on $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Let $\overset{\circ}{\mathcal{A}}_j = \ker \phi_j$ $(j = 1, 2)$. The algebraic free product $\mathcal{A}_1 \star \mathcal{A}_2$ as a linear space is a quotient of

$$\mathbb{C} \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus (\mathcal{A}_1 \otimes \mathcal{A}_2) \oplus (\mathcal{A}_2 \otimes \mathcal{A}_1) \oplus (\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_1) \oplus \dots$$

by a subspace in order to identify the units in $\mathbb{C}$, $\mathcal{A}_1$ and $\mathcal{A}_2$. As vector spaces, $\mathcal{A}_j = \mathbb{C} \oplus \overset{\circ}{\mathcal{A}}_j$ $(j = 1, 2)$ and thus, as a vector space,

$$\mathcal{A}_1 \star \mathcal{A}_2 = \mathbb{C} \oplus \overset{\circ}{\mathcal{A}}_1 \oplus \overset{\circ}{\mathcal{A}}_2 \oplus (\overset{\circ}{\mathcal{A}}_1 \otimes \overset{\circ}{\mathcal{A}}_2) \oplus (\overset{\circ}{\mathcal{A}}_2 \otimes \overset{\circ}{\mathcal{A}}_1) \oplus (\overset{\circ}{\mathcal{A}}_1 \otimes \overset{\circ}{\mathcal{A}}_2 \otimes \overset{\circ}{\mathcal{A}}_1) \oplus \dots$$

The projection onto $\mathbb{C}$ associated with the above direct sum is a free product $\phi_1 \star \phi_2$ (see [1]).

For a ∗-algebra $\mathcal{B}$ we denote by $\Sigma^2(B)$ the set of finite sums of elements of the form x^*x with $x \in \mathcal{B}$. This is, clearly, a convex cone which is called the cone of Hermitian squares. Denote by $S(\mathcal{A}_j)$ the set of states on $\mathcal{A}_j$, i.e., the set of functionals $f\colon A_j \to \mathbb{C}$ such that $f(x) \geq 0$ for all $x \in \Sigma^2(\mathcal{A}_j)$ and $f(1) = 1$.

Put $C(\mathcal{A}_1, \mathcal{A}_2) = \{a \in \Sigma^2(\mathcal{A}_1 \star \mathcal{A}_2)|$ for all $\phi_1 \in S(\mathcal{A}_1), \phi_2 \in S(\mathcal{A}_2)$, and every $x \in \mathcal{A}_1 \star \mathcal{A}_2, (\phi_1 \star \phi_2)(xax^*) \geq 0\}$.

Theorem 5. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two C^*-representable ∗-algebras generated by unitaries. Then the set $C(\mathcal{A}_1, \mathcal{A}_2)$ is an algebraically admissible cone in the free product $\mathcal{A}_1 \star \mathcal{A}_2$ and the unit e of $\mathcal{A}_1 \star \mathcal{A}_2$ is Archimedean order unit.*

In particular, $\mathcal{A}_1 \star \mathcal{A}_2$ is C^*-representable.

Proof. Clearly $C = C(\mathcal{A}_1, \mathcal{A}_2)$ is a cone and for every its element a and every $x \in \mathcal{A}_1 \star \mathcal{A}_2$, $xax^* \in C$.

Let us show that $e \in C$, $C \cap (-C) = 0$ and e is an Archimedean order unit. Any product of the form $c_1 c_2 \dots c_l$ with $c_t \in \overset{\circ}{\mathcal{A}}_{d_t}$ $(d_t = 1, 2)$ and such that $d_t \neq d_{t+1}$ for all $t = 1, \dots, l-1$ will be called a word. Consider two words $a = a_1 a_2 \dots a_n$ and $b = b_1 b_2 \dots b_m$ where $a_k \in \overset{\circ}{\mathcal{A}}_{i_k}$, and $b_r \in \overset{\circ}{\mathcal{A}}_{j_r}$. Assume that $(\phi_1 \star \phi_2)(ab^*) \neq 0$ for some unital functional ϕ_1 and ϕ_2. We claim that $m = n$, $i_1 = j_1$, ..., $i_n = j_n$, and

(1) $$(\phi_1 \star \phi_2)(ab^*) = \phi_{i_1}(a_1 b_1^*) \cdot \dots \cdot \phi_{i_n}(a_n b_n^*).$$

We will assume that $n \leq m$. If $i_n \neq j_m$, then ab^* is a word since in the expression $a_1 \dots a_n b_m^* \dots b_1^*$, the adjacent terms belong to different spaces $\overset{\circ}{\mathcal{A}}_r$. Thus $(\phi_1 \star \phi_2)(ab^*) = 0$. This contradiction shows that $i_n = j_m$. Then $a_n b_m^* = \phi_{i_n}(a_n b_m^*) + c_n$ with some $c_n \in \overset{\circ}{\mathcal{A}}_{i_n}$. Compute

$$a_{n-1} a_n b_m^* b_{m-1} = \phi_{i_n}(a_n b_m^*) a_{n-1} b_{m-1}^* + a_{n-1} c_n b_{m-1}^*.$$

Since $a_{n-1} c_n b_{m-1}^*$ is a word, the element

$$a_1 \dots a_{n-1} c_n b_{m-1}^* \dots b_1^*$$

is also a word. Hence $i_{n-1} = j_{m-1}$ and $a_{n-1} b_{m-1}^* = \phi_{i_{n-1}}(a_{n-1} b_{m-1}^*) + c_{n-1}$ for some $c_{n-1} \in \overset{\circ}{\mathcal{A}}_{i_{n-1}}$. Hence,

$$a_{n-1} a_n b_m^* b_{m-1} = \alpha_{n-1} \alpha_n e + \alpha_n c_{n-1} + a_{n-1} c_n b_{m-1}^*$$

where $\alpha_k = \phi_{i_k}(a_k b^*_{m-n+k})$.

By induction we get

$$a_1 a_2 \ldots a_n b^*_m b^*_{m-1} \ldots b^*_1 = \big[\alpha_1 \ldots \alpha_n + \alpha_2 \ldots \alpha_n c_1$$
$$+ \alpha_3 \ldots \alpha_n a_1 c_2 b^*_{m-n+1} + \alpha_4 \ldots \alpha_n a_1 a_2 c_3 b^*_{m-n+2} b^*_{m-n+1} + \ldots$$
$$+ a_1 \ldots a_{n-1} c_n b^*_{m-1} b^*_{m-2} \ldots b^*_{m-n+1} \big] b^*_{m-n} \ldots b^*_1,$$

with some $c_r \in \overset{\circ}{\mathcal{A}}_{i_r}$.

Thus $(\phi_1 \star \phi_2)(ab^*) \neq 0$ implies that $m = n$ and that equality (1) holds.

Every $x \in \mathcal{A}_1 \star \mathcal{A}_2$ is of the form

$$(2) \qquad\qquad x = \alpha e + \sum_{n \geq 1} \sum_{(i_1 i_2 \ldots i_n)} \sum_j a^{(j)}_{i_1} \ldots a^{(j)}_{i_n}$$

for some $a^{(j)}_{i_r} \in \overset{\circ}{\mathcal{A}}_{i_r}$. Applying formula (1) we get

$$(3) \qquad (\phi_1 \star \phi_2)(xx^*) = \sum_n \sum_{(i_1 i_2 \ldots i_n)} \sum_{j,s} \phi_{i_1}(a^{(j)}_{i_1} a^{(s)*}_{i_1}) \cdot \ldots \cdot \phi_{i_n}(a^{(j)}_{i_n} a^{(s)*}_{i_n}).$$

By the GNS construction we can consider ϕ_j as a vector states corresponding to some representations $\pi_j : \mathcal{A}_j \to B(H_j)$, i.e., $\phi_j(x) = \langle \pi_j(x)\xi_j, \xi_j \rangle$ for some unit vector $\xi_j \in H_j$.

For each multi-index $(i_1 \ldots i_n)$, the sum

$$(4) \qquad\qquad \sum_{j,s} \phi_{i_1}(a^{(j)}_{i_1} a^{(s)*}_{i_1}) \cdot \ldots \cdot \phi_{i_n}(a^{(j)}_{i_n} a^{(s)*}_{i_n})$$

is just $\langle \pi(yy^*)\xi, \xi \rangle = (\phi_{i_1} \otimes \ldots \otimes \phi_{i_n})(y^*y)$, where $\pi = \pi_{i_1} \otimes \ldots \otimes \pi_{i_n}$ is a $*$-representation of $\mathcal{A}_{i_1} \otimes \ldots \otimes \mathcal{A}_{i_n}$, $\xi = \xi_{i_1} \otimes \ldots \otimes \xi_n$ and $y = \sum_j a^{(j)}_{i_1} \otimes \ldots \otimes a^{(j)}_{i_n}$. Hence (2) is non-negative and consequently $(\phi_1 \star \phi_2)(xx^*) \geq 0$. This proves that $e \in C$.

Assume that there exists a non-zero $a \in C \cap (-C)$. Then for every $y \in \mathcal{A}_1 \star \mathcal{A}_2$, $(\phi_1 \star \phi_2)(y^*ay) = 0$ for all $\phi_j \in S(\mathcal{A}_j)$. Since $a \in \Sigma^2(\mathcal{A}_1 \star \mathcal{A}_2)^+$, $a = \sum x^*_j x_j$ for some $x_j \in \mathcal{A}_1 \star \mathcal{A}_2$. Since $(\phi_1 \star \phi_2)(y^*x^*_j x_j y) \geq 0$ for all j we can assume that a is of the form x^*x. Since A_j is generated by unitary elements and the product of unitaries is unitary, we have that A_j is linearly generated by unitaries. Hence we can also assume that x is written in the form (2) with $a^{(j)}_s$ being non-scalar unitary elements. If $\alpha \neq 0$ then $(\phi_1 \star \phi_2)(x^*x) \geq |\alpha|^2 > 0$. Otherwise, by (3) we can assume that $x = \sum_j a^{(j)}_{i_1} \ldots a^{(j)}_{i_n}$ for some multi-index $(i_1 \ldots i_n)$ and such that all the summands of x are linearly independent. Taking $y = a^{(1)*}_{i_n} \ldots a^{(1)n*}_{i_1}$ we get that xy has a representation (2) with the summand corresponding to $n = 0$ being the unit e. Hence $(\phi_1 \star \phi_2)(y^*x^*xy) \geq (\phi_1 \star \phi_2)(e) = 1$. The obtained contradiction proves that $C \cap (-C) = \{0\}$. The C^*-representability of $\mathcal{A}_1 \star \mathcal{A}_2$ now follows from Theorem 4. $\qquad\square$

3. C^*-REPRESENTABILITY OF $*$-ALGEBRAS

For a $*$-algebra $\mathcal{A}$ the *reducing ideal* denoted by $\mathrm{Rad}(\mathcal{A})$ is the intersection of the kernels of all $*$-representations of $\mathcal{A}$ on Hilbert spaces. There is a connection between the problem of finding reducing ideals and finding closures of ideals in the C^*-algebras which was presented in [5] in case of $*$-algebras generated by unitaries. Here we consider a more general class of algebraically bounded $*$-algebras. Recall that a $*$-algebra A is called *algebraically bounded* if any $x \in A$ is *bounded*, i.e., there is $C > 0$ such that $C - x^*x \in \Sigma^2(A)$. It is know that A is algebraically bounded if it is generated by bounded elements (see for example [7]).

Given an algebraically bounded $*$-algebra $\mathcal{A}$ one can find a corepresentation, i.e., a $*$-isomorphism $\phi \colon \mathcal{A} \to \mathcal{F}/\mathcal{J}$ where $\mathcal{F}$ is a free $*$-algebra with a generating set $\{X_\lambda\}_{\lambda \in \Lambda}$

and $\mathcal{J}$ is a ∗-ideal of $\mathcal{F}$. Let x_λ denote $\phi^{(-1)}(X_\lambda)$. Consider the algebraically bounded ∗-algebra

$$\mathcal{F}_1 = \mathbb{C}\langle\{X_\lambda\} \cup \{Y_\lambda\} \mid X_\lambda^* X_\lambda + Y_\lambda^* Y_\lambda = c(\lambda)\rangle,$$

where $c(\lambda)$ is chosen such that $c(\lambda) - x_\lambda^* x_\lambda \in \Sigma^2(\mathcal{A})$. Such $c(\lambda)$ exists for every λ since $\mathcal{A}$ is algebraically bounded. Let $G = \{X_\lambda\} \cup \{Y_\lambda\})$ be equipped with well-ordering $\prec$. One can easily check that the defining relation above constitute a Gröbner basis for the associative algebra $\mathcal{F}_1$ with respect to degree-lexicographic order induced on the free semigroup generated by G by the order $\prec$. It follows that the map $\mathcal{F} \ni X_\lambda \to X_\lambda \in \mathcal{F}_1$ extends to an embedding $i\colon \mathcal{F} \to \mathcal{F}_1$. Denote by $\mathcal{J}_1$ the ideal of $\mathcal{F}_1$ generated by $i(\mathcal{J})$ and by $\widetilde{i}\colon \mathcal{F}/\mathcal{J} \to \mathcal{F}_1/\mathcal{J}_1$ the induced embedding. Let $\tau\colon \mathcal{A} \to \mathcal{F}_1/\mathcal{J}_1$ denote the injective ∗-homomorphism $\widetilde{i} \circ \phi$.

The pair $(\mathcal{F}_1/\mathcal{J}_1, \tau)$ has the property that every ∗-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ can be extended to a ∗-representation $\widetilde{\pi}$ of $\mathcal{F}_1/J_1$ on the same $\mathcal{H}$. Indeed, put $\widetilde{\pi}(X_\lambda) = \pi(X_\lambda)$ and $\widetilde{\pi}(Y_\lambda) = (c(\lambda) - \pi(X_\lambda)^*\pi(X_\lambda))^{1/2}$. It is clear from the definition of $c(\lambda)$ that $c(\lambda) - \pi(X_\lambda)^*\pi(X_\lambda)$ is a positive operator, hence, the square root exists. Hence $\widetilde{\pi}$ can be extended to a ∗-homomorphism of $\mathcal{F}_1$ which coincides with π on the ∗-subalgebra generated by X_λ. Thus $\widetilde{\pi}(i(\mathcal{J})) = \{0\}$ and $\widetilde{\pi}$ induces a ∗-representation of $\mathcal{F}_1/\mathcal{J}_1$ which will also be denoted by $\widetilde{\pi}$.

Let $C^*(\mathcal{F}_1)$ be the universal enveloping C^*-algebra of $\mathcal{F}_1$. Then there is the canonical ∗-homomorphism $\gamma\colon \mathcal{F}_1 \to C^*(\mathcal{F}_1)$ with a dense image and such that every ∗-homomorphism π of $\mathcal{F}_1$ can be extended to a ∗-homomorphism $\hat{\pi}$ of $C^*(\mathcal{F}_1)$, i.e.,

$$\hat{\pi}(\gamma(f)) = \pi(f)$$

for all $f \in \mathcal{F}_1$. In particular, γ induce a topology on $\mathcal{F}_1$. Let S^{cl} denote the closure in this topology of a subset $S \subseteq \mathcal{F}_1$.

Proposition 6. $\mathrm{Rad}(\mathcal{A}) = (\mathcal{F} \cap \mathcal{J}^{cl})/\mathcal{J}$.

Proof. Let $q\colon \mathcal{F}_1 \to \mathcal{F}_1/\mathcal{J}_1$ be the canonical epimorphism and let $\pi\colon \mathcal{F}_1/\mathcal{J}_1 \to C^*(\mathcal{F}_1/\mathcal{J}_1)$ denote the canonical ∗-homomorphism into the enveloping C^*-algebra $C^*(\mathcal{F}_1/\mathcal{J}_1)$. By the universal property of an enveloping C^*-algebra it follows that $\mathrm{Rad}(\mathcal{A}) = \ker \pi$ and, by the same property, there is an extension of q to a surjective ∗-homomorphism $\hat{q}\colon C^*(\mathcal{F}_1) \to C^*(\mathcal{F}_1/\mathcal{J}_1)$. Since $\mathcal{J}_1 \subseteq \ker \hat{q}$ and the letter is closed we have that $\mathcal{J}_1^{cl} \subseteq \ker \hat{q}$.

To show the converse inclusion note that the quotient $C^*(\mathcal{F}_1)/\mathcal{J}_1{}^{cl}$ is a C^*-algebra. It can be regarded as a C^*-subalgebra in $\mathbf{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Hence the quotient map $C^*(\mathcal{F}_1) \to C^*(\mathcal{F}_1)/\mathcal{J}_1{}^{cl}$ can be viewed as a ∗-representation $\eta\colon C^*(\mathcal{F}_1) \to \mathbf{B}(\widetilde{\mathcal{H}})$. The restriction of η to $\mathcal{F}_1$ annulates $\mathcal{J}_1$ and thus can be regarded as a ∗-representation $\widetilde{\eta}$ of $\mathcal{F}_1/\mathcal{J}_1$. We will denote by the same symbol its unique extension to $C^*(\mathcal{F}_1/\mathcal{J}_1)$. A moment reflection reveals that for every $f \in \mathcal{F}_1$ we have $\widetilde{\eta}(\hat{q})(f) = f + \mathcal{J}_1{}^{cl}$ and $\eta(f) = f + \mathcal{J}_1{}^{cl}$. Hence the following diagram is commutative:

$$
\begin{array}{ccc}
C^*(\mathcal{F}_1) & \xrightarrow{\hat{q}} & C^*(\mathcal{F}_1/\mathcal{J}_1) \\
& \eta \searrow & \downarrow \widetilde{\eta} \\
& & \mathbf{B}(\mathcal{H})
\end{array}
$$

Thus $\ker \hat{q} \subseteq \ker \eta = \mathcal{J}_1{}^{cl}$ which gives $\ker \hat{q} = \mathcal{J}_1{}^{cl}$. Thus $\mathrm{Rad}(\mathcal{F}_1/\mathcal{J}_1) = (\mathcal{J}_1{}^{cl} \cap \mathcal{F}_1)/\mathcal{J}_1$. Then $\ker \pi = (i(\mathcal{F}) \cap \ker \hat{q})/\mathcal{J} = (\mathcal{F} \cap J_1{}^{cl})/\mathcal{J}$. □

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