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ls-PONOMAREV-SYSTEMS AND COMPACT IMAGES OF LOCALLY SEPARABLE METRIC SPACES

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Abstract. We introduce the notion of an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$, and give necessary and sufficient conditions such that the mapping f is a compact (compact-covering, sequence-covering, pseudo-sequence-covering, sequentially-quotient) mapping from a locally separable metric space M onto a space X. As applications of these results, we systematically get characterizations of certain compact images of locally separable metric spaces.

1. Introduction

Finding characterizations of certain images of metric spaces is of a considerable interest in general topology. In the past, many nice results have been obtained in [8], [10], [11], [16], [17], [19]. Related to characterizing images of metric spaces, many topologists were engaged in a research of characterizations of compact images of locally separable metric spaces, and some noteworthy results have been shown. In [9], Y. Ikeda, C. Liu and Y. Tanaka characterized quotient compact images of locally separable metric spaces. After that, Y. Ge characterized pseudo-sequence-covering compact images of locally separable metric spaces in [5]. In general, it is difficult to obtain nice characterizations of compact images of locally separable metric spaces (under covering-mappings) instead of metric domains. It is known that the key to prove these results is to construct covering-mappings and compact mappings from locally separable metric spaces. In [14], V. I. Ponomarev proved that every first countable space can be characterized as an open image of a subspace of Baire's zero-dimensional space. After that, S. Lin and P. Yan generalized "Ponomarev's method" to establish a system $(f, M, X, \{\mathcal{P}_n\})$, called a *Ponomarev-system*, to characterize images of metric spaces in [12]. Recently, Y. Tanaka and Y. Ge investigated the Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, and characterized certain compact-covering (or sequence-covering) quotient compact images of metric spaces in terms of weak bases or symmetric spaces, and considered relations between these compact-covering images and sequence-covering images in [18]. Moreover, for a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, Y. Ge has obtained necessary and sufficient conditions such that the mapping f is a compact-covering (pseudo-sequence-covering, sequentially-quotient, compact) mapping from a metric space M onto a space X in [7].

From the above, the following question naturally arises.

Question 1.1. *Find a consistent method to construct a covering-mapping (compact mapping) onto a space X from some locally separable metric space?*

In this paper, same as the Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, where M is a metric space, we introduce the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$, where M is a locally

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separable metric space, and give necessary and sufficient conditions such that the mapping f is a compact (compact-covering, sequence-covering, pseudo-sequence-covering, sequentially-quotient) mapping from a locally separable metric space M onto a space X. As applications of these results, we systematically get characterizations of certain compact images of locally separable metric spaces.

Throughout this paper, all spaces are T_1 and regular, all mappings are continuous and onto, a convergent sequence includes its limit point, $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \longrightarrow Y$ be a mapping, $x \in X$, and $\mathcal{P}, \mathcal{Q}$ be families of subsets of X, we denote $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $st(x, \mathcal{P}) = \bigcup \mathcal{P}_x$, $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$, and $\mathcal{P} \cap \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ converging to x is *eventually* (resp., *frequently*) in A if $\{x_n : n \geq n_0\} \cup \{x\} \subset A$ for some $n_0 \in \mathbb{N}$ (resp., $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset A$ for some subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$).

For terms which are not defined here, please refer to [4] and [17].

2. Results

Definition 2.1. Let $\mathcal{P}$ be a family of subsets of a space X, and K be a subset of X. Assume that $\mathcal{P}$ is closed under finite intersections.

(1) For each $x \in X$, $\mathcal{P}$ is a *network at x in X*, if $\mathcal{P} \subset \mathcal{P}_x$, and if $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

(2) $\mathcal{P}$ is a *cfp-cover for K in X*, if for each compact subset H of K, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\}$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. Note that such an $\mathcal{F}$ is a *full cover* in the sense of [3]. If $K = X$, then a cfp-cover for K in X is a *cfp-cover for X* [20].

(3) $\mathcal{P}$ is a *cs-cover for K in X* (resp., *cs^*-cover for K in X*), if for each convergent sequence S in K, S is eventually (resp., frequently) in some $P \in \mathcal{P}$. If $K = X$, then a cs-cover for K in X (resp., cs^*-cover for K in X) is a *cs-cover for X* [21] (resp., *cs^*-cover for X* [18]).

(4) $\mathcal{P}$ is a *wcs-cover for K in X*, if for each convergent sequence S in K, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}_x$ such that S is eventually in $\bigcup \mathcal{F}$. If $K = X$, then a wcs-cover for K in X is a *wcs-cover* [6].

(5) A cfp-cover (resp., cs-cover, wcs-cover, cs^*-cover) for X is abbreviated to a *cfp-cover* (resp., *cs-cover, wcs-cover, cs^*-cover*).

Remark 2.2. For each subset K of X, if $\mathcal{P}$ is a cfp-cover (resp., cs-cover, wcs-cover, cs^*-cover), then $\mathcal{P}$ is a cfp-cover (resp., cs-cover, wcs-cover, cs^*-cover) for K in X.

Lemma 2.3. *Let $\mathcal{P}$ be a countable family of subsets of a space X. Then the following are equivalent for a convergent sequence S in X.*

(1) *$\mathcal{P}$ is a cfp-cover for S in X.*
(2) *$\mathcal{P}$ is a wcs-cover for S in X.*
(3) *$\mathcal{P}$ is a cs^*-cover for S in X.*

Proof. (1) ⇒ (2) ⇒ (3). It is obvious.

(3) ⇒ (1). It follows from [19, Lemma 3]. □

Definition 2.4. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is a *compact-covering* mapping [13], if every compact subset of Y is the image of some compact subset of X.

(2) f is a *sequence-covering* mapping [15], if for every convergent sequence S of Y, there is a convergent sequence L of X such that $f(L) = S$.

(3) f is a *pseudo-sequence-covering* mapping [9], if every convergent sequence of Y is the image of some compact subset of X.

(4) f is a *subsequence-covering* mapping [11], if for every convergent sequence S of Y, there is a compact subset K of X such that $f(K)$ is a subsequence of S.

(5) f is a *sequentially-quotient* mapping [2], if for every convergent sequence S of Y, there is a convergent sequence L of X such that $f(L)$ is a subsequence of S.

(6) f is a *compact* mapping [1], if $f^{-1}(y)$ is compact for every $y \in Y$.

The following lemma is clear, where certain covers are preserved under covering-mappings.

Lemma 2.5. *Let $f : X \longrightarrow Y$ be a mapping, and $\mathcal{P}$ be a cover for X. Then the following hold.*

(1) *If $\mathcal{P}$ is a cs-cover for X and f is sequence-covering, then $f(\mathcal{P})$ is a cs-cover for Y.*
(2) *If $\mathcal{P}$ is a cfp-cover for X and f is compact-covering, then $f(\mathcal{P})$ is a cfp-cover for Y.*
(3) *If $\mathcal{P}$ is a wcs-cover for X and f is pseudo-sequence-covering, then $f(\mathcal{P})$ is a wcs-cover for Y.*
(4) *If $\mathcal{P}$ is a cs*-cover for X and f is sequentially-quotient, then $f(\mathcal{P})$ is a cs*-cover for Y.*

The next result concerns preservation of certain covers but no need of covering-properties of mappings.

Lemma 2.6. *Let $f : X \longrightarrow Y$ be a mapping, and $\mathcal{P}$ be a cover for X. Then the following hold.*

(1) *If $\mathcal{P}$ is a cs-cover for a convergent sequence S in X, then $f(\mathcal{P})$ is a cs-cover for $f(S)$ in Y.*
(2) *If $\mathcal{P}$ is a cfp-cover for a compact subset K in X, then $f(\mathcal{P})$ is a cfp-cover for $f(K)$ in Y.*
(3) *If $\mathcal{P}$ is a wcs-cover for a convergent sequence S in X, then $f(\mathcal{P})$ is a wcs-cover for $f(S)$ in Y.*
(4) *If $\mathcal{P}$ is a cs*-cover for a convergent sequence S in X, then $f(\mathcal{P})$ is a cs*-cover for $f(S)$ in Y.*

Proof. (1). Let L be a convergent sequence in $f(S)$. Then $K = f^{-1}(L) \cap S$ is a convergent sequence in S satisfying that $f(K) = L$. Since $\mathcal{P}$ is a cs-cover for S in X, K is eventually in some $P \in \mathcal{P}$. This implies that L is eventually in $f(P)$. Therefore, $f(\mathcal{P})$ is a cs-cover for $f(S)$ in Y.

(2). Let L be a compact subset of $f(K)$. Then $H = f^{-1}(L) \cap K$ is a compact subset of K satisfying that $f(H) = L$. Since $\mathcal{P}$ is a cfp-cover for K in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\}$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. This implies that $f(\mathcal{F})$ is a finite subfamily of $f(\mathcal{P})$ such that $L \subset \bigcup\{f(C_F) : f(F) \in f(\mathcal{F})\}$, where $f(C_F)$ is closed and $f(C_F) \subset f(F)$ for every $f(F) \in f(\mathcal{F})$. Therefore, $f(\mathcal{P})$ is a cfp-cover for $f(K)$ in Y.

(3). Let L be a convergent sequence in $f(S)$ converging to y in Y. Then $K = f^{-1}(L) \cap S$ is a convergent sequence in S converging to some $x \in f^{-1}(y)$, and $f(K) = L$. Since $\mathcal{P}$ is a wcs-cover for S in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}_x$ such that K is eventually in $\bigcup \mathcal{F}$. Then $f(\mathcal{F})$ is a finite subfamily of $(f(\mathcal{P}))_y$ and L is eventually in $\bigcup f(\mathcal{F})$. It implies that $f(\mathcal{P})$ is a wcs-cover for $f(S)$ in Y.

(4). Let L be a convergent sequence in $f(S)$. Then $K = f^{-1}(L) \cap S$ is a convergent sequence in S satisfying that $f(K) = L$. Since $\mathcal{P}$ is a cs^*-cover for S in X, K is frequently some $P \in \mathcal{P}$. Then L is frequently in $f(P)$. Therefore, $f(\mathcal{P})$ is a cs^*-cover for $f(S)$ in Y. $\square$

Definition 2.7. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a cover sequence for a space X. $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is *a point-star network for* X [12], if $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for every $x \in X$.

Definition 2.8. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a point-star network for X. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, and endowed A_n with discrete topology. Put

$$M = \Big\{ a = (\alpha_n) \in \prod_{n \in \mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X \Big\}.$$

Then M, which is a subspace of the product space $\prod_{n \in \mathbb{N}} A_n$, is a metric space, x_a is unique, and $x_a = \bigcap_{n \in \mathbb{N}} P_{\alpha_n}$ for every $a \in M$. Define $f : M \longrightarrow X$ by $f(a) = x_a$, then f is a mapping and $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev-system* [12].

Remark 2.9. There are two ways to define the Ponomarev-system in [12]. The Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ requires that $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a point-star network for X, and the Ponomarev-system $(f, M, X, \mathcal{P})$ requires that $\mathcal{P}$ is a strong network for X (i.e., for each $x \in X$, there exists a countable $\mathcal{P}(x) \subset \mathcal{P}$ such that $\mathcal{P}(x)$ is a network at x in X). In this paper, we use the definition of Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, where $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a point-star network for X.

In [18, Lemma 2.2] and [7, Theorem 2.7], the authors have investigated the Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ and obtained conditions such that the mapping f is a compact mapping (covering-mapping) from a metric space M onto a space X. Now, based on certain covers for a convergent sequence (compact subset) of a space, we get the following.

Lemma 2.10. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a Ponomarev-system. Then the following hold.*

(1) *$\mathcal{P}_n$ is a cs-cover for a convergent sequence S in X for each $n \in \mathbb{N}$ if and only if there exists a convergent sequence L in M such that $S = f(L)$.*
(2) *$\mathcal{P}_n$ is a cfp-cover for a compact set K in X for each $n \in \mathbb{N}$ if and only if there exists a compact subset L of M such that $K = f(L)$.*
(3) *$\mathcal{P}_n$ is a wcs-cover for a convergent sequence S in X for each $n \in \mathbb{N}$ if and only if there exists a compact subset L of M such that $S = f(L)$.*
(4) *$\mathcal{P}_n$ is a cs^*-cover for a convergent sequence S in X for each $n \in \mathbb{N}$ if and only if there exists a convergent sequence L in M such that $f(L)$ is a convergent subsequence of S.*

Proof. (1). *Necessity.* For each $n \in \mathbb{N}$, let each $\mathcal{P}_n$ be a *cs*-cover for a convergent sequence S in X. As in the proof of [18, Lemma 2.2(ii)], $S = f(L)$ for some convergent sequence L in M.

Sufficiency. Let S be a convergent sequence in X and $S = f(L)$ for some convergent sequence L in M. Then, as in the proof (2) of [7, Theorem 2.7], S is eventually in some P_{α_n} for each $n \in \mathbb{N}$. It implies that $\mathcal{P}_n$ is a *cs*-cover for S in X.

(2). As in the proof of [7, Theorem 2.7(3)].

(3). It follows from Lemma 2.3 and (2).

(4). As in the proof of [7, Theorem 2.7(2)]. □

Definition 2.11. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a cover for a space X such that each X_λ has a sequence cover $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$.

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cover* for X, if $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ consisting of countable covers $\mathcal{P}_{\lambda,n}$.

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *point-finite double point-star cover* for X, if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cover for X, and for each $\lambda \in \Lambda$ and $n \in \mathbb{N}$, both $\{X_\lambda : \lambda \in \Lambda\}$ and $\mathcal{P}_{\lambda,n}$ are point-finite.

Definition 2.12. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cover for X.

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cs-cover* for X, if for each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is eventually in X_λ and, for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ.

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cfp-cover* for X, if for each compact subset K of X, there exists a finite subset Λ_K of Λ such that $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$ and, for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, K_λ is compact and $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ.

(3) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star wcs-cover* for X, if for each convergent sequence S in X, there exists a finite subset Λ_S of Λ such that $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$ and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, S_λ is a convergent sequence and $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ.

(4) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cs*-cover* for X, if for each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is frequently in X_λ and, for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a subsequence S_λ of S in X_λ.

(5) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *point-finite double point-star cs-cover* (resp., *point-finite double point-star cfp-cover*, *point-finite double point-star wcs-cover*, *point-finite double point-star cs*-cover*) for X, if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover (resp., double point-star cfp-cover, double point-star wcs-cover, double point-star cs^*-cover) for X, and for each $\lambda \in \Lambda$ and $n \in \mathbb{N}$, both $\{X_\lambda : \lambda \in \Lambda\}$ and $\mathcal{P}_{\lambda,n}$ are point-finite.

Remark 2.13. If $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cover (resp., double point-star cfp-cover, double point-star cs-cover, double point-star wcs-cover, double point-star cs^*-cover) for X, then $\{X_\lambda : \lambda \in \Lambda\}$ is a cover (resp, cfp-cover, cs-cover, wcs-cover, cs^*-cover) for X, and each $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ.

Definition 2.14. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cover for a space X, and $(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\})$ be the Ponomarev-system for each $\lambda \in \Lambda$. Since each $\mathcal{P}_{\lambda,n}$ is countable, M_λ is a separable metric space. Put $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, and $f = \bigoplus_{\lambda\in\Lambda} f_\lambda$. Then M is a locally separable metric space, and f is a mapping from a locally separable metric space M onto X. The system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an *ls-Ponomarev-system.*

Y. Ge has proved a necessary and sufficient condition such that the mapping f is a compact mapping from a metric space M onto a space X, where $(f, M, X, \{\mathcal{P}_n\})$ is a Ponomarev-system in [7, Lemma 2.7]. The following result is a necessary and sufficient condition such that the mapping f is a compact mapping from a locally separable metric space M onto a space X, where $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an ls-Ponomarev-system.

Theorem 2.15. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. Then f is a compact mapping if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X.*

Proof. Necessity. Let f be a compact mapping. For each $x \in X$, since $f^{-1}(x)$ is compact, $\{\lambda \in \Lambda : f^{-1}(x) \cap M_\lambda \neq \emptyset\}$ is finite. Then $\{\lambda \in \Lambda : x \in X_\lambda\}$ is finite, i.e., $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite. On the other hand, for each $\lambda \in \Lambda$, since $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is compact, f_λ is compact. Then each $\mathcal{P}_{\lambda,n}$ is point-finite by [7, Theorem 2.7(1)]. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a point-finite double point-star cover for X. For each $x \in X$, $\Lambda_x = \{\lambda \in \Lambda : x \in X_\lambda\}$ is finite by point-finiteness of $\{X_\lambda : \lambda \in \Lambda\}$. Since each $\mathcal{P}_{\lambda,n}$ is point-finite, $f_\lambda^{-1}(x)$ is compact by [7, Theorem 2.7(1)]. It implies that $f^{-1}(x) = \bigcup\{f_\lambda^{-1}(x) : \lambda \in \Lambda_x\}$ is compact, i.e., f is a compact mapping. □

Corollary 2.16. *The following are equivalent for a space X.*

(1) *X is a compact image of a locally separable metric space.*

(2) *X has a point-finite double point-star cover.*

Proof. (1) ⇒ (2). Let $f : M \longrightarrow X$ be a compact mapping from a locally separable metric M onto X. Since M is a locally separable metric space, $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ where each M_λ is separable by [4, 4.4.F]. Since each M_λ is a separable metric space, M_λ has a sequence of open countable covers $\{\mathcal{B}_{\lambda,n} : n \in \mathbb{N}\}$ such that for every compact subset K of M_λ and any open set U in M_λ with $K \subset U$, there exists $n \in \mathbb{N}$ satisfying $st(K, \mathcal{B}_{\lambda,n}) \subset U$ by [4, 5.4.E]. Let $\mathcal{C}_{\lambda,n}$ be a locally finite open refinement of each $\mathcal{B}_{\lambda,n}$. Then, for each $\lambda \in \Lambda$, $\{\mathcal{C}_{\lambda,n} : n \in \mathbb{N}\}$ is a sequence of locally finite open countable covers for M_λ such that for every compact subset K of M_λ and any open set U in M_λ with $K \subset U$, there exists $n \in \mathbb{N}$ satisfying $st(K, \mathcal{C}_{\lambda,n}) \subset U$. For each $\lambda \in \Lambda$ and $n \in \mathbb{N}$, put $X_\lambda = f(M_\lambda)$, and $\mathcal{P}_{\lambda,n} = f(\mathcal{C}_{\lambda,n})$. We have the following claims (a)–(e).

(a) $\{X_\lambda : \lambda \in \Lambda\}$ is a cover for X.

(b) Each $\mathcal{P}_{\lambda,n}$ is countable.

(c) For each $\lambda \in \Lambda$, $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ.

Let $x \in U$ with U open in X_λ. Then $x \in V$ with V open in X and $V \cap X_\lambda = U$. Since f is compact, $f^{-1}(x)$ is compact. Then $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is a compact subset of M_λ and $f_\lambda^{-1}(x) \subset V_\lambda$ with $V_\lambda = f^{-1}(V) \cap M_\lambda$ open in M_λ. Therefore, there exists $n \in \mathbb{N}$ such that $st(f_\lambda^{-1}(x), \mathcal{C}_{\lambda,n}) \subset V_\lambda$. It implies that $st(x, \mathcal{P}_{\lambda,n}) \subset f(f^{-1}(V) \cap M_\lambda) \subset V \cap X_\lambda = U$. Then $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ.

(d) $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite.

For each $x \in X$, since f is compact, $f^{-1}(x)$ is compact. Then $f^{-1}(x)$ meets only finitely many M_λ's. It implies that $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite.

(e) Each $\mathcal{P}_{\lambda,n}$ is point-finite.

For each $x \in X_\lambda$, since f is compact, $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is a compact subset of M_λ. Then $f_\lambda^{-1}(x)$ meets only finitely many members of $\mathcal{C}_{\lambda,n}$ by locally finiteness of $\mathcal{C}_{\lambda,n}$ for each $n \in \mathbb{N}$. It implies that x meets only finitely many members of $\mathcal{P}_{\lambda,n}$ for each $n \in \mathbb{N}$. Then each $\mathcal{P}_{\lambda,n}$ is point-finite.

From (a)-(e) we get that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X_λ.

(2) ⇒ (1). By Theorem 2.15. □

In [7] and [18], the authors have proved conditions such that the mapping f is a covering-mapping from a metric space M onto a space X, where $(f, M, X, \{\mathcal{P}_n\})$ is a Ponomarev-system. Next, we give necessary and sufficient conditions such that the mapping f is a covering-mapping from a locally separable metric space M onto a space X, where $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an ls-Ponomarev-system.

Theorem 2.17. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. Then the following hold.*

(1) *f is sequence-covering if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover for X.*

(2) *f is compact-covering if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cfp-cover for a space X.*

(3) *f is pseudo-sequence-covering if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X.*

(4) *f is sequentially-quotient if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs*-cover for X.*

Proof. (1). *Necessity.* Let f be sequence-covering. For each convergent sequence S in X, $S = f(L)$ for some convergent sequence L in M. Then L is eventually in some M_λ. Therefore, S is eventually in X_λ. Put $S_\lambda = f_\lambda(L_\lambda)$, where $L_\lambda = L \cap M_\lambda$ is a convergent sequence. It follows from Lemma 2.10 that each $\mathcal{P}_{\lambda,n}$ is a cs-cover for S_λ in X_λ. Then each $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cs-cover for X. For each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is eventually in X_λ and, for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ. It follows from Lemma 2.10 that there exists a convergent sequence L_λ in M_λ such that $S_\lambda = f_\lambda(L_\lambda) = f(L_\lambda)$. Since $S - S_\lambda$ is finite, $S - S_\lambda = f(F)$ for some finite subset F of M. Put $L = F \cup L_\lambda$, then L is a convergent sequence in M and $S = f(L)$. It implies that f is sequence-covering.

(2). *Necessity.* Let f be compact-covering. For each compact subset K of X, $K = f(L)$ for some compact subset L of M. Since L is compact, $\Lambda_K = \{\lambda \in \Lambda : L \cap M_\lambda \neq \emptyset\}$ is a finite subset of Λ and each $L_\lambda = L \cap M_\lambda$ is compact. For each $\lambda \in \Lambda_K$, put $K_\lambda = f_\lambda(L_\lambda)$. Then K_λ is compact, $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$, and each $\mathcal{P}_\lambda$ is a cfp-cover for K_λ in X_λ by Lemma 2.10. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cfp-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cfp-cover for X. For each compact subset K of X, there exists a finite subset Λ_K of Λ such that $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$ and, for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, K_λ is compact and $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ. It follows from Lemma 2.10 that there exists a compact subset L_λ of M_λ such that $K_\lambda = f_\lambda(L_\lambda) = f(L_\lambda)$. Put $L = \bigcup\{L_\lambda : \lambda \in \Lambda_K\}$. Then L is a compact subset of M and $K = f(L)$. It implies that f is compact-covering.

(3). *Necessity.* Let f be pseudo-sequence-covering. For each convergent sequence S in X, $S = f(L)$ for some compact subset L of M. Note that S is also a compact subset of X. Then, as in the proof of necessity of (2), there is a finite subset Λ_S of Λ such that $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$ and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, S_λ is compact and $\mathcal{P}_{\lambda,n}$ is a cfp-cover for S_λ in X_λ. For each $\lambda \in \Lambda_S$ and each $n \in \mathbb{N}$, we have that S_λ is a convergent sequence, and then, $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ by Lemma 2.3. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star wcs-cover for X. For each convergent sequence S in X, there exists a finite subset Λ_S of Λ such that $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$ and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, S_λ is a convergent sequence and $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ. It follows from Lemma 2.10 that there exists a compact subset L_λ in M_λ such that $S_\lambda = f_\lambda(L_\lambda) = f(L_\lambda)$. Put $L = \bigcup\{L_\lambda : \lambda \in \Lambda_S\}$. Then L is a compact subset of M and $S = f(L)$. It implies that f is pseudo-sequence-covering.

(4). *Necessity.* Let f be sequentially-quotient. For each convergent sequence S in X, there exists some convergent sequence L of M such that $H = f(L)$ is a convergent subsequence of S. Then, as in the proof necessity of (1), H is eventually in some X_λ and each $\mathcal{P}_{\lambda,n}$ is a cs-cover for $H \cap X_\lambda$ in X_λ. Therefore, S is frequently in X_λ and each $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a convergent subsequence $S_\lambda = H \cap X_\lambda$ of S in X_λ. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs^*-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cs^*-cover for X. For each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is frequently in X_λ and, for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a subsequence S_λ of S in X_λ. It follows from Lemma 2.10 that there exists a convergent sequence L_λ in M_λ such that $f_\lambda(L_\lambda)$ is a convergent subsequence of S_λ. Note that $f_\lambda(L_\lambda) = f(L_\lambda)$ is also a convergent subsequence of S. It implies that f is sequentially-quotient. □

In [5] and [18], the authors have characterized compact images of locally separable metric spaces by means of certain point-star networks. From the above theorems, we systematically get characterizations of compact images of locally separable metric spaces under certain covering-mappings by means of double point-star covers as follows.

Corollary 2.18. *The following hold for a space X.*

(1) *X is a sequence-covering compact image of a locally separable metric space if and only if X has a point-finite double point-star cs-cover.*
(2) *X is a compact-covering compact image of a locally separable metric space if and only if X has a point-finite double point-star cfp-cover.*
(3) *X is a pseudo-sequence-covering compact image of a locally separable metric space if and only if X has a point-finite double point-star wcs-cover.*
(4) *X is a sequentially-quotient compact image of a locally separable metric space if and only if X has a point-finite double point-star cs^*-cover.*

Proof. (1). *Necessity.* Let $f : M \longrightarrow X$ be a sequence-covering compact mapping from a locally separable metric M onto X. By notations and arguments in the proof (1) $\Rightarrow$ (2) of Corollary 2.16, it suffices to show that the double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover for X.

For each convergent sequence S in X, since f is sequence-covering, $S = f(L)$ for some convergent sequence L in M. Then L is eventually in some M_λ. It implies that S is eventually in X_λ. Since $L_\lambda = L \cap M_\lambda$ is a convergent sequence in M_λ and each $\mathcal{C}_{\lambda,n}$ is a cs-cover for L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S_\lambda = f(L_\lambda)$ in X_λ by Lemma 2.6. Then $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover for X.

Sufficiency. By Theorem 2.15 and Theorem 2.17.(1).

(2). *Necessity.* Let $f : M \longrightarrow X$ be a compact-covering compact mapping from a locally separable metric M onto X. By notations and arguments in the proof (1) $\Rightarrow$ (2) of Corollary 2.16, it suffices to show that the double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cfp-cover for X.

For each compact subset K of X, since f is compact-covering, $K = f(L)$ for some compact subset L of M. Put $\Lambda_K = \{\lambda \in \Lambda : L \cap M_\lambda \neq \emptyset\}$, then Λ_K is finite, and $L_\lambda = L \cap M_\lambda$ is compact for each $\lambda \in \Lambda_K$ by compactness of L. For each $\lambda \in \Lambda_K$, put $K_\lambda = f(L_\lambda)$. Then $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$ and each K_λ is compact. For each $\lambda \in \Lambda_K$ and each $n \in \mathbb{N}$, since $\mathcal{C}_{\lambda,n}$ is a cfp-cover for L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ by Lemma 2.6. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cfp-cover for X.

Sufficiency. By Theorem 2.15 and Theorem 2.17.(2).

(3). *Necessity.* Let $f : M \longrightarrow X$ be a pseudo-sequence-covering compact mapping from a locally separable metric M onto X. By notations and arguments in the proof (1) $\Rightarrow$ (2) of Corollary 2.16, it suffices to show that the double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X.

For each convergent sequence S in X, since f is pseudo-sequence-covering, $S = f(L)$ for some compact subset L of M. Put $\Lambda_S = \{\lambda \in \Lambda : L \cap M_\lambda \neq \emptyset\}$, then Λ_S is finite, and $L_\lambda = L \cap M_\lambda$ is compact by compactness of L. For each $\lambda \in \Lambda_S$, put $S_\lambda = f(L_\lambda)$, then $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$ and each S_λ is compact. Since S_λ is a compact subset of a convergent sequence S, S_λ is a convergent sequence. On the other hand, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, since $\mathcal{C}_{\lambda,n}$ is a cfp-cover for a compact subset L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for S_λ in X_λ by Lemma 2.6. Then $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ by Lemma 2.3. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double wcs-cover for X.

Sufficiency. By Theorem 2.15 and Theorem 2.17.(3).

(4). *Necessity.* Let $f : M \longrightarrow X$ be a sequentially-quotient compact mapping from a locally separable metric M onto X. By notations and arguments in the proof (1) $\Rightarrow$ (2) of Corollary 2.16, it suffices to show that the double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs^*-cover for X.

For each convergent sequence S in X, since f is sequentially-quotient, there exists a convergent sequence L in M such that $f(L)$ is a convergent subsequence of S. Since L is eventually in some M_λ, $L_\lambda = L \cap M_\lambda$ is a convergent sequence. Then $S_\lambda = f(L_\lambda)$ is a convergent subsequence of S, and hence, S is frequently in X_λ. On the other hand, since each $\mathcal{C}_{\lambda,n}$ is a cs^*-cover for a convergent sequence L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for S_λ in X_λ by Lemma 2.6. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs^*-cover for X.

Sufficiency. By Theorem 2.15 and Theorem 2.17.(4). □

Remark 2.19. Since subsequence-covering mappings and sequentially-quotient mappings are equivalent for metric domains, "sequentially-quotient" in Theorem 2.17.(4) and Corollary 2.18.(4) can be replaced by "subsequence-covering".

In [5], the author proved that a space X is a sequentially-quotient compact image of a locally separable metric space if and only if X is a pseudo-sequence-covering compact image of a locally separable metric space. Now, we get this result again by the following lemma.

Lemma 2.20. *Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cover for X such that $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite. Then the following are equivalent.*

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ *is a double point-star wcs-cover for X.*
(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ *is a double point-star cs^*-cover for X.*

Proof. (1) ⇒ (2). It is obvious.

(2) ⇒ (1). Let S be a convergent sequence in X converging to x. Then there exists $\lambda \in \Lambda$ such that S is frequently in X_λ and each $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a convergent subsequence S_λ of S in X_λ. Put

$$\Lambda'_S = \{\lambda \in \Lambda : \text{ each } \mathcal{P}_{\lambda,n} \text{ is a } cs^*\text{-cover for some subsequence } S_\lambda \text{ of } S \text{ in } X_\lambda\}.$$

Since $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite, the limit point x of S meets only finitely many X_λ's. Then Λ_S is finite. We shall prove that S is eventually in $\bigcup\{S_\lambda : \lambda \in \Lambda'_S\}$. If not, there exists a subsequence L of S such that $L \subset S - \bigcup\{S_\lambda : \lambda \in \Lambda'_S\}$. Since $L \cup \{x\}$ is also a convergent sequence in X, $L \cup \{x\}$ is frequently in some X_α, and each $\mathcal{P}_{\alpha,n}$ is a cs^*-cover for some convergent subsequence S_α of $L \cup \{x\}$. Since S_α is a convergent subsequence of S, $\alpha \in \Lambda'_S$. It is a contradiction. Then S is eventually in $\bigcup\{S_\lambda : \lambda \in \Lambda'_S\}$. Since $S - \bigcup\{S_\lambda : \lambda \in \Lambda'_S\}$ is finite, $S - \bigcup\{S_\lambda : \lambda \in \Lambda'_S\} = \bigcup\{S_\lambda : \lambda \in \Lambda''_S\}$, where Λ''_S is also a finite subset of Λ and each S_λ is a finite subset of X_λ. Put $\Lambda_S = \Lambda'_S \cup \Lambda''_S$, then $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$, where Λ_S is a finite subset of Λ and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, S_λ is a convergent sequence and $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for S_λ in X_λ. It follows from Lemma 2.3 that each $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ. Then $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X. □

Corollary 2.21 (Theorem 2.2, [5])**.** *The following are equivalent for a space X.*

(1) *X is a pseudo-sequence-covering compact image of a locally separable metric space,*
(2) *X is a subsequence-covering compact image of a locally separable metric space,*
(3) *X is a sequentially-quotient compact image of a locally separable metric space.*

Proof. It is obvious from Corollary 2.18 and Lemma 2.20. □

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