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ON THE SPECTRUM OF A MODEL OPERATOR IN FOCK SPACE

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Abstract. A model operator H associated to a system describing four particles in interaction, without conservation of the number of particles, is considered. We describe the essential spectrum of H by the spectrum of the channel operators and prove the Hunziker-van Winter-Zhislin (HWZ) theorem for the operator H. We also give some variational principles for boundaries of the essential spectrum and interior eigenvalues.

1. Introduction

In the present paper we consider a model operator H associated to a system describing four particles in interaction, without conservation of the number of particles, acting in the *four-particle cut subspace* of the Fock space. For the study of location of the essential spectrum of H, we introduce the channel operators and prove that the essential spectrum of H is the union of spectra of the channel operators. Since the channel operators have a more simple structure than H, this fact plays an important role in the subsequent investigations of the essential spectrum of the operator. The two-particle, three-particle and four-particle branches of the essential spectrum of H are singled out. We also prove the HWZ theorem on the location of the essential spectrum of H. A variational approach to find boundaries of the essential spectrum and some interior eigenvalues is given at the end of the paper.

The well-known methods for an investigation of the location of essential spectra of Schrödinger operators are the Weyl criterion for the one particle problem and the HWZ theorem for multiparticle problems, a modern proof of which is based on the Ruelle-Simon partition of unity. The theorem on the location of the essential spectrum of multi-particle Hamiltonians was named the HWZ theorem in [6, 21] to the honor of Hunziker [11], van Winter [25] and Zhislin [26]. A lattice analogue of this theorem for the four-particle Schrödinger operator is proved in [2, 17]. In [19] by means of the limit operators method the essential spectrum of discrete Schrödinger operators on lattice $\mathbf{Z}^N$ is studied. This method has been applied by one of the authors to the effective description of the location of the essential spectrum of electromagnetic Schrödinger operators on $\mathbf{R}^N$ in [18].

The systems considered above have a fixed number of quasi-particles. In statistical physics [15], solid-state physics [16] and the theory of quantum fields [9], one considers systems, where the number of quasi-particles is bounded, but not fixed. Often, the number of particles can be arbitrarily large as in cases involving photons, in other cases, such as scattering of spin waves on defects, scattering massive particles and chemical reactions, there are only participants at any given time, though their number can change.

Recall that the study of systems describing N particles in interaction, without conservation of the number of particles is reduced to the investigation of the spectral properties

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of self-adjoint operators acting in the *cut subspace* $\mathcal{H}^{(N)}$ of the Fock space, consisting of $n \leq N$ particles [9, 15, 16, 22]. In [22] geometric and commutator techniques have been developed in order to find the location of the spectrum and to prove the absence of singular continuous spectrum for Hamiltonians without conservation of the number of the particles.

The perturbation problem of an operator (the Friedrichs model), with point and continuous spectrum (which acts in $\mathcal{H}^{(2)}$) has played a considerable role in the study of spectral problems connected to the quantum theory of fields [9].

The location and the structure of the essential spectrum of the lattice model operators acting in $\mathcal{H}^{(3)}$ were well studied in [3, 4, 13, 14, 20, 23]. In these papers some basic results of standard three body problem was discussed for these operators. Therefore, there arises a natural question: does the lattice model operator in $\mathcal{H}^{(n)}$, $n > 3$, satisfies some basic results of standard n-body problem, $n > 3$? For example, is there an analog of a HWZ theorem for such operators? Therefore, first we consider a case $n = 4$ to understand some spectral properties of a model operator acting in $\mathcal{H}^{(n)}$, $n > 3$. In the present paper we prove a HWZ theorem, and we will discuss the case where n, $n > 4$, is an arbitrary number in next papers.

The paper is organized as follows. Section 1 is an introduction to the whole work. In Section 2 the model operator H is described as a bounded self-adjoint operator in $\mathcal{H}^{(4)}$ and the main results of the present paper are formulated. In Section 3 we study spectrum of channel operators by considering the spectrum of corresponding families of operators. In Section 4 we obtain an analogue of the Faddeev-Yakubovskii type system of integral equations for the eigenvectors of H. Section 5 is devoted to the proof of the main results of the present paper (Theorems 2.1 and 2.2). In Section 6 we apply some results from the classical variational theory and the variational theory of the spectrum of operator pencils to the model operator H.

Throughout the present paper we adopt the following convention. Denote by $\mathbf{T}^\nu$ the ν-dimensional torus, the cube $(-\pi, \pi]^\nu$ with appropriately identified sides. The torus $\mathbf{T}^\nu$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the ν-dimentional space $\mathbf{R}^\nu$ modulo $(2\pi\mathbf{Z})^\nu$.

2. The model operator and statements of the main results

Let us introduce some notations used in this work. Let $\mathbf{C}$ be the field of complex numbers, $(\mathbf{T}^\nu)^n$, $n = 1, 2, 3$, be the Cartesian n-th power of $\mathbf{T}^\nu$ and $L_2((\mathbf{T}^\nu)^n)$, $n = 1, 2, 3$, the Hilbert space of square-integrable (complex-valued) functions defined on $(\mathbf{T}^\nu)^n$, $n = 1, 2, 3$. Denote

$$\mathcal{H}_0 = \mathbf{C}, \quad \mathcal{H}_1 = L_2(\mathbf{T}^\nu), \quad \mathcal{H}_2 = L_2((\mathbf{T}^\nu)^2), \quad \mathcal{H}_3 = L_2((\mathbf{T}^\nu)^3),$$

$$\mathcal{H}^{(n,m)} = \bigoplus_{i=n}^{m} \mathcal{H}_i, \quad 0 \leq n < m \leq 3.$$

The Hilbert space $\mathcal{H}^{(4)} \equiv \mathcal{H}^{(0,3)}$ is called the *"four-particle cut subspace"* of the Fock space.

Let H_{ij} be the annihilation (creation) operators [9] defined in the Fock space for $i < j$ $(i > j)$. In this paper we consider the case where the number of annihilations and creations of the particles of the considered system equals 1. This means that $H_{ij} \equiv 0$ for all $|i - j| > 1$. So, a model operator H associated to a system describing four particles in interaction, without conservation of the number of particles, acts in the Hilbert space

$\mathcal{H}^{(0,3)}$ as a matrix operator,

$$H = \begin{pmatrix} H_{00} & H_{01} & 0 & 0 \\ H_{10} & H_{11} & H_{12} & 0 \\ 0 & H_{21} & H_{22} & H_{23} \\ 0 & 0 & H_{32} & H_{33} \end{pmatrix}. \tag{2.1}$$

Let its components $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i,\ \ i,j = 0,1,2,3$, be defined by the rule

$$(H_{00}f_0)_0 = w_0 f_0, \quad (H_{01}f_1)_0 = \int_{\mathbf{T}^\nu} v_1(s) f_1(s)\,ds, \quad (H_{10}f_0)_1(p) = v_1(p) f_0,$$

$$(H_{11}f_1)_1(p) = w_1(p) f_1(p), \quad (H_{12}f_2)_1(p) = \int_{\mathbf{T}^\nu} v_2(s) f_2(p,s)\,ds,$$

$$(H_{21}f_1)_2(p,q) = v_2(q) f_1(p), \quad H_{22} = H_{22}^0 - V_{21} - V_{22},$$

$$(H_{22}^0 f_2)_2(p,q) = w_2(p,q) f_2(p,q), \quad (V_{21}f_2)_2(p,q) = v_{21}(p) \int_{\mathbf{T}^\nu} v_{21}(s) f_2(s,q)\,ds,$$

$$(V_{22}f_2)_2(p,q) = v_{22}(q) \int_{\mathbf{T}^\nu} v_{22}(s) f_2(p,s)\,ds, \quad (H_{23}f_3)_2(p,q) = \int_{\mathbf{T}^\nu} v_3(s) f_3(p,q,s)\,ds,$$

$$(H_{32}f_2)_3(p,q,t) = v_3(t) f_2(p,q), \quad (H_{33}f_3)_3(p,q,t) = w_3(p,q,t) f_3(p,q,t).$$

Here $f_i \in \mathcal{H}_i$, $i = 0,1,2,3$, w_0 is a real number, $v_i(\cdot)$, $i = 1,2,3$, $v_{2j}(\cdot)$, $j = 1,2$, $w_1(\cdot)$ are real-valued continuous functions on $\mathbf{T}^\nu$ and $w_2(\cdot,\cdot)$ (resp. $w_3(\cdot,\cdot,\cdot)$) is a real-valued continuous function on $(\mathbf{T}^\nu)^2$ resp. $(\mathbf{T}^\nu)^3$.

Under these assumptions, the operator H is bounded and self-adjoint in $\mathcal{H}^{(0,3)}$.

To formulate the main results of the present paper we introduce the following channel operators H_n, $n = 1,3$, (resp. H_2) acting in $\mathcal{H}^{(2,3)}$ (resp. $\mathcal{H}^{(1,3)}$) by the following formula:

$$H_1 = \begin{pmatrix} H_{22}^0 - V_{21} & H_{23} \\ H_{32} & H_{33} \end{pmatrix}, \ H_2 = \begin{pmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22}^0 - V_{22} & H_{23} \\ 0 & H_{32} & H_{33} \end{pmatrix}, \ H_3 = \begin{pmatrix} H_{22}^0 & H_{23} \\ H_{32} & H_{33} \end{pmatrix}.$$

Now we give the main results of the paper (for the proof see Section 5).

The essential spectrum of the operator H can be precisely described as well as in the following theorem.

Theorem 2.1. *The essential spectrum $\sigma_{\mathrm{ess}}(H)$ of the operator H is the union of spectra of the channel operators H_1, H_2 and H_3.*

Let $\sigma(H_n)$, $n = 1,2,3$, be the spectrum of the operator H_n, $n = 1,2,3$.

The following theorem shows that the least element of the essential spectrum of H belongs to the spectrum of channel operator H_1 or H_2, and it is analogues of HWZ theorem for H.

Theorem 2.2. *The following equality $\min \sigma_{\mathrm{ess}}(H) = \min\{\min \sigma(H_1), \min \sigma(H_2)\}$ holds.*

3. The spectrum of the channel operators

In this section we describe the spectrum of the channel operators H_n, $n = 1,2$, (resp. H_3) by the spectrum of the family of operators $h_n(p)$, $p \in \mathbf{T}^\nu$, $n = 1,2$, (resp. $h_3(p,q)$, $p,q \in \mathbf{T}^\nu$) defined below. We notice that some spectral properties of the operators similar to the family of operators $h_1(p)$, $p \in \mathbf{T}^\nu$ and $h_3(p,q)$, $p,q \in \mathbf{T}^\nu$, i.e., the generalized Friedrichs models have been studied in [1, 3, 4, 12].

First we consider the operator H_3 that commutes with any multiplication operator $U_\alpha^{(3)}$ by a bounded function $\alpha(\cdot,\cdot)$ on $(\mathbf{T}^\nu)^2$,

$$U_\alpha^{(3)} \begin{pmatrix} g_2(p,q) \\ g_3(p,q,t) \end{pmatrix} = \begin{pmatrix} \alpha(p,q) g_2(p,q) \\ \alpha(p,q) g_3(p,q,t) \end{pmatrix}, \quad \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} \in \mathcal{H}^{(2,3)}.$$

Therefore the decomposition of the space $\mathcal{H}^{(2,3)}$ into the direct integral

$$\mathcal{H}^{(0,1)} = \int_{(\mathbf{T}^\nu)^2} \oplus \mathcal{H}^{(2,3)} dp\, dq$$

yields the decomposition into the direct integral

$$H_3 = \int_{(\mathbf{T}^\nu)^2} \oplus h_3(p,q)\, dp\, dq, \tag{3.1}$$

where a family of the generalized Friedrichs models $h_3(p,q)$, $p,q \in \mathbf{T}^\nu$ acts in $\mathcal{H}^{(0,1)}$ as

$$h_3(p,q) = \begin{pmatrix} h_{00}^{(3)}(p,q) & h_{01}^{(3)} \\ h_{10}^{(3)} & h_{11}^{(3)}(p,q) \end{pmatrix}.$$

Here

$$(h_{00}^{(3)}(p,q) f_0)_0 = w_2(p,q) f_0, \quad (h_{01}^{(3)} f_1)_0 = \int_{\mathbf{T}^\nu} v_3(s) f_1(s)\, ds,$$

$$(h_{10}^{(3)} f_0)_1(t) = v_3(t) f_0, \quad (h_{11}^{(3)}(p,q) f_1)_1(t) = w_3(p,q,t) f_1(t).$$

In analogy with the operator H_3 one can give the decomposition

$$H_n = \int_{\mathbf{T}^\nu} \oplus h_n(p)\, dp, \quad n = 1,2, \tag{3.2}$$

where the family of operators $h_1(p)$, $p \in \mathbf{T}^\nu$, (resp. $h_2(p)$, $p \in \mathbf{T}^\nu$) acts in $\mathcal{H}^{(1,2)}$ (resp. $\mathcal{H}^{(0,2)}$) by

$$h_1(p) = \begin{pmatrix} h_{11}^{(1)}(p) & h_{12}^{(1)} \\ h_{21}^{(1)} & h_{22}^{(1)}(p) \end{pmatrix}, \quad \text{resp.} \quad h_2(p) = \begin{pmatrix} h_{00}^{(2)}(p) & h_{01}^{(2)} & 0 \\ h_{10}^{(2)} & h_{11}^{(2)}(p) & h_{12}^{(1)} \\ 0 & h_{21}^{(1)} & h_{22}^{(1)}(p) \end{pmatrix}$$

with the entries

$$(h_{11}^{(1)}(p) f_1)_1(q) = w_2(p,q) f_1(q) - v_{21}(q) \int_{\mathbf{T}^\nu} v_{21}(s) f_1(s)\, ds,$$

$$(h_{12}^{(1)} f_2)_1(q) = \int_{\mathbf{T}^\nu} v_3(s) f_2(q,s)\, ds,$$

$$(h_{21}^{(1)} f_1)_2(q,t) = v_3(t) f_1(q), \quad (h_{22}^{(1)}(p) f_2)_2(q,t) = w_3(p,q,t) f_2(q,t),$$

$$(h_{00}^{(2)}(p) f_0)_0 = w_1(p) f_0, \quad (h_{01}^{(2)} f_1)_0 = \int_{\mathbf{T}^\nu} v_2(s) f_1(s)\, ds, \quad (h_{10}^{(2)} f_0)_1(q) = v_2(q) f_0,$$

$$(h_{11}^{(2)}(p) f_1)_1(q) = w_2(p,q) f_1(q) - v_{22}(q) \int_{\mathbf{T}^\nu} v_{22}(s) f_1(s)\, ds.$$

Let us introduce the notations

$$m = \min_{p,q,t \in \mathbf{T}^\nu} w_3(p,q,t), \quad M = \max_{p,q,t \in \mathbf{T}^\nu} w_3(p,q,t),$$

$$\sigma_{\rm two}(H_n) = \bigcup_{p \in \mathbf{T}^\nu} \sigma_{\rm disc}(h_n(p)), \quad n = 1,2,$$

$$\sigma_{\rm four}(H_n) = [m; M], \quad \sigma_{\rm three}(H_n) = \bigcup_{p,q \in \mathbf{T}^\nu} \sigma_{\rm disc}(h_3(p,q)), \quad n = 1,2,3.$$

The spectrum of the operators H_n, $n = 1, 2, 3$ can be precisely described as in the following theorem.

Theorem 3.1. *The following equalities hold:*

(i) $\sigma(H_n) = \sigma_{\mathrm{two}}(H_n) \cup \sigma_{\mathrm{three}}(H_n) \cup \sigma_{\mathrm{four}}(H_n)$, $n = 1, 2$;
(ii) $\sigma(H_3) = \sigma_{\mathrm{three}}(H_3) \cup \sigma_{\mathrm{four}}(H_3)$.

Before proving the Theorem 3.1, we note that the sets $\sigma_{\mathrm{two}}(H_1) \cup \sigma_{\mathrm{two}}(H_2)$, $\sigma_{\mathrm{three}}(H_3)$ and $\sigma_{\mathrm{four}}(H_3)$ may replace the sets $\sigma_{\mathrm{two}}(H)$, $\sigma_{\mathrm{three}}(H)$ and $\sigma_{\mathrm{four}}(H)$, which are called two-particle, three-particle and four-particle branches of the essential spectrum of H, respectively.

We starts the proof of the Theorem 3.1 with the following auxiliary statements.

Let the operator $h_3^0(p,q)$, $p, q \in \mathbf{T}^\nu$ act in $\mathcal{H}^{(0,1)}$ as

$$h_3^0(p,q) = \begin{pmatrix} 0 & 0 \\ 0 & h_{11}^{(3)}(p,q) \end{pmatrix}, \quad p, q \in \mathbf{T}^\nu.$$

The perturbation $h_3(p,q) - h_3^0(p,q)$, $p, q \in \mathbf{T}^\nu$ of the operator $h_3^0(p,q)$, $p, q \in \mathbf{T}^\nu$ is a self-adjoint operator of rank 2. Therefore in accordance with the invariance of the essential spectrum under finite rank perturbations the essential spectrum $\sigma_{\mathrm{ess}}(h_3(p,q))$ of $h_3(p,q)$, $p, q \in \mathbf{T}^\nu$ fills the following interval on the real axis:

$$\sigma_{\mathrm{ess}}(h_3(p,q)) = [m_3(p,q); M_3(p,q)],$$

where the numbers $m_3(p,q)$ and $M_3(p,q)$ are defined by

$$m_3(p,q) = \min_{t \in \mathbf{T}^\nu} w_3(p,q,t), \quad M_3(p,q) = \max_{t \in \mathbf{T}^\nu} w_3(p,q,t).$$

For any fixed $p, q \in \mathbf{T}^\nu$ we define an analytic function $\Delta_3(p,q\,;\cdot)$ (the Fredholm determinant associated with the operator $h_3(p,q)$, $p, q \in \mathbf{T}^\nu$) in $\mathbf{C} \setminus \sigma_{\mathrm{ess}}(h_3(p,q))$ by

$$\Delta_3(p,q\,;z) = w_2(p,q) - z - \int_{\mathbf{T}^\nu} \frac{v_3^2(s)ds}{w_3(p,q,s) - z}.$$

The following lemma established a connection between of eigenvalues of $h_3(p,q)$, $p, q \in \mathbf{T}^\nu$ and zeroes of the function $\Delta_3(p,q\,;\cdot)$, $p, q \in \mathbf{T}^\nu$.

Lemma 3.2. *For any fixed $p, q \in \mathbf{T}^\nu$ the number $z \in \mathbf{C} \setminus \sigma_{\mathrm{ess}}(h_3(p,q))$ is an eigenvalue of the operator $h_3(p,q)$, $p, q \in \mathbf{T}^\nu$ if and only if $\Delta_3(p,q\,;z) = 0$.*

Proof. Let for any fixed $p, q \in \mathbf{T}^\nu$ the number $z \in \mathbf{C} \setminus \sigma_{\mathrm{ess}}(h_3(p,q))$ be an eigenvalue of the operator $h_3(p,q)$, $p, q \in \mathbf{T}^\nu$ and $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$ be the corresponding eigenvector, i.e., the equation $h_3(p,q)f = zf$ or the system of equations

$$\tag{3.3} \begin{cases} (w_2(p,q) - z)f_0 + \int_{\mathbf{T}^\nu} v_3(s) f_1(s)\,ds = 0, \\ v_3(t) f_0 + (w_3(p,q,t) - z) f_1(t) = 0 \end{cases}$$

has a nontrivial solution, $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$.

Since $z \in \mathbf{C} \setminus \sigma_{\mathrm{ess}}(h_3(p,q))$ from the second equation of the system (3.3) we find

$$\tag{3.4} f_1(t) = -\frac{v_3(t) f_0}{w_3(p,q,t) - z}.$$

Substituting the expression (3.4) for f_1 into the first equation of the system (3.3) we get that the system of equations (3.3) has a nontrivial solution if and only if $\Delta_3(p,q;z) = 0$. □

From Lemma 3.2 the following equality immediately follows:

$$\tag{3.5} \sigma_{\mathrm{disc}}(h_3(p,q)) = \{z \in \mathbf{C} \setminus \sigma_{\mathrm{ess}}(h_3(p,q)) : \Delta_3(p,q\,;z) = 0\}, \quad p, q \in \mathbf{T}^\nu.$$

Using definitions of the operators $h_1(p)$, $p \in \mathbf{T}^\nu$ and $h_2(p)$, $p \in \mathbf{T}^\nu$ we obtain that for any $p \in \mathbf{T}^\nu$ the equality $\sigma_{\rm ess}(h_1(p)) = \sigma_{\rm ess}(h_2(p))$ holds.

For any fixed $p \in \mathbf{T}^\nu$ we define an analytic function $\Delta_1(p\,;\cdot)$ (resp. $\Delta_2(p\,;z)$) (the Fredholm determinant associated with the operator $h_1(p)$, $p \in \mathbf{T}^\nu$, (resp. $h_2(p)$, $p \in \mathbf{T}^\nu$)) in $\mathbf{C} \setminus \sigma_{\rm ess}(h_1(p))$ by

$$\Delta_1(p\,;z) = 1 - \int_{\mathbf{T}^\nu} \frac{v_{21}^2(s)\,ds}{\Delta_3(p,s;z)}$$

, resp.

$$\Delta_2(p\,;z) = \left(1 - \int_{\mathbf{T}^\nu} \frac{v_{22}^2(s)\,ds}{\Delta_3(p,s;z)}\right)\left(w_1(p) - z - \int_{\mathbf{T}^\nu} \frac{v_2^2(s)\,ds}{\Delta_3(p,s;z)}\right) - \left(\int_{\mathbf{T}^\nu} \frac{v_2(s)v_{22}(s)\,ds}{\Delta_3(p,s;z)}\right)^2.$$

Analogously to (3.5) one can derive the equalities

$$\sigma_{\rm disc}(h_n(p)) = \{z \in \mathbf{C} \setminus \sigma_{\rm ess}(h_n(p)) : \Delta_n(p\,;z) = 0\}, \quad p \in \mathbf{T}^\nu, \quad n = 1, 2. \tag{3.6}$$

Proof of Theorem 3.1. The assertions of the Theorem 3.1 follows from the representations (3.1), (3.2) and the theorem on the spectrum of decomposable operators (see [21]) and the equalities (3.5), (3.6). □

Corollary 3.3. *The following inclusion $\sigma(H_3) \subset \sigma(H_1) \cup \sigma(H_2)$ holds.*

The proof of Corollary 3.3 immediately follows from Theorem 3.1.

4. The Faddeev-Yakubovskii type system of integral equations and the operator $T(z)$

In this section we derive an analog of the Faddeev-Yakubovskii type system of integral equations for the eigenvectors corresponding to the eigenvalues lying outside of the essential spectrum of the operator H, which plays a crucial role in our analysis of the spectrum of H.

Let us introduce the notations

$$\overline{\mathcal{H}}_0 = \mathcal{H}_0, \quad \overline{\mathcal{H}}_1 = \overline{\mathcal{H}}_2 = \overline{\mathcal{H}}_3 = \mathcal{H}_1 \quad \text{and} \quad \overline{\mathcal{H}} = \bigoplus_{n=0}^{3} \overline{\mathcal{H}}_n.$$

For each $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ let the operator matrices $A(z)$ and $K(z)$ act in the Hilbert space $\overline{\mathcal{H}}$ as

$$A(z) = \begin{pmatrix} A_{00}(z) & 0 & 0 & 0 \\ 0 & A_{11}(z) & 0 & A_{13}(z) \\ 0 & 0 & A_{22}(z) & 0 \\ 0 & A_{31}(z) & 0 & A_{33}(z) \end{pmatrix},$$

$$K(z) = \begin{pmatrix} K_{00}(z) & K_{01}(z) & 0 & 0 \\ K_{10}(z) & 0 & K_{12}(z) & 0 \\ 0 & K_{21}(z) & 0 & K_{23}(z) \\ 0 & 0 & K_{32}(z) & 0 \end{pmatrix},$$

where $A_{ij}(z) : \overline{\mathcal{H}}_j \to \overline{\mathcal{H}}_i$, $i, j = 0, 1, 2, 3$, is the multiplication operator by the function $a_{ij}(p\,;z)$,

$$a_{00}(p\,;z) \equiv 1, \quad a_{11}(p\,;z) = w_1(p) - z - \int_{\mathbf{T}^\nu} \frac{v_2^2(s)\,ds}{\Delta_3(p,s\,;z)},$$

$$a_{13}(p\,;z) \equiv a_{31}(p\,;z) = \int_{\mathbf{T}^\nu} \frac{v_2(s)v_{22}(s)\,ds}{\Delta_3(p,s\,;z)},$$

$$a_{22}(p\,;z) = 1 - \int_{\mathbf{T}^\nu} \frac{v_{21}^2(s)\,ds}{\Delta_3(s,p\,;z)}, \quad a_{33}(p\,;z) = 1 - \int_{\mathbf{T}^\nu} \frac{v_{22}^2(s)\,ds}{\Delta_3(p,s\,;z)},$$

and the operators $K_{ij}(z) : \overline{\mathcal{H}}_j \to \overline{\mathcal{H}}_i,\ i,j = 0,1,2,3$, are defined by

$$(K_{00}(z)\psi_0)_0 = (w_0 - z + 1)\psi_0, \quad K_{01}(z) \equiv H_{01}, \quad K_{10}(z) \equiv -H_{10},$$

$$(K_{12}(z)\psi_2)_1(p) = -v_{21}(p)\int_{\mathbf{T}^\nu} \frac{v_2(s)\psi_2(s)\,ds}{\Delta_3(p,s\,;z)},\ (K_{21}(z)\psi_1)_2(p) = -v_2(p)\int_{\mathbf{T}^\nu} \frac{v_{21}(s)\psi_1(s)\,ds}{\Delta_3(s,p\,;z)},$$

$$(K_{23}(z)\psi_3)_2(p) = v_{22}(p)\int_{\mathbf{T}^\nu} \frac{v_{21}(s)\psi_3(s)\,ds}{\Delta_3(s,p\,;z)},\ (K_{32}(z)\psi_2)_3(p) = v_{21}(p)\int_{\mathbf{T}^\nu} \frac{v_{22}(s)\psi_2(s)\,ds}{\Delta_3(p,s\,;z)}.$$

We note that for each $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ the operators $K_{ij}(z)$, $i,j = 0,1,2$, belong to the Hilbert-Schmidt class and therefore $K(z)$ is a compact operator.

Lemma 4.1. *For each $z \in z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ the operator $A(z)$ is bounded and invertible and the inverse operator $A^{-1}(z)$ is given by*

$$A^{-1}(z) = \begin{pmatrix} B_{00}(z) & 0 & 0 & 0 \\ 0 & B_{11}(z) & 0 & B_{13}(z) \\ 0 & 0 & B_{22}(z) & 0 \\ 0 & B_{31}(z) & 0 & B_{33}(z) \end{pmatrix},$$

where $B_{ij}(z) : \overline{\mathcal{H}}_j \to \overline{\mathcal{H}}_i,\ i,j = 0,1,2,3$, is the multiplication operator by the function $b_{ij}(p\,;z)$

$$b_{00}(p\,;z) \equiv 1, \quad b_{11}(p\,;z) = \frac{a_{33}(p\,;z)}{\Delta_2(p\,;z)}, \quad b_{13}(p\,;z) = b_{31}(p\,;z) = -\frac{a_{13}(p\,;z)}{\Delta_2(p\,;z)},$$

$$b_{22}(p\,;z) = \frac{1}{\Delta_1(p\,;z)}, \quad b_{33}(p\,;z) = \frac{a_{11}(p\,;z)}{\Delta_2(p\,;z)}.$$

Proof. By the definition, $A(z)$ is the multiplication operator by the matrix $A(p\,;z)$, where

$$A(p\,;z) = \begin{pmatrix} a_{00}(p\,;z) & 0 & 0 & 0 \\ 0 & a_{11}(p\,;z) & 0 & a_{13}(p\,;z) \\ 0 & 0 & a_{22}(p\,;z) & 0 \\ 0 & a_{31}(p\,;z) & 0 & a_{33}(p\,;z) \end{pmatrix}. \tag{4.1}$$

Obviously, for each $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$, the matrix-valued function $A(\cdot\,;z)$ is a continuous on $\mathbf{T}^\nu$. This implies that $A(z)$ is bounded. Since $\det A(p\,;z) = \Delta_1(p\,;z)\Delta_2(p\,;z)$ and $z \notin \sigma(H_1)\cup\sigma(H_2)\cup\sigma(H_3)$, we have that $\det A(p\,;z)) \neq 0$. Therefore for each $p \in \mathbf{T}^\nu$ and $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ the matrix $A(p\,;z)$ is invertible and its inverse is a matrix of the form (4.1), where we $a_{ij}(p\,;z)$ is replaced by $b_{ij}(p\,;z)$.

Then, for each $z \in \mathbf{C}\setminus(\sigma(H_1)\cup\sigma(H_2)\cup\sigma(H_3))$, the matrix-valued function $A^{-1}(\cdot\,;z)$ is continuous on $\mathbf{T}^\nu$. Let $A^{-1}(z)$ be the operator of multiplication by the matrix $A^{-1}(p\,;z)$ acting in $\overline{\mathcal{H}}$. It is easy to show that $A^{-1}(z)$ is the inverse of $A(z)$. □

Since for each $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ the operator $A(z)$ is invertible, for such z we can define the operator $T(z) = A^{-1}(z)K(z)$.

The following lemma established a connection between eigenvalues of H and $T(z)$.

Lemma 4.2. *The number $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ is an eigenvalue of the operator H if and only if the number $\lambda = 1$ is an eigenvalue of the operator $T(z)$.*

Proof. Let $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ be an eigenvalue of the operator H and $f = (f_0, f_1, f_2, f_3) \in \mathcal{H}^{(0,3)}$ be the corresponding eigenvector, that is, the equation $Hf = zf$

or the system of equations

$$
\begin{aligned}
&((H_{00} - zI_0)f_0)_0 + (H_{01}f_1)_0 = 0,\\
&(H_{10}f_0)_1(p) + ((H_{11} - zI_1)f_1)_1(p) + (H_{12}f_2)_1(p) = 0,\\
&(H_{21}f_1)_2(p,q) + ((H_{22} - zI_2)f_2)_2(p,q) + (H_{23}f_3)_2(p,q) = 0,\\
&(H_{32}f_2)_3(p,q,t) + ((H_{33} - zI_3)f_3)_3(p,q,t) = 0
\end{aligned}
\tag{4.2}
$$

has a nontrivial solution $f = (f_0, f_1, f_2, f_3) \in \mathcal{H}^{(0,3)}$, where I_i, $i = \overline{0,3}$, is an identity operator in $\mathcal{H}_i$, $i = \overline{0,3}$. Since $z \notin \sigma_{\mathrm{four}}(H_3)$, from the fourth equation of system (4.2) for f_3 we have

$$
f_3(p,q,t) = -\frac{v_3(t)f_2(p,q)}{w_3(p,q,t) - z}.
\tag{4.3}
$$

Substituting the expression (4.3) for f_3 into the third equation of the system (4.2) we obtain that the system of equations

$$
\begin{aligned}
&((H_{00} - zI_0)f_0)_0 + (H_{01}f_1)_0 = 0,\\
&(H_{10}f_0)_1(p) + ((H_{11} - zI_1)f_1)_1(p) + (H_{12}f_2)_1(p) = 0,\\
&(H_{21}f_1)_2(p,q) + ((H_{22} - zI_2 - H_{23}R_{33}(z)H_{32})f_2)_2(p,q) = 0
\end{aligned}
\tag{4.4}
$$

has a nontrivial solution if and only if the system of equations (4.2) has a nontrivial solution, where $R_{33}(z)$ is the resolvent of H_{33}.

Since $z \notin \sigma_{\mathrm{three}}(H_3)$ from the third equation of system (4.4) for f_2 we have

$$
f_2(p,q) = -\frac{v_2(q)f_1(p)}{\Delta_3(p,q\,;z)} + \frac{v_{21}(p)c_1(q) + v_{22}(q)c_2(p)}{\Delta_3(p,q\,;z)},
\tag{4.5}
$$

where

$$
c_1(q) = \int_{\mathbf{T}^\nu} v_{21}(s)f_2(s,q)ds, \quad c_2(p) = \int_{\mathbf{T}^\nu} v_{22}(s)f_2(p,s)ds.
\tag{4.6}
$$

Substituting the expression (4.5) for f_2 into the second equation of the system (4.4) and the equalities (4.6) we obtain that the equation

$$
A(z)\psi = K(z)\psi, \quad \psi = (f_0, f_1, c_1, c_2) \in \overline{\mathcal{H}}
\tag{4.7}
$$

has a nontrivial solution if and only if the system of equations (4.4) has a nontrivial solution.

By the Lemma 4.1, for each $z \in \mathbf{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ the operator $A(z)$ is invertible and hence the equation $\psi = A^{-1}(z)K(z)\psi$ or $\psi = T(z)\psi$ has a nontrivial solution if and only if the equation (4.7) has a nontrivial solution. □

Remark 4.3. We point out that the equation $T(z)g = g$ is an analogue of the Faddeev-Yakubovskii type system of integral equations for eigenvectors of the operator H.

5. The proof of the main results

In this section applying the Weyl criterion and the Faddeev-Yakubovskii type system of integral equations we prove Theorem 2.1, then the proof of Theorem 2.2 will follow from Theorems 2.1 and 3.1. We remark that the proof of Theorem 2.1 is based on ideas and results given in [23].

Proof of Theorem 2.1. The inclusion $\sigma(H_3) \subset \sigma_{\mathrm{ess}}(H)$ can be proved quite similarly to the corresponding inclusion of [13]. We prove that $\sigma(H_1) \cup \sigma(H_2) \subset \sigma_{\mathrm{ess}}(H)$.

By Theorem 3.1 we have $\sigma(H_1) \cup \sigma(H_2) = \sigma_{\mathrm{two}}(H_1) \cup \sigma_{\mathrm{two}}(H_2) \cup \sigma(H_3)$.

Let z_0 be an arbitrary point of $\sigma(H_1)\cup\sigma(H_2)$. There are two cases possible: 1) $z_0\in\sigma(H_3)$, 2) $z_0\notin\sigma(H_3)$. If $z_0\in\sigma(H_3)$, then $z_0\in\sigma_{\mathrm{ess}}(H)$. Let

$$z_0\in(\sigma_{\mathrm{two}}(H_1)\cup\sigma_{\mathrm{two}}(H_2))\setminus\sigma(H_3).$$

By the definition of $\sigma_{\mathrm{two}}(H_1)\cup\sigma_{\mathrm{two}}(H_2)$, there exists a point $p_0\in\mathbf{T}^\nu$ such that

$$\Delta_1(p_0\,;z_0)\Delta_2(p_0\,;z_0)=0.$$

Then the system of homogenous linear equations

$$\begin{gathered}
l_0=0,\\
\left(w_1(p_0)-z_0-\int_{\mathbf{T}^\nu}\frac{v_2^2(s)\,ds}{\Delta_3(p_0,s\,;z_0)}\right)l_1+\int_{\mathbf{T}^\nu}\frac{v_2(s)v_{22}(s)\,ds}{\Delta_3(p_0,s\,;z_0)}l_3=0,\\
\left(1-\int_{\mathbf{T}^\nu}\frac{v_{21}^2(s)\,ds}{\Delta_3(p_0,s\,;z_0)}\right)l_2=0,\\
\int_{\mathbf{T}^\nu}\frac{v_2(s)v_{22}(s)\,ds}{\Delta_3(p_0,s\,;z_0)}l_1+\left(1-\int_{\mathbf{T}^\nu}\frac{v_{22}^2(s)\,ds}{\Delta_3(p_0,s\,;z_0)}\right)l_3=0
\end{gathered}\tag{5.1}$$

has an infinite number of solutions in $\mathbf{C}^4$, where $\mathbf{C}^4$ is the Cartesian fourth power of $\mathbf{C}$. It is easy to verify that there exists a nontrivial solution $\mathbf{l}=(0,l_1,l_2,l_3)\in\mathbf{C}^4$ of the system of equations (5.1) satisfying one of the following conditions:

1. If $\Delta_2(p_0\,;z_0)=0$, then either $l_1\neq0$ and $l_2=0$ or $l_1=0$, $l_2=0$ and $l_3\neq0$.
2. If $\Delta_1(p_0\,;z_0)=0$, then $l_2\neq0$ and $l_1=l_3=0$.

The system of equations (5.1) can be written in the form

$$A(p_0\,;z_0)l=0,\quad l=(0,l_1,l_2,l_3)\in\mathbf{C}^4.$$

Let $\chi_{V_n}(\cdot)$ be the characteristic function of the set

$$V_n(p_0)=\left\{p\in\mathbf{T}^\nu:\frac{1}{n+1}<|p-p_0|<\frac{1}{n}\right\},\quad n=1,2,\dots,$$

and $\mu(V_n(p_0))$ be the Lebesgue measure of the set $V_n(p_0)$.

We choose a sequence of orthogonal vector-functions $\{f^{(n)}\}$ defined by

$$f^{(n)}=\begin{pmatrix}0\\ f_1^{(n)}(p)\\ f_2^{(n)}(p,q)\\ f_3^{(n)}(p,q,t)\end{pmatrix},$$

where

$$f_1^{(n)}(p)=\psi_1^{(n)}(p),\quad p\in\mathbf{T}^\nu,$$

$$f_2^{(n)}(p,q)=-\frac{v_2(q)\psi_1^{(n)}(p)}{\Delta_3(p,q\,;z_0)}+\frac{v_{21}(p)\psi_2^{(n)}(q)+v_{22}(q)\psi_3^{(n)}(p)}{\Delta_3(p,q\,;z_0)},\quad p,q\in\mathbf{T}^\nu,$$

$$f_3^{(n)}(p,q,t)=-\frac{v_3(t)f_2^{(n)}(p,q)}{w_3(p,q,t)-z_0},\quad p,q,t\in\mathbf{T}^\nu,$$

$$\psi_i^{(n)}(p)=l_ik_n(p)\chi_{V_n}(p)(\mu(V_n(p_0)))^{-1/2},\quad i=1,2,3.$$

Here $\{k_n\} \subset L_2(\mathbf{T}^\nu)$ is found from the orthogonality condition for $\{f^{(n)}\}$, i.e.,
(5.2)

$$(f^{(n)}, f^{(m)}) = \frac{l_2}{\sqrt{\mu(V_n(p_0))}\sqrt{\mu(V_m(p_0))}} \int_{V_n(p_0)} \int_{V_m(p_0)} \left(1 + \int_{\mathbf{T}^\nu} \frac{v_3^2(t)\, dt}{(w_3(p,q,t)-z_0)^2}\right)$$
$$\times \left[\frac{v_{21}(p)(l_3 v_{22}(q) - l_1 v_2(q))}{\Delta_3^2(p,q\,;z_0)} + \frac{v_{21}(q)(l_3 v_{22}(p) - l_1 v_2(p))}{\Delta_3^2(p,q\,;z_0)}\right]$$
$$\times\, k_n(p) k_m(q)\, dp dq = 0, \quad n \neq m.$$

The existence of $k_n(p)$ follows from the following proposition.

Proposition 5.1. *There exists an orthonormal system $\{k_n\} \subset L_2(\mathbf{T}^\nu)$ satisfying the conditions $\operatorname{supp} k_n \subset V_n(p_0)$ and (5.2).*

The Proposition can be proved similarly to the corresponding proposition in [23].

We resume the proof of Theorem 2.1. A simple computations shows that

$$\|f^{(n)}\|^2_{\mathcal{H}^{(0,3)}} \geq \frac{\xi_0}{\mu(V_n(p_0))} \tag{5.3}$$

for all $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of positive integers and

$$\xi_0 = \min\left\{l_1^2, \frac{l_2^2\|v_{21}\|^2_{\mathcal{H}_1}}{\max_{p,q\in\mathbf{T}^\nu}|\Delta_3(p,q\,;z_0)|^2}, \frac{l_3^2\|v_{22}\|^2_{\mathcal{H}_1}}{\max_{p,q\in\mathbf{T}^\nu}|\Delta_3(p,q\,;z_0)|^2}\right\} > 0.$$

We set $\widetilde{f}^{(n)} = f^{(n)}/\|f^{(n)}\|_{\mathcal{H}^{(0,3)}}$. It is clear that the system $\{\widetilde{f}^{(n)}\}$ is orthonormal.

We consider the operator $(H - z_0)\widetilde{f}^{(n)}$ and estimate its norm as

$$\|(H-z_0)\widetilde{f}^{(n)}\|_{\mathcal{H}^{(0,3)}} \leq \|A(z_0)\widetilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}} + \|K(z_0)\widetilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}}$$

with

$$\widetilde{\psi}^{(n)} = \left(0, \frac{\psi_1^{(n)}}{\|f^{(n)}\|_{\mathcal{H}^{(0,3)}}}, \frac{\psi_2^{(n)}}{\|f^{(n)}\|_{\mathcal{H}^{(0,3)}}}, \frac{\psi_3^{(n)}}{\|f^{(n)}\|_{\mathcal{H}^{(0,3)}}}\right).$$

From the definition of $\widetilde{\psi}^{(n)}$ and the inequality (5.3) it follows that

$$\|\widetilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}} \leq \frac{1}{\xi_0}\, \|\mathbf{l}\|^2_{\mathbf{C}^4}\,. \tag{5.4}$$

Since for any $n \neq m$ the supports of the functions $\widetilde{\psi}^{(n)}$, the inequality (5.4) implies that $\{\widetilde{\psi}^{(n)}\} \subset \overline{\mathcal{H}}$ is a bounded orthogonal system.

The operator $K(z_0)$ is compact and hence $\|K(z_0)\widetilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}} \to 0$ as $n \to \infty$.

We next estimate $\|A(z_0)\widetilde{\psi}^{(n)}\|_{\overline{\mathcal{H}}}$. Applying the Schwarz inequality we have

$$\|A(z_0)\widetilde{\psi}^{(n)}\|^2_{\overline{\mathcal{H}}} \leq M^2 \sup_{p\in V_n(p_0)} \|A(p\,;z_0)\mathbf{l}\|^2_{\mathbf{C}^4} \ \text{ with } \ M^2 = \max\left\{\frac{2}{\xi_0}, \frac{\|v_{21}\|^2_{\mathcal{H}_1}}{\xi_0}, \frac{\|v_{22}\|^2_{\mathcal{H}_1}}{\xi_0}\right\}.$$

The continuity of the matrix-valued function $A(\cdot\,;z_0)$ implies that

$$\sup_{p\in V_n(p_0)} \|A(p\,;z_0)\mathbf{l}\|_{\mathbf{C}^4} \to 0 \quad \text{as} \quad n\to\infty.$$

Therefore, for the sequence of orthonormal vector-functions $\{\widetilde{f}^{(n)}\}$ it follows that

$$\|(H-z_0)\widetilde{f}^{(n)}\|_{\mathcal{H}^{(0,3)}} \to 0 \quad \text{as} \quad n \to \infty$$

and hence by Weyl's criterion we have that $z_0 \in \sigma_{\text{ess}}(H)$. Since z_0 is an arbitrary point of $\sigma(H_1)\cup\sigma(H_2)$, it follows that $\sigma(H_1)\cup\sigma(H_2) \subset \sigma_{\text{ess}}(H)$.

Thus we have proved that $\sigma(H_1)\cup\sigma(H_2)\cup\sigma(H_3) \subset \sigma_{\text{ess}}(H)$.

Now we prove the converse inclusion, that is, $\sigma_{\rm ess}(H) \subset \sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)$. Since for any $z \in \mathbf{C}\backslash(\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ the operator $K(z)$ is compact and $A^{-1}(z)$ is bounded, we have that $f(z) = A^{-1}(z)K(z)$ is a compact-valued analytic function in $\mathbf{C}\backslash(\sigma(H_1)\cup\sigma(H_2)\cup\sigma(H_3))$. From the self-adjointness of H and Lemma 4.2 it follows that the operator $(\mathbf{I} - f(z))^{-1}$ exists for all $Imz \neq 0$, where $\mathbf{I}$ is the identity operator in $\overline{\mathcal{H}}$. In accordance with the analytic Fredholm theorem, we conclude that the operator-valued function $(\mathbf{I}-f(z))^{-1}$ exists on $\mathbf{C}\backslash(\sigma(H_1)\cup\sigma(H_2)\cup\sigma(H_3))$ everywhere except at a discrete set S, where it has finite-rank residues. Hence, with $\sigma_{\rm disc}(H)$ denoting the discrete spectrum of H, we have $\sigma(H)\setminus\sigma(H_1)\cup\sigma(H_2)\cup\sigma(H_3) \subset \sigma_{\rm disc}(H) = \sigma(H)\setminus\sigma_{\rm ess}(H)$, i.e., $\sigma_{\rm ess}(H) \subset (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$. The Theorem 2.1 is completely proved. $\square$

The proof of Theorem 2.2 follows from Theorems 2.1 and 3.1.

6. Block operator matrices and a variational technique

In this section we give a variational technique to find the boundaries of $\sigma_{\rm ess}(H)$ and eigenvalues from some interior part of the discrete spectrum of $\sigma(H)$. Define

$$A = \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ H_{12} & 0 \end{pmatrix}, \quad B^* = \begin{pmatrix} 0 & H_{21} \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} H_{22} & H_{23} \\ H_{32} & H_{33} \end{pmatrix},$$

where

$$A : \mathcal{H}^{(0,1)} \to \mathcal{H}^{(0,1)}, \qquad C : \mathcal{H}^{(2,3)} \to \mathcal{H}^{(2,3)}, \qquad B : \mathcal{H}^{(2,3)} \to \mathcal{H}^{(0,1)}$$

and

$$B^* : \mathcal{H}^{(0,1)} \to \mathcal{H}^{(2,3)}.$$

Then the operator H acting in Hilbert space $\mathcal{H}^{(0,1)} \bigoplus \mathcal{H}^{(2,3)}$ defined by (2.1) can be written as a symmetric operator matrix in the form

$$H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \tag{6.1}$$

For any subspace $S \subset \mathcal{H}^{(0,3)}$ define $S^1 = \{x \in S : \|x\| = 1\}$ and let

$$\lambda_i(H) = \inf_{S\in S_i} \max_{S^1}(Hx, x) \quad \text{and} \quad \mu_i(H) = \sup_{S\in S_i} \min_{S^1}(Hx, x),$$

where S_i denotes the set of all subspaces of dimension i. These numbers, in general are not eigenvalues of H. Now define the boundaries of the spectrum and essential spectrum of H as

$$a(H) = \min \sigma(H), \;\; b(H) = \max \sigma(H), \;\; a_{\rm ess}(H) = \min \sigma_{\rm ess}(H), \;\; b_{\rm ess}(H) = \max \sigma_{\rm ess}(H).$$

Notice that, although we are mainly interested in spectral properties of the operator H, but the facts given below valid for a self-adjoint operator and symmetric operator matrices of the form (6.1).

It is known from the classical Courant-Hilbert-Weyl variational theory that, if

$$a(H) < a_{\rm ess}(H)$$

then the spectrum of H in $[a(H), a_{\rm ess}(H))$ is discrete. Moreover, if eigenvalues in $[a(H), a_{\rm ess}(H))$ arranged in increasing order, including multiplicities, then they are equal to

$$\lambda_i(H) = \inf_{S\in S_i} \max_{S^1}(Hx, x), \quad i = 1, 2, \ldots,$$

where the inf is attained, i.e., there exists a subspace $S \in S_i$ such that

$$\lambda_i(H) = \max_{S^1}(Hx, x).$$

Now we give some results from the classical variational theory to obtain $a_{\rm ess}(H) = \min\sigma_{\rm ess}(H)$. First, if $a_{\rm ess}(H) = a(H)$ then this means that $\min\sigma_{\rm ess}(H) = \min\sigma(H)$ and for this reason we let $a(H) < a_{\rm ess}(H)$. Two cases are possible (see [24], Theorem 1, p. 12):

1) The first case: $\lambda_1(H), \lambda_2(H), \ldots, \lambda_n(H)$ are attained but $\lambda_{n+1}(H)$ is not attained, i.e., $\lambda_i(H) = \min_{S\in S_i}\max_{S^1}(Hx, x)$, for $i = 1, 2, \ldots, n$, $\lambda_{n+1}(H) = \inf_{S\in S_{n+1}} \max_{S^1}(Hx, x)$ and no subspace $S \in S_{n+1}$ such that $\lambda_{n+1}(H) = \max_{S^1}(Hx, x)$.

Then there are only n eigenvalues of H in $[a(H), a_{\rm ess}(H))$, which can be described by

$$\lambda_i(H) = \min_{S\in S_i}\max_{S^1}(Hx, x), \quad i = 1, 2, \ldots, n, \tag{6.2}$$

and

$$\min\sigma_{\rm ess}(H) := a_{\rm ess}(H) = \lambda_{n+1} = \lambda_{n+2} = \cdots \tag{6.3}$$

2) The second case: all of the numbers $\lambda_i(H)$ are attained, i.e.,

$$\lambda_i(H) = \min_{S\in S_i}\max_{S^1}(Hx, x), \quad i = 1, 2, \ldots, n, \ldots$$

Then in this case the spectrum of H in $[a(H), a_{\rm ess}(H))$ consists of a countable number of eigenvalues $\lambda_i(H)$ and

$$a_{\rm ess}(H) = \lim_{n\to\infty}\lambda_n(H). \tag{6.4}$$

The same results hold for $b_{\rm ess}(H) = \max\sigma_{\rm ess}(H)$ but we need to replace $\lambda_i(H)$ with $\mu_i(H)$.

Note that classical variational principles are applicable mainly to describe the discrete spectrum at the end parts of $\sigma(H)$. Deeper results can be obtained if we use the following operator function (operator pencil) technique. Here we give a method (see [5, 7, 8]) which allows to find eigenvalues in some interior parts of the discrete spectrum of symmetric block operator matrices (particularly, for H) of the form (6.1).

First we give a short information on operator functions (see [5, 7, 8, 10]). Denote by $S(\mathcal{H})$ the space of bounded symmetric operators on a Hilbert space $\mathcal{H}$. Let $L(\lambda)$ be an operator-valued function (or simply an operator function), defined on an interval $[\alpha, \beta]$ with values in $S(\mathcal{H})$. So, $L : [\alpha, \beta] \to S(\mathcal{H})$. A typical example is an operator polynomial of the form $L(\lambda) = \lambda^n A_n + \lambda^{(n-1)}A_{n-1} + \cdots + A_0$, where $A_i \in S(\mathcal{H})$, $i = 0, 1, \ldots, n$. Polynomial operator functions are often called operator pencils. We denote the spectrum and the essential spectrum of L by $\sigma(L)$ and $\sigma_{\rm ess}(L)$, respectively. Define the family of functions $\varphi_x(\lambda)$ depending on $x \in \mathcal{H}\setminus\{0\}$ by $\varphi_x(\lambda) := (L(\lambda)x, x)$. In the variational theory of operator functions it is always supposed that the following two conditions are satisfied:

I) The equation $\varphi_x(\lambda) = 0$ has at most one solution on $[\alpha, \beta]$ (which is denoted by $p(x)$) and $\varphi_x(\lambda)$ is decreasing (or increasing) at $p(x)$, i.e.,

$$\varphi_x(\lambda) < 0 \Leftrightarrow \lambda > p(x),$$

$$\varphi_x(\lambda) > 0 \Leftrightarrow \lambda < p(x).$$

II) $\kappa_\alpha(L) := \max\dim\{E \big| \varphi_x(\lambda) < 0, \ x \in E\setminus\{0\}\} < +\infty$.

By these conditions, if the equation $\varphi_x(\lambda) = 0$ has no solution on $[\alpha, \beta]$ for some x then either $\varphi_x(\lambda) < 0$ or $\varphi_x(\lambda) > 0$ for all $\lambda \in [\alpha, \beta]$. Clearly, the functional $p(x)$ in general is not defined for all $x \neq 0$ and in this case we define the extended functional $p(x)$ by

$$p(x) = \begin{cases} \lambda_0, & \text{if } \varphi_x(\lambda_0) = 0, \\ +\infty, & \text{if } \ \varphi_x(\lambda) > 0, \ \lambda \in [\alpha, \beta], \\ -\infty, & \text{if } \varphi_x(\lambda) < 0, \ \lambda \in [\alpha, \beta]. \end{cases}$$

Now the same formulas (6.2), (6.3) and (6.4) (under the same conditions) hold (see [5], [10]) for the operator function $L(\lambda)$ if we replace (Hx, x) by $p(x)$ and define

$$\lambda_i(L) = \inf_{S \in S_i} \sup_{S^1} p(x) \quad \text{and} \quad \mu_i(L) = \sup_{S \in S_i} \inf_{S^1} p(x).$$

This means that in the spectral theory of operator functions the functional $p(x)$, which is called a Rayleigh functional, plays the same role as the quadratic form $\frac{(Hx,\,x)}{(x,\,x)}$ of the operator H in the operator theory.

Let $I^{(n,m)}$ be an identity operator in $\mathcal{H}^{(n,m)}$, $0 \le n < m \le 3$.

Now we give a connection between the spectrum of H of the form (6.1) and the spectrum of the operator function defined below (see [5, 7, 8] for details). The first important step is the following.

Theorem 6.1. ([5, 7, 8]). *The spectrum of H of the form (6.1) outside of the spectrum C coincides with the spectrum of the operator function*

$$L(\lambda) = A - \lambda I^{(0,1)} - B(C - \lambda I^{(2,3)})^{-1} B^*.$$

Finally, we give a theorem which shows how one can obtain eigenvalues from some interior parts of the discrete spectrum of H by using the Rayleigh functional $p(x)$ for $L(\lambda)$ (see [5], pp. 204–205).

Theorem 6.2. *Let H be an operator matrix of the form (6.1) and $\sigma(C) < \sigma_{\text{ess}}(A)$. Then,*

1) *the spectrum of H in $(b(C),\, a_{\text{ess}}(A))$ is discrete with only possible accumulation point at $a_{\text{ess}}(A)$.*

2) *For $L(\lambda) = A - \lambda I^{(0,1)} - B(C - \lambda I^{(2,3)})^{-1} B^*$ we have*

$$\sigma(L) \cap (b(C),\, a_{\text{ess}}(A))] = \sigma(H) \cap (b(C),\, a_{\text{ess}}(A))].$$

3) *If $\lambda_i(H),\ i = 1, 2, 3, \ldots$ are eigenvalues of H in $(b(C),\, a_{\text{ess}}(A))$, then*

$$\lambda_i(H) = \inf_{S \in S_i} \max_{S^1} p(x),$$

where $p(x)$ is the Rayleigh functional for $L(\lambda)$.

A sketch of the proof. Evidently, it is enough two show that the operator function $L(\lambda)$ satisfies the condition **I)** and **II)** on $(b(C),\, a_{\text{ess}}(A))$. It follows from the Hilbert identity $R_\lambda(C) - R_\mu(C) = (\lambda - \mu) R_\lambda(C) R_\mu(C)$ that $R'_\lambda(C) = R^2_\lambda(C)$, where

$$R_\lambda(C) := (C - \lambda I^{(2,3)})^{-1}$$

is the resolvent of the operator C. Using this fact we have

$$L'(\lambda) = -I^{(1,2)} - B(C - \lambda I^{(2,3)})^{-2} B^* \ll 0$$

for all $\lambda \in (b(C),\, a_{\text{ess}}(A)]$. Here $L'(\lambda) \ll 0$ means that $(L'(\lambda)x,\, x) \le \delta(x,\, x)$ for all x and some $\delta > 0$.

In fact by the spectral theorem for a self-adjoint operator we can write

$$C = \int_{\sigma(C)} s \, dE_C(s),$$

where $E_C(S)$ is the spectral measure of the operator C. Then

$$C - \lambda I^{(2,3)} = \int_{\sigma(C)} (s - \lambda) \, dE_C(s) < 0,$$

because $s - \lambda < 0$ for $\lambda > b(C)$. Notice that the inequality $C - \lambda I^{(2,3)} < 0$ also follows from the fact that the spectrum of a bounded operator is a subset of the closure of its numerical range. Now we have $L'(\lambda) \ll 0$ and the condition **I)** follows from this inequality. On the other hand for $\alpha = b(C)$ we can write

$$\{x \Big| (L(\alpha)x, x) < 0\} \subset \{x \Big| ((A - \alpha I^{(0,1)})x, x)) < 0\}.$$

It follows from this that $\kappa_\alpha(\lambda) \leq N(\alpha, A)$, where $N(\alpha, A)$ is the spectral distribution function of A. The condition $\alpha = b(C) < \sigma_{\mathrm{ess}}(A)$ means that $N(\alpha, A)$ is finite and by the inequality $\kappa_\alpha(\lambda) \leq N(\alpha, A)$ we get

$$\kappa_\alpha(\lambda) = \max\{E \Big| L(\alpha)x, x) < 0, \ x \in E \setminus \{0\} < +\infty,$$

i.e., the condition **II)** is satisfied. Consequently, the eigenvalues of the operator H in the interval $(b(C), a_{\mathrm{ess}}(A))$ can be characterized by variational principles for the operator function $L(\lambda)$ (see [5, 7, 8, 10]). More precisely,

$$\lambda_i(H) = \min_{S \in S_i} \sup_{S^1} p(x), \quad i = 1, 2, \dots,$$

where $p(x)$ is the Rayleigh functional of the operator function

$$L(\lambda) = A - \lambda I^{(0,1)} - B(C - \lambda I^{(2,3)})^{-1} B^*$$

on $[b(C), a_{\mathrm{ess}}(A)]$.

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