Methods of Functional Analysis and Topology
Vol. 15 (2009), no. 4, pp. 361–368

SOME RESULTS ON THE UNIFORM BOUNDEDNESS THEOREM IN LOCALLY CONVEX CONES

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Abstract. Walter Roth studied one form of the uniform boundedness theorem in [3]. We investigate some other versions of the uniform boundedness theorem for barreled and upper-barreled locally convex cones. Finally, we show some applications of this theorem.

1. Introduction

The general theory of locally convex cones as developed in [1] deals with preordered cones. We review some of the main concepts and refer to [1] for details.

A *cone* is a set $\mathbf{P}$ endowed with an addition and a scalar multiplication for non-negative real numbers. The addition is associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold. We have $1a = a$ and $0a = 0$ for all $a \in \mathbf{P}$. The *cancellation law*, sating that $a + c = b + c$ implies $a = b$, however, is not required in general. It holds if and only if the cone $\mathbf{P}$ may be embedded into a real vector space. A *preordered cone* (*ordered cone* for short) is a cone with a preorder, that is, a reflexive transitive relation $\leq$ which is compatible with the algebraic operations.

For cones $\mathbf{P}$ and $\mathbf{Q}$ a mapping $t : \mathbf{P} \to \mathbf{Q}$ is called a *linear operator* if $t(a + b) = t(a) + t(b)$ and $t(\alpha a) = \alpha t(a)$ hold for $a, b \in \mathbf{P}$ and $\alpha \geq 0$. A *linear functional* on $\mathbf{P}$ is a linear operator $\mu : \mathbf{P} \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

A subset $\mathbf{V}$ of the preordered cone $\mathbf{P}$ is called an *(abstract) 0-neighborhood system*, if the following properties hold:

(v_1) $0 < v$ for all $v \in \mathbf{V}$;
(v_2) for all $u, v \in \mathbf{V}$ there is a $w \in \mathbf{V}$ with $w \leq u$ and $w \leq v$;
(v_3) $u + v \in \mathbf{V}$ and $\alpha v \in \mathbf{V}$ whenever $u, v \in \mathbf{V}$ and $\alpha > 0$.

For every $a \in \mathbf{P}$ and $v \in \mathbf{V}$ we define

$$v(a) = \{b \in \mathbf{P} \ : \ b \leq a + v\}, \quad \text{resp.} \quad (a)v = \{b \in \mathbf{P} \ : \ a \leq b + v\},$$

to be a neighborhood of a in the *upper*, resp. *lower* topologies on $\mathbf{P}$. Their common refinement is called *symmetric* topology. We denote the neighborhoods of the symmetric topology as $v(a) \cap (a)v$ or $v(a)v$ for $a \in \mathbf{P}$ and $v \in \mathbf{V}$. We call $(\mathbf{P}, \mathbf{V})$ is a *full locally convex cone*, if each element of $\mathbf{P}$ is *bounded blow*, i.e. for every $a \in \mathbf{P}$ and $v \in \mathbf{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. Each subcone of $\mathbf{P}$, not necessarily containing $\mathbf{V}$, is called a *locally convex cone*. An element a of locally convex cone $(\mathbf{P}, \mathbf{V})$ is called bounded if it is also *upper bounded*, i.e. for every $v \in \mathbf{V}$ there is a $\rho > 0$ such that $a \leq \rho v$. On $\mathbf{P}$ we define the *global preorder* $\preceq$ as: $a \preceq b$ if and only if $a \leq b + v$ for all $v \in \mathbf{V}$.

2000 *Mathematics Subject Classification.* 46A03, 46A08, 46A30.
Key words and phrases. Locally convex cone, bounded, internally bounded, uniform boundedness theorem.

Let $\mathbf{P}$ be a cone. For $U,V \subseteq \mathbf{P}^2$, by $U \circ V$ we mean the set of all $(a,b) \in \mathbf{P}^2$ such that there is a $c \in \mathbf{P}$ with $(a,c) \in U$ and $(c,b) \in V$, $U^{-1} = \{(b,a) \ : \ (a,b) \in U\}$ and $\triangle = \{(a,a) : a \in \mathbf{P}\}$ }. A collection $\mathcal{U}$ of convex subsets $U \subset \mathbf{P}^2 = \mathbf{P} \times \mathbf{P}$ is called a *convex quasiuniform structure*, if the following properties hold:

$(U1)$ $\triangle \subset U$ for every $U \in \mathcal{U}$;
$(U2)$ for all $U,V \in \mathcal{U}$ there is a $W \in \mathcal{U}$ such that $W \subset U \cap V$;
$(U3)$ $\lambda U \circ \mu U \subset (\lambda+\mu)U$ for all $U \in \mathcal{U}$ and $\lambda,\mu > 0$;
$(U4)$ $\lambda U \in \mathcal{U}$ for all $U \in \mathcal{U}$ and $\lambda > 0$.

To every quasiuniform structure $\mathcal{U}$ on $\mathbf{P}$ we associate a preorder defined by $a \le b$ if and only if $(a,b) \in U$ for all $U \in \mathcal{U}$ and, two topologies: the neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathbf{P} : (b,a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in \mathbf{P} : (a,b) \in U\}, \quad U \in \mathcal{U}.$$

The topology associated with the uniform structure $\mathcal{U}_s = \{U \cap U^{-1} : U \in \mathcal{U}\}$ is the common refinement of the upper and lower topologies.

The notions of an (abstract) 0-neighborhood system $\mathbf{V}$ and a convex quasiuniform structure $\mathcal{U}$ for a cone $\mathbf{P}$ are equivalent in the following sense.

For a locally convex cone $(\mathbf{P},\mathbf{V})$, and each $v \in \mathbf{V}$, we put

$$\tilde{v} = \{(a,b) \in \mathbf{P} \times \mathbf{P} : a \le b + v\}.$$

The collection $\widetilde{\mathbf{V}} = \{\tilde{v} : v \in \mathbf{V}\}$ is a convex quasiuniform structure on $\mathbf{P}$, which induces the global preorder on $\mathbf{P}$ and the same upper, lower and symmetric topologies. On the other hand: if $\mathbf{P}$ is a cone with a convex quasiuniform structure $\mathcal{U}$, then one can find a preorder and an (abstract) 0-neighborhood system $\mathbf{V}$ such that the convex quasiuniform structure $\widetilde{\mathbf{V}}$ is equivalent to $\mathcal{U}$ (see [1], I.5.5).

If $\mathbf{P}$ with $\mathbf{V}$ is a locally convex cone, then each element of $\mathbf{P}$ is bounded below, and hence

$$\text{for all } a \in \mathbf{P} \text{ and } \tilde{v} \in \widetilde{\mathbf{V}} \text{ there is some } \rho > 0 \text{ such that } (0,a) \in \rho\tilde{v}.$$

Conversely, if a convex quasiuniform structure $\mathcal{U}$ on $\mathbf{P}$ satisfies also

$$(U5) \quad \text{for all } a \in \mathbf{P} \text{ and } U \in \mathcal{U} \text{ there is some } \rho > 0 \text{ such that } (0,a) \in \rho U,$$

then the generated preordered cone would be locally convex.

For locally convex cones $\mathbf{P}$ and $\mathbf{Q}$, with convex quasiuniform structures $\mathcal{U}$ and $\mathcal{V}$, respectively, a linear mapping $t : \mathbf{P} \to \mathbf{Q}$ is called *uniformly continuous (u-continuous)* if for every $V \in \mathcal{V}$ one can find a $U \in \mathcal{U}$ such that $(a,b) \in U$ implies $(t(a),t(b)) \in V$, or $T(U) \subset V$, $T = t \times t$. If $\mathbf{V}$ and $\mathbf{W}$ are (abstract) 0-neighborhood systems on $\mathbf{P}$ and $\mathbf{Q}$, respectively, t is u-continuous if and only if for every $w \in \mathbf{W}$ there is a $v \in \mathbf{V}$ such that $(a,b) \in \tilde{v}$ implies $(t(a),t(b)) \in \tilde{w}$, or equivalently, $t(a) \le t(b) + w$ whenever $a \le b + v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathbf{P}$ and $\mathbf{Q}$.

Endowed with the (abstract) 0-neighborhood system $\boldsymbol{\varepsilon} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. The u-continuous linear functionals on the locally convex cone $(\mathbf{P},\mathbf{V})$ form a cone with the usual addition and scalar multiplication of functions. This cone is called the *dual cone* of $\mathbf{P}$ and denoted by $\mathbf{P}^*$.

For a locally convex cone $(\mathbf{P},\mathbf{V})$, the polar v° of $v \in \mathbf{V}$ consists of all linear functionals μ on $\mathbf{P}$ satisfying $\mu(a) \le \mu(b) + 1$ whenever $a \le b + v$ for $a,b \in \mathbf{P}$. We have $\mathbf{P}^* = \cup\{v^\circ : v \in \mathbf{V}\}$.

In this paper, we modify the uniform boundedness theorem which studied by W. Roth in [3] and prove other forms of this theorem for the locally convex cones in section 2. These forms are more similar to the case of the convex topological vector spaces. We use the definition of the upper-barreled cone that is stated by using of convex quasiuniform structure and follows of the convex topological spaces (see [2]).

2. Uniform boundedness theorems for locally convex cones

One form of uniform boundedness theorem for the locally convex cones studied in [3]. We use upper-barreled notion to study another version of this theorem which is more similar to the case of the convex topological vector spaces.

In [3] a barrel and a barreled cone has been defined as following.

Definition 2.1. Let $(\mathbf{P}, \mathbf{V})$ be a locally convex cone. A *barrel* is a convex subset B of $\mathbf{P}^2$ with the following properties :

- $(B1)$ For every $b \in \mathbf{P}$ there is a $v \in \mathbf{V}$ such that for every $a \in v(b)v$ there is a $\lambda > 0$ such that $(a, b) \in \lambda B$.
- $(B2)$ For all $a, b \in \mathbf{P}$ such that $(a, b) \notin B$ there is a $\mu \in \mathbf{P}^*$ such that $\mu(c) \leq \mu(d) + 1$ for all $(c, d) \in B$ and $\mu(a) > \mu(b) + 1$.

Definition 2.2. A locally convex cone $(\mathbf{P}, \mathbf{V})$ is said to be *barreled* if for every barrel $B \subseteq \mathbf{P}^2$ and every $b \in \mathbf{P}$ there is a $v \in \mathbf{V}$ and a $\lambda > 0$ such that $(a, b) \in \lambda B$ for all $a \in v(b)v$.

We shall say that the locally convex cone $(\mathbf{P}, \mathbf{V})$ has the *strict separation property (SP)*, if: for all $a, b \in \mathbf{P}$ and $v \in \mathbf{V}$ such that $a \not\leq b + \rho v$ for some $\rho > 1$, there is a linear functional $\mu \in v^\circ$ such that $\mu(a) > \mu(b) + 1$ (see [1], page 32).

The uniform boundedness theorem stated in [3] is as following:

Theorem 2.3 ([3], Theorem 3.1)**.** *Let $(\mathbf{P}, \mathbf{V})$ and $(\mathbf{Q}, \mathbf{W})$ be locally convex cones, and let $\mathcal{T}$ be a family of u-continuous linear operators from $\mathbf{P}$ to $\mathbf{Q}$. Suppose that for every $b \in \mathbf{P}$ and $w \in \mathbf{W}$ there is a $v \in \mathbf{V}$ such that for every $a \in v(b)v$ there is a $\lambda > 0$ such that*

$$T(a) \leq T(b) + \lambda w \quad \text{for all} \quad T \in \mathcal{T}.$$

If $(\mathbf{P}, \mathbf{V})$ is barreled and $(\mathbf{Q}, \mathbf{W})$ has the strict separation property, then for every internally bounded set $E \subseteq \mathbf{P}$, every $b \in E$ and $w \in \mathbf{W}$ there is a $v \in \mathbf{V}$ and a $\lambda > 0$ such that

$$T(a) \leq T(b) + \lambda w \quad \text{for all} \quad T \in \mathcal{T}$$

and all $a \in v(b') \cap (b'')v$ for some $b', b'' \in E$.

A locally convex cone $(\mathbf{P}, \mathbf{V})$ is said to be *tightly covered by its bounded elements* if for all $a, b \in \mathbf{P}$ and $v \in \mathbf{V}$ such that $a \notin v(b)$ (or $a \not\leq b + v$) there is some bounded element $a' \in \mathbf{P}$ such that $a' \preceq a$ and $a' \notin v(b)$ ([1], II.2.13).

Theorem II.2.14 of [1] states that every locally convex cone which is tightly covered by its bounded elements has strict separation property. Therefore if $(\mathbf{Q}, \mathbf{W})$ is tightly covered by its bounded elements in Theorem 2.3, we obtain the same result. But there is a direct and simple proof for this case as following:

Theorem 2.4. *Under the conditions of Theorem 2.3 if $(\mathbf{P}, \mathbf{V})$ is barreled and $(\mathbf{Q}, \mathbf{W})$ is tightly covered by its bounded elements, then for every internally bounded set $E \subseteq \mathbf{P}$, every $b \in E$ and $w \in \mathbf{W}$ there is a $v \in \mathbf{V}$ and a $\lambda > 0$ such that*

$$T(a) \leq T(b) + \lambda w \quad \text{for all} \quad T \in \mathcal{T}$$

and all $a \in v(b') \cap (b'')v$ for some $b', b'' \in E$.

Proof. Let $\widetilde{\mathbf{W}}$ be the convex quasiuniform structure related to the (abstract) 0-neighborhood system $\mathbf{W}$, $\tilde{w} \in \widetilde{\mathbf{W}}$ and $t = T \times T$. Put

$$B = \cap\{t^{-1}(\tilde{w}) \ : \ T \in \mathcal{T}\}.$$

Since $\mathbf{Q}$ is tightly covered by its bounded elements, then $\tilde{w}$ is a barrel in $\mathbf{Q}$ (cf. [2], Proposition 4.3) and $t^{-1}(\tilde{w})$ is a barrel in $\mathbf{P}$ for each $t = T \times T$ (cf. [2], Lemma 4.4). Now we show that B is a barrel in $\mathbf{P}$.

$(B1)$ Let $b \in \mathbf{P}$, $v \in \mathbf{V}$ as in the assumption and $a \in v(b)v$. By the condition in the hypothesis there is a $\lambda > 0$ such that

$$(T(a), T(b)) \in \lambda\tilde{w}$$

for all $T \in \mathcal{T}$. So

$$(a, b) \in \lambda B.$$

$(B2)$ Let $(a, b) \notin B$, so $(a, b) \notin t^{-1}(\tilde{w})$ for some t. Since $t^{-1}(\tilde{w})$ is a barrel, there is a $\mu \in \mathbf{P}^*$ such that $\mu(a) > \mu(b) + 1$ and $\mu(c) \le \mu(d) + 1$ for all $(c, d) \in t^{-1}(\tilde{w})$ and then for all $(c, d) \in B$.

Thus by Lemma 2.2 of [3] for every $b \in E$ there is a neighborhood $v \in \mathbf{V}$ and a $\lambda > 0$ such that $(a, b) \in \lambda B$ for all $a \in v(b') \cap (b'')v$ for some $b', b'' \in E$. By the definition of B this shows that

$$(T(a), T(b)) \in \lambda\tilde{w}$$

for all $T \in \mathcal{T}$. Hence

$$T(a) \le T(b) + \lambda w \quad \text{for all} \quad T \in \mathcal{T}.$$

□

We bring the definition of the upper-barreled cones which was defined in [2].

Definition 2.5. Let $(\mathbf{P}, \mathbf{V})$ be a locally convex cone and $\widetilde{\mathbf{V}}$ be the convex quasiuniform structure generated by $\mathbf{V}$. $\mathbf{P}$ is called *upper-barreled* if for every barrel $B \subseteq \mathbf{P}^2$, there is a $\tilde{v} \in \widetilde{\mathbf{V}}$ such that $\tilde{v} \subseteq B$.

It is easy to see that every upper-barreled cone is barreled. So in the Theorems 2.3 and 2.4 we can consider $(\mathbf{P}, \mathbf{V})$ to be upper-barreled.

We do not know if there is a barreled locally convex cone which is not upper-barreled.

Now we state some other cases of the uniform boundedness theorem for locally convex cones.

We recall the definition of the uniform equicontinuity for locally convex cones in the following (see [1], page 69).

Definition 2.6. Let $(\mathbf{P}, \mathbf{V})$ and $(\mathbf{Q}, \mathbf{W})$ be locally convex cones. A set $\mathcal{T}$ of linear operators $T : \mathbf{P} \to \mathbf{Q}$ is called *uniformly equicontinuous (u-equicontinuous)*, if for every $w \in \mathbf{W}$ there is a $v \in \mathbf{V}$ such that for all $a, b \in \mathbf{P}$ and $T \in \mathcal{T}$

$$a \le b + v \quad \text{implies} \quad T(a) \le T(b) + w,$$

i.e.

$$(a, b) \in \tilde{v} \quad \text{implies} \quad (T(a), T(b)) \in \tilde{w}.$$

If $\mathcal{T}$ is a u-equicontinuous family of linear functionals, then $\mathcal{T} \subseteq v^\circ$ for some $v \in \mathbf{V}$. In particular, for each $v \in \mathbf{V}$, the set v° is u-equicontinuous.

Let $(\mathbf{P}, \mathbf{V})$ and $(\mathbf{Q}, \mathbf{W})$ be locally convex cones. We saw that if $T : \mathbf{P} \to \mathbf{Q}$ is u-continuous and $E \subseteq \mathbf{P}$ is a bounded set, then $T(E)$ is bounded i.e. for each $w \in \mathbf{W}$ there is a $\lambda > 0$ such that

$$0 \le T(a) + \lambda w \quad \text{and} \quad T(a) \le \lambda w \quad \text{for all} \quad T(a) \in T(E).$$

Clearly this λ depends on, both w and T. Hence for a fixed $w \in \mathbf{W}$ and different T's we shall have different λ's. But

Proposition 2.7. *If $\mathcal{T}$ is a u-equicontinuous family of linear operators from $(\mathbf{P}, \mathbf{V})$ to $(\mathbf{Q}, \mathbf{W})$ and $E \subseteq \mathbf{P}$ is (internally) bounded, then $\mathbf{Q}$ has an (internally) bounded subset B such that $T(E) \subseteq B$ for all $T \in \mathcal{T}$.*

Proof. Put $B = \cup\{T(E) \ : \ T \in \mathcal{T}\}$. For $w \in \mathbf{W}$, since $\mathcal{T}$ is u-equicontinuous, there is a $v \in \mathbf{V}$ such that for all $a, b \in \mathbf{P}$ and $T \in \mathcal{T}$

$$a \leq b + v \quad \text{implies} \quad T(a) \leq T(b) + w.$$

Since E is bounded, there is a $\lambda > 0$ such that for all $a \in E$

$$0 \leq a + \lambda v \quad \text{and} \quad a \leq \lambda v.$$

Now if $b \in B$ is arbitrary and $a \in E$ such that $b = T(a)$, we have

$$0 \leq T(a) + \lambda w = b + \lambda w \quad \text{and} \quad b = T(a) \leq \lambda w.$$

Hence B is bounded and $T(E) \subseteq B$ for all $T \in \mathcal{T}$. The internally bounded case is similarly proved. □

This means in particular that if $E \subseteq \mathbf{P}$ is (internally) bounded and $\mathcal{T}$ is u-equicontinuous, then $\{T(a) \ : \ a \in E \ , \ T \in \mathcal{T}\}$ is also (internally) bounded. Also u-equicontinuity of $\mathcal{T}$ implies the conclusions of Theorems 2.3 and 2.4.

Theorem 2.8. *Let $(\mathbf{P}, \mathbf{V})$ and $(\mathbf{Q}, \mathbf{W})$ be locally convex cones, and $\mathcal{T}$ be a family of u-continuous linear operators from $\mathbf{P}$ to $\mathbf{Q}$. Suppose that for each $b \in \mathbf{P}$ and $w \in \mathbf{W}$ there is a $v \in \mathbf{V}$ such that for every $a \in v(b)v$ there is a $\lambda > 0$ such that*

$$T(a) \leq T(b) + \lambda w \quad \textit{for all} \quad T \in \mathcal{T}.$$

If $(\mathbf{P}, \mathbf{V})$ is upper-barreled and $(\mathbf{Q}, \mathbf{W})$ has the strict separation property, then $\mathcal{T}$ is u-equicontinuous.

Proof. Let $w \in \mathbf{W}$ and set

$$B = \{(a, b) \in \mathbf{P}^2 \ : \ T(a) \leq T(b) + \rho w \quad \text{for all } \rho > 1 \text{ and } T \in \mathcal{T}\}.$$

We prove that B is a barrel for $\mathbf{P}$ ([3], proof of Theorem 3.1).

$(B1)$ For $b \in \mathbf{P}$ choose $v \in \mathbf{V}$ as in the assumption of the theorem. Then for every $a \in v(b)v$ there is a $\lambda > 0$ such that $T(a) \leq T(b) + \lambda w$ for all $T \in \mathcal{T}$, that is $(a, b) \in \lambda B$.

$(B2)$ For $a, b \in \mathbf{P}$ such that $(a, b) \notin B$ there is a $T \in \mathcal{T}$ such that $T(a) \not\leq T(b) + \rho w$ for some $\rho > 1$. Hence there is a $t \in v^\circ$ such that $t(T(a)) > t(T(b)) + 1$ ($\mathbf{Q}$ has (SP) property). Set $\mu = T^*(t) \in \mathbf{P}^*$. By [1], II.2.15, we have $\mu(a) = T^*(t)(a) = t(T(a))$ for all $a \in \mathbf{P}$. The functional μ fulfills the requirement of $(B2)$.

Now since $\mathbf{P}$ is upper-barreled, $2\tilde{v} \subseteq B$ for some $v \in \mathbf{V}$. Hence if $a, b \in \mathbf{P}$ and $a \leq b + v$, i.e. $(a, b) \in \tilde{v}$, then $T(a) \leq T(b) + \frac{\rho}{2} w$ for all $\rho > 1$ and $T \in \mathcal{T}$. Putting $\rho = 2$, we get $T(a) \leq T(b) + w$. This shows that $\mathcal{T}$ is u-equicontinuous. □

Since every locally convex cone which is tightly covered by its bounded elements has strict separation property, so we have

Corollary 2.9. *Under the conditions of Theorem 2.8, if $(\mathbf{P}, \mathbf{V})$ is upper-barreled and $(\mathbf{Q}, \mathbf{W})$ is tightly covered by its bounded elements, then $\mathcal{T}$ is u-equicontinuous.*

Theorem 2.10. *Let $(\mathbf{P}, \mathbf{V})$ and $(\mathbf{Q}, \mathbf{W})$ be locally convex cones, and $\mathcal{T}$ be a family of u-continuous linear operators from $\mathbf{P}$ to $\mathbf{Q}$ such that the set $\{T(x) \ : \ T \in \mathcal{T}\}$ is bounded for each $x \in \mathbf{P}$. If $(\mathbf{P}, \mathbf{V})$ is upper-barreled and $\mathbf{Q}$ is tightly covered by its bounded elements, then $\mathcal{T}$ is u-equicontinuous.*

Proof. Let $w \in \mathbf{W}$, $t = T \times T$ and $B = \cap\{t^{-1}(\tilde{w}) \ : \ T \in \mathcal{T}\}$. We show that B is a barrel. (B_2) holds as in the proof of the Theorem 2.4. We investigate (B_1): Let $b \in \mathbf{P}$, $v \in \mathbf{V}$ and $a \in v(b)v$. Since

$$\{T(a) \ : \ T \in \mathcal{T}\} \quad \text{and} \quad \{T(b) \ : \ T \in \mathcal{T}\}$$

are bounded, there is a $\lambda > 0$ such that

$$(T(a), 0) \in \lambda\tilde{w} \quad \text{and} \quad (0, T(b)) \in \lambda\tilde{w}$$

for all $T \in \mathcal{T}$. We have
$$(T(a), T(b)) \in 2\lambda\tilde{w}$$
for all $T \in \mathcal{T}$ and then
$$(a, b) \in 2\lambda B.$$

Now since $\mathbf{P}$ is upper-barreled, there is a $\tilde{v} \in \widetilde{\mathbf{V}}$ such that $\tilde{v} \subseteq B$ and then
$$t(\tilde{v}) \subseteq \tilde{w}$$
for all $t = T \times T$, $T \in \mathcal{T}$. □

Proposition 2.11. *Let $(\mathbf{P}, \mathbf{V})$ and $(\mathbf{Q}, \mathbf{W})$ be locally convex cones, and $(T_s)_{s\in S}$ be a net of u-continuous linear operators from $\mathbf{P}$ to $\mathbf{Q}$ which converges pointwise to T under the symmetric topologies on $\mathbf{P}$ and $\mathbf{Q}$. (i.e. $T_s(a) \to T(a)$ for each $a \in \mathbf{P}$). If (T_s) is u-equicontinuous, then T is u-continuous.*

Proof. Let $w \in \mathbf{W}$. There is a $v \in \mathbf{V}$ such that
$$a \le b + v \quad \text{implies} \quad T_s(a) \le T_s(b) + \frac{w}{3}$$
for all $a, b \in \mathbf{P}$ and $s \in S$. Now let $a, b \in \mathbf{P}$ and $a \le b + v$. Since (T_s) converges pointwise to T, there is an $s_0 \in S$ such that
$$T_s(a) \in \frac{w}{3}(T(a))\frac{w}{3} \quad \text{and} \quad T_s(b) \in \frac{w}{3}(T(b))\frac{w}{3}$$
for $s \ge s_0$. These render
$$T(a) \le T_s(a) + \frac{w}{3} \quad \text{and} \quad T_s(b) \le T(b) + \frac{w}{3}.$$
We have
$$T(a) \le T(b) + w.$$
□

Let $(\mathbf{P}, \mathbf{V})$ and $(\mathbf{Q}, \mathbf{W})$ be two locally convex cones. $\mathbf{P} \times \mathbf{Q} = \{(a, b) : a \in \mathbf{P}, b \in \mathbf{Q}\}$ is a pre-ordered cone with
$$(a, b) \le (a', b') \quad \text{if and only if} \quad a \le a' \quad \text{and} \quad b \le b'.$$
We consider $\mathbf{V} \times \mathbf{W} = \{(v, w) : v \in \mathbf{V}, w \in W\}$ as an (abstract) 0-neighborhood system for $\mathbf{P} \times \mathbf{Q}$.

Suppose $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are cones and L maps $\mathbf{P} \times \mathbf{Q}$ into $\mathbf{R}$. L is said to be *bilinear* if L is linear in each coordinates.

Theorem 2.12. *Let $(\mathbf{P}, \mathbf{V})$, $(\mathbf{Q}, \mathbf{W})$ and $(\mathbf{R}, \mathbf{U})$ be locally convex cones with the symmetric topologies, and $L : \mathbf{P} \times \mathbf{Q} \to \mathbf{R}$ be a bilinear mapping which is u-continuous with respect to the first coordinate and continuous with respect to the second coordinate. Let $(p_s)_{s\in S}$ and $(q_s)_{s\in S}$ be nets in $\mathbf{P}$ and $\mathbf{Q}$ respectively such that $p_s \to p_0$ and $q_s \to q_0$. Suppose for every $p \in \mathbf{P}$, every $q \in \mathbf{Q}$ and every $u \in \mathbf{U}$ there is a $v \in \mathbf{V}$ such that for every $a \in v(p)v$ there is a $\lambda > 0$ such that*
$$L(a, q_s) \le L(p, q_s) + \lambda u$$
for all $s \in S$. If $(\mathbf{P}, \mathbf{V})$ is upper-barreled and $(\mathbf{R}, \mathbf{U})$ has the strict separation property, then $L(p_s, q_s) \to L(p_0, q_0)$ in $\mathbf{R}$.

Proof. Let $u \in \mathbf{U}$ and define $b_s : \mathbf{P} \to \mathbf{R}$ as
$$b_s(p) = L(p, q_s) \quad p \in \mathbf{P}, \quad s \in S.$$
By 2.8 $(b_s)_{s\in S}$ is u-equicontinuous, so there is a $v \in \mathbf{V}$ such that
$$p_1 \le p_2 + v \quad \text{implies} \quad b_s(p_1) \le b_s(p_2) + u$$

for all $p_1, p_2 \in \mathbf{P}$ and $s \in S$, i.e. if $p_1 \in v(p_2)v$, then $L(p_1, q_s) \in u(L(p_2, q_s))u$ for all $s \in S$. Since L is continuous in $\mathbf{Q}$, there is a $w \in \mathbf{W}$ such that

$$q_1 \in w(q_2)w \quad \text{implies} \quad L(p, q_1) \in u(L(p, q_2))u$$

for all $p \in \mathbf{P}$ and $q_1, q_2 \in \mathbf{Q}$. Since $q_s \to q_0$, there is an s_0 such that $q_s \in w(q_0)w$ for all $s \geq s_0$ and so $L(p, q_s) \in u(L(p, q_0))u$. Now if $s \geq s_0$ and $(p_s, q_s) \in (v, w)((p_0, q_0))(v, w)$ i.e. $p_s \in v(p_0)v$ and $q_s \in w(q_0)w$, then $L(p_s, q_s) \in u(L(p_0, q_0))u$. □

Corollary 2.13. *Let* $(\mathbf{P}, \mathbf{V})$, $(\mathbf{Q}, \mathbf{W})$ *and* $(\mathbf{R}, \mathbf{U})$ *be locally convex cones, and* $L : \mathbf{P} \times \mathbf{Q} \to \mathbf{R}$ *be a bilinear mapping which is u-continuous w.r.t. the first coordinate and continuous w.r.t. the second coordinate. Let* $(p_s)_{s \in S}$ *and* $(q_s)_{s \in S}$ *be nets in* $\mathbf{P}$ *and* $\mathbf{Q}$ *respectively such that* $p_s \to p_0$ *and* $q_s \to q_0$ *with respect to the symmetric topologies. Under the conditions of Theorem 2.12, if the set*

$$\{L(p, q_s) \ : \ s \in S\}$$

is bounded for each $p \in \mathbf{P}$, *then* $L(p_s, q_s) \to L(p_0, q_0)$ *in* $\mathbf{R}$.

Proof. By 2.10 $(b)_s$ is u-equicontinuous and conclusion follows as the proof of Theorem 2.12. □

Now we bring an example (Example 3.2 of [3]), and illustrate the Theorem 2.3.

Example 2.14. Let $\mathbf{P}$ be the cone of all lower semicontinuous $\overline{\mathbb{R}}$-valued functions f on the interval $[0, 1]$ such that $f(x) = +\infty$ for all x in a neighborhood of 1, endowed with the pointwise algebraic operations and order and the (abstract) 0-neighborhood system $\boldsymbol{\varepsilon} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$. Thus $(\mathbf{P}, \mathbf{V})$ is a locally convex cone, but not a full cone, as the constant functions are not contained in $\mathbf{P}$. By Theorem 2.3, Lemma 2.4 and Example 3.2 of [3], the conditions of the Theorem hold. Now for each $n \in \mathbb{N}$ we set $t_n(x) = nx^n$ and define linear operators $T_n : \mathbf{P} \to \mathbf{P}$ by $T_n(f) = t_n f$. These operators are u-continuous, as $f \leq g + \epsilon$ implies that $T_n(f) \leq T_n(g) + n\epsilon$ for all $f, g \in \mathbf{P}$. For an internally bounded set $B \subseteq \mathbf{P}$ there is $0 < \delta < 1$ such that $g(x) = +\infty$ for all $\delta \leq x \leq 1$ and all $g \in B$. We choose a function $g \in B$ and the neighborhood $w = 1 \in \boldsymbol{\varepsilon}$. Then for every function $f \in \mathbf{P}$ such that $f \leq g' + 1$ and $g'' \leq f + 1$ for some $g', g'' \in B$ we realize that $f(x) = +\infty$ holds for all $\delta \leq x \leq 1$ with the same $0 < \delta < 1$ from above. Therefore

$$T_n(f) \leq T_n(g) + n\delta^n,$$

and as $n\delta^n \to 0$ as $n \to \infty$, there is a $\lambda > 0$ such that

$$T_n(f) \leq T_n(g) + \lambda$$

for such functions $f \in \mathbf{P}$ and $n \in \mathbb{N}$. The constant $\lambda > 0$ depends however on the set B and may not be chosen independently.

As it was noted in [2], every full locally convex cone $(\mathbf{P}, \mathbf{V})$ is upper-barreled. Since, if $B \subseteq \mathbf{P}^2$ is a barrel, for $0 \in \mathbf{P}$ there is a $v \in \mathbf{V}$ such that for every $a \in v(0)v$ there is a $\lambda_a > 0$ such that $(a, 0) \in \lambda_a B$. Since $v \in \mathbf{P}$ and $v \in v(0)v$, there is a $\lambda > 0$ such that $(v, 0) \in \lambda B$. Now let $(a, b) \in \tilde{v}$ and $(a, b) \notin \lambda B$. Then there is a $\mu \in \mathbf{P}^*$ such that $\mu(a) > \mu(b) + \lambda$ and $\mu(v) \leq \lambda$. But $a \leq b + v$ implies $\mu(a) \leq \mu(b) + \mu(v) \leq \mu(b) + \lambda$, which is a contradiction. Hence $\tilde{v} \subseteq \lambda B$ or $(\frac{1}{\lambda}v)\tilde{} \subseteq B$.

Now by the above consideration we illustrate Theorems 2.4, 2.8 and 2.10 and Proposition 2.7 with the following example.

Example 2.15. Let $\mathbb{E}$ be a normed vector space with unit ball $\mathbb{B}$. Let $\mathbf{P} = \{a + \rho\mathbb{B} : a \in \mathbb{E} \ , \ \rho \geq 0\}$, endowed with the usual addition, usual multiplication of sets by non-negative scalars and the set inclusion as order. The (abstract) 0-neighborhood system $\mathbf{V}$ is given by the positive multiples of the unit ball $\mathbb{B}$. Then $(\mathbf{P}, \mathbf{V})$ is a full locally convex cone (since $\mathbf{V} \subseteq \mathbf{P}$) and so is upper-barreled. We consider all $\mu \oplus r$, where $r \geq 0$ and μ is a linear functional on $\mathbb{E}$ such that $||\mu|| \leq r$, if we define $(\mu \oplus r)(a + \rho\mathbb{B}) = \mu(a) + r\rho$. These functionals from $\mathbf{P}$ to $\overline{\mathbb{R}}$ are u-continuous. On the other hand $(\overline{\mathbb{R}}, \boldsymbol{\varepsilon})$ is a locally convex cone which is tightly covered by its bounded elements (and so has the strict separation property). Let $\mathcal{T}$ be the set of all $\mu \oplus r$ which $r \leq 1$. The Theorem 2.4 is verified similar to the Theorem 2.3 in the Example 2.14. It is easy to see that $\mathcal{T}$ is u-equicontinuous. For Proposition 2.7, let F be a bounded set in $\mathbf{P}$. There is a $\lambda > 0$ such that

$$F \subseteq \lambda\mathbb{B} \quad \text{and} \quad \{0\} \subseteq F + \lambda\mathbb{B}.$$

We have

$$\mu(a) + r\rho \leq r\lambda \quad \text{and} \quad 0 \leq \mu(a) + r\rho + r\lambda$$

for all $\mu \oplus r \in \mathcal{T}$ and $a + \rho\mathbb{B} \in F$. Since $r \leq 1$,

$$-\lambda \leq \mu(a) + r\rho \leq \lambda.$$

This shows that $(\mu \oplus r)(F) \subseteq [-\lambda, \lambda]$ and $[-\lambda, \lambda]$ is a bounded set in $\overline{\mathbb{R}}$.

References

1. K. Keimel and W. Roth, *Ordered Cones and Approximation*, Lecture Notes in Mathematics, Vol. 1517, Springer-Verlag, Heidelberg—Berlin—New York, 1992.
2. A. Ranjbari and H. Saiflu, *Projective and inductive limits in locally convex cones*, J. Math. Anal. Appl. **332** (2007), 1097–1108.
3. W. Roth, *A uniform boundedness theorem for locally convex cones*, Proc. Amer. Math. Soc. **126** (1998), 1973–1982.

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Received 07/04/2008; Revised 17/06/2008