Methods of Functional Analysis and Topology
Vol. 15 (2009), no. 4, pp. 356–360

POINTS OF JOINT CONTINUITY OF SEPARATELY CONTINUOUS MAPPINGS

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ABSTRACT. Let X be a Baire space, Y be a compact Hausdorff space and $f : X \times Y \to \mathbb{R}$ be a separately continuous mapping. For each $y \in Y$, we define a game $G(Y, \{y\})$ between players O and P, to show that if in this game either O player has a winning strategy or X is α-favorable and P player does not have a winning strategy, then for each countable subset E of Y, there exists a dense G_δ subset D of X such that f is jointly continuous on $D \times E$.

1. INTRODUCTION

Let X and Y be topological spaces and $f : X \times Y \to \mathbb{R}$. We say that f is *separately continuous*, if for every $(x_0, y_0) \in X \times Y$, the maps

$$x \longmapsto f(x, y_0), \quad y \longmapsto f(x_0, y)$$

are continuous. If f is continuous in every (x_0, y_0) with respect to the product topology, then f is said to be *jointly continuous*.

R. Baire [1] proved that every separately continuous mapping $f : \mathbb{R}^2 \to \mathbb{R}$ is jointly continuous on a subset $D \times \mathbb{R}$ of $\mathbb{R}^2$ where D is a dense G_δ subset of $\mathbb{R}$. It is natural to ask the following:

If X and Y are topological spaces and $f : X \times Y \to \mathbb{R}$ is a separately continuous mapping, can one find a dense G_δ subset $D \subset X$ such that $f : D \times Y \to \mathbb{R}$ is jointly continuous?

Several partial results have been obtained under some geometrical restrictions on the topological spaces X and Y (see e.g. [3]–[5], [10]–[16], [18]–[19]). For example, I. Namioka [14] has shown that the above result holds if X is strongly countably complete (Ĉech-complete) and Y is a compact space. It was expected that the above question must have positive answer for every Baire space X and compact space Y. However, M. Talagrand [19] provided an example of a separately continuous mapping $f : X \times Y \to \mathbb{R}$, where X is an α-favorable (hence it is Baire) and Y is compact such that for each $x \in X$, f is not jointly continuous in some point of $\{x\} \times Y$. The result of M. Talagrand raises the following question:

What are compact spaces Y such that for every Baire (or α-favorable) space X and separately continuous mapping $f : X \times Y \to \mathbb{R}$, f is jointly continuous at each point of a dense subset of $X \times Y$?

In the next section, we will use two person game $G(Y, E)$ between players P and O, where Y is a compact space and $E \subset Y$ to show the following:

If $f : X \times Y \to \mathbb{R}$ is a separately continuous mapping,

i) X is a Baire space and for every $y \in Y$, O has a winning strategy on $G(Y, \{y\})$, or

2000 *Mathematics Subject Classification.* Primary 54C30, 54C05; Secondary 54D30.
Key words and phrases. Separate continuity, joint continuity, topological game.
The research is financially supported by Ferdowsi University of Mashhad.

ii) X is an α-favorable space and Y is a compact Hausdorff space such that for every $y \in Y$, P does not have a winning strategy for the game $G(Y, \{y\})$,

then for every countable subset E of Y, there exists a dense G_δ subset D_E of X such that f is jointly continuous on $D_E \times E$.

2. Topological games and existence of a winning strategy

We start this section by introducing the following topological game, which is known as *"Banach-Mazur game"* (or *"Choquet game"* see [2] or [17]).

Let X be a topological space. The Banach-Mazur game $BM(X)$ is played by two players α and β, who select alternately, nonempty open subsets of X. β starts a game by selecting a nonempty open subset V_1 of X. In return, α-player replies by selecting some nonempty open subset W_1 of V_1. At the n-th stage of the game, $n \geq 1$, the player β chooses a nonempty open subset $V_n \subset W_{n-1}$ and α answers by choosing a nonempty open subset W_n of V_n. Proceeding in this fashion, the players generate a sequence $(V_n, W_n)_{n=1}^{\infty}$ which is called a *play.* The player α is said to have *won* the play $(V_n, W_n)_{n=0}^{\infty}$ if $\bigcap_{n\geq 1} V_n = \bigcap_{n\geq 1} W_n \neq \emptyset$; otherwise the player β is said to have won this play. A *partial play* is a finite sequence of sets consisting of the first few moves of a play. A *strategy* for player α is a rule by means of which the player makes his choices. Here is a more formal definition of the notion strategy. A strategy s for the player α is a sequence of mappings $s = \{s_n\}$, which is inductively defined as follows:

The domain of s_1 is the set of all open subsets of X and s_1 assigns to each nonempty open set $V_1 \subset X$, a nonempty open subset $W_1 = s_1(V_1)$ of V_1. In general, if a partial play $(V_1, \ldots, W_{n-1})$ has already been specified, where $W_i = s_i(V_1, \ldots, V_i)$, $1 \leq i \leq n-1$. Then the domain of s_n would be the set

$$\{(V_1, W_1, \ldots, W_{n-1}, V) : V \subset W_{n-1} \text{ can be the next move of } \beta\text{-player}\}$$

and it assigns to each choice $V_n \subset W_{n-1}$ some nonempty open subset

$$W_n = s_n(V_1, W_1, \ldots, W_{n-1}, V_n)$$

of V_n.

An *s-play* is a play in which α selects his moves according to the strategy s. The strategy s for the player α is said to be a *winning strategy* if every s-play is won by α. A space X is called *α-favorable* if there exists a winning strategy for α in $BM(X)$.

It is easy to verify that every α-favorable space X is a Baire space, that is, a space in which the intersection of countably many dense and open subsets is dense in the space. There are examples of Baire spaces which are not α-favorable (see for example [10]). It is known that X is a Baire space if and only if the player β does not have a winning strategy in the game $BM(X)$ (see [18] Theorems 1 and 2).

Let Y be a compact Hausdorff space and $E \subset Y$, G. Gruenhage [7] introduced the following two person game $G(Y, E)$:

Player O goes first by selecting a nonempty open neighborhood U_1 of E. P answers by choosing a point $y_1 \in U_1$.

In general, in step n, if selections $U_1, y_1, \ldots, U_n, y_n$ have already been specified, O selects a nonempty open set $U_n \supset E$ and then P chooses a point $y_n \in U_n$.

We say O wins the game $g = (U_n, y_n)_{n\geq 1}$ if $y_n \to E$ (i.e. every neighborhood of E contains all but finitely many y_n). If

$$g_1 = (U_1, y_1), \ \ldots \ , \ g_n = (U_1, y_1, \ldots, U_n, y_n)$$

are the first "n" move of some play (of the game), we call g_n the n^{th} (*partial play*) of the game.

By a strategy s for the player O, we mean a sequence of mappings $\{s_n\}$ which is defined inductively as follows:

$s_1(\emptyset)$ is an open neighborhood U_1 of E. In step n, s_n assigns to the partial play $g_{n-1} = (U_i, y_i)_{i\leq n-1}$ an open set $U_n \supset E$. If s is a strategy for O, a play in which O selects his moves according to the strategy s is called an s-play. The strategy s is said to be a winning one if every s-play is won by O. The game $G(Y,E)$ is called O-favorable, if there exists winning strategy for the player O. Otherwise $G(Y,E)$ is said to be O-unfavorable. Similarly, winning strategy for the player P, P-favorable and P-unfavorable $G(Y,E)$ can be defined.

The main result of this paper is based on the following lemma:

Lemma 2.1. *Let $f : X \times Y \to \mathbb{R}$ be a separately continuous mapping. If either*

(1) *X is a Baire space and Y is a compact Hausdorff space such that for some $y_0 \in Y$, O has a winning strategy in the game $G(Y, \{y_0\})$, or*

(2) *X is an α-favorable space and Y is a compact Hausdorff space such that for some $y_0 \in Y$, P does not have a winning strategy for the game $G(Y, \{y_0\})$,*

then for each $\varepsilon > 0$, and every open subset U of X, there exists a nonempty open subset V of U and an open neighborhood H of y_0 such that

$$\mathrm{diam} f(V \times H) \leq \varepsilon.$$

Proof. Suppose that for some $\varepsilon > 0$ and open subset U of X,

$$\mathrm{diam} f(V \times H) > \varepsilon,$$

for all nonempty open subset V of U and every open neighborhood H of y_0. Take some $x_0 \in U$ and an open subset U_1 of U such that

$$|f(x, y_0) - f(x_0, y_0)| < \varepsilon/12, \quad \forall x \in U_1.$$

Let U_1 be the first choice of β-player in $BM(X)$. If H_1 is the first choice of O-player in $G(Y, \{y_0\})$, we take some nonempty open subset H_1' of H_1 such that

$$|f(x_0, y) - f(x_0, y_0)| < \varepsilon/12, \quad \forall y \in H_1'.$$

Let $V_1 \subset U_1$ be the answer of α-player to U_1. Then by our assumption

$$\mathrm{diam} f(V_1 \times H_1') > \varepsilon.$$

Therefore there exists some $(x_1, y_1) \in V_1 \times H_1'$ such that $|f(x_1, y_1) - f(x_0, y_0)| > \varepsilon/2$. The answer of β-player to (U_1, V_1) will be an open subset U_2 of V_1 which satisfies the following

$$|f(x, y_1) - f(x_0, y_0)| > \varepsilon/4 \quad \text{and} \quad |f(x, y_1) - f(x_1, y_1)| < \varepsilon/12.$$

Let y_1 be the first choice of P-player. If H_2 is the answer of O-player to (H_1, y_1), we take a nonempty open subset H_2' of H_2 such that

$$|f(x_i, y) - f(x_i, y_0)| < \varepsilon/12 \quad \text{for each} \quad y \in H_2' \quad \text{and} \quad i = 0, 1.$$

In general, if the partial play $(U_1, V_1, \ldots, U_n, V_n)$ in $BM(X)$ together with finite sequence $\{x_i\}_{1\leq i<n}$, where $x_i \in V_i$ and the partial play $(H_1, x_1, \ldots .H_n)$ have already been selected. We choose an open neighborhood H_n' of H_n, such that for all $y \in H_n'$,

$$|f(x_i, y) - f(x_i, y_0)| < \varepsilon/12, \quad \forall i = 1, \ldots, n-1.$$

By our assumption,

$$\mathrm{diam} f(V_n \times H_n') > \varepsilon.$$

Therefore, there is some $(x_n, y_n) \in V_n \times H_n'$ such that

$$|f(x_n, y_n) - f(x_{n-1}, y_{n-1})| > \varepsilon/2.$$

Let y_n be the answer of the player O to $(H_1, y_1, \ldots .H_n)$ and take a nonempty open subset U_{n+1} of V_n such that for all $x \in U_{n+1}$,

$$|f(x, y_n) - f(x_{n-1}, y_{n-1})| > \varepsilon/4 \quad \text{and} \quad |f(x, y_i) - f(x_n, y_i)| < \varepsilon/12, \quad \text{for} \quad i = 0, \ldots, n.$$

In this way by induction on n, a strategy for β in $BM(X)$ and a strategy for P in $G(Y, \{y_0\})$ is defined.

If (i) holds, then the strategy for β-player is not a winning one, thus some play $(U_i, V_i)_{i\geq 1}$ is won by α. Moreover, O has a winning strategy against the strategy of P defined above.

If (ii) holds, then α has a winning strategy against the strategy β defined above and since the strategy of P is not a winning one, O wins some play $(H_i, y_i)_{i\geq 1}$ described above.

Therefore, in either case the related plays $(U_i, V_i)_{i\geq 1}$ and $(H_i, y_i)_{i\geq 1}$ are won by α and O respectively.

Let $x \in \bigcap_{i\geq 1} V_i$ and H be a neighborhood of y_0 such that

$$|f(x, y) - f(x, y_0)| < \varepsilon/12 \quad \text{for all} \quad y \in H.$$

Since O is the winner of the play $(H_i, y_i)_{i\geq 1}$, there exists some n_0 such that for all $n \geq n_0$, we have $y_n \in H$. Since $x \in \bigcap_{i\geq 1} V_i$,

$$|f(x, y_{n+1}) - f(x_n, y_n)| > \varepsilon/4, \quad \forall n \in \mathbb{N}.$$

However, our construction shows that if $n \geq n_0$, we have

$$\begin{aligned} &|f(x, y_{n+1}) - f(x_n, y_n)| \\ &\quad \leq |f(x, y_{n+1}) - f(x, y_0)| + |f(x, y_0) - f(x_n, y_0)| + |f(x_n, y_0) - f(x_n, y_n)| \\ &\quad < \varepsilon/12 + \varepsilon/12 + \varepsilon/12 = \varepsilon/4. \end{aligned}$$

This contradiction proves the Lemma. □

We call E a W-set in Y if O has a winning strategy in the game $G(Y, E)$. A space Y in which each point of Y is a W-set is called a W-space. One can easily see that first countable spaces are W-spaces and that W-spaces are Frechet (i.e., if $y \in \overline{A}$, then there exists a sequence $\{a_n\} \subset A$ with $a_n \to y$). We also define Y to be a w-space if for every $y \in Y$, P fails to have a winning strategy in $G(Y, \{y\})$. It is natural to ask the following:

Is there a w-space which is not a W-space?

A. Hajnal and I. Juhasz (see [6] or [9]) have shown that if ∞ is the one point compactification of an Aronszajn tree T, with the interval topology, then neither P nor O has a winning strategy in $G(T \cup \{\infty\}, \{\infty\})$.

Now, we are ready to state the main result of the paper.

Theorem 2.2. *Let Y be a compact Hausdorff space and E be a countable subset of Y. If either*

(1) *X is a Baire space and E is a W-space, or*
(2) *X is an α-favorable space and E is a w-space,*

then for every separably continuous mapping $f : X \times Y \to \mathbb{R}$, there is a dense G_δ subset A_E of X such that f is jointly continuous at each point of $A_E \times E$.

Proof. For each $y \in E$ and $n \in \mathbb{N}$, define

$$A_{n,y} = \{x \in X : \exists \text{ neighborhoods } V \text{ of } x \text{ and } H \text{ of } y; \operatorname{diam} f(V \times H) \leq \frac{1}{n}\}.$$

It is clear that each $A_{n,y}$ is a open subset of X. Under either conditions (1) or (2), by Lemma 2.1, each $A_{n,y}$ is dense in X. Put

$$A_E = \bigcap_{n\in\mathbb{N} \& y\in Y} A_{n,y}.$$

Since E is countable and X is a Baire space, A_E is dense G_δ subset of X. Clearly f is jointly continuous on in each point of $A_E \times E$. □

Proposition 2.3. *Let $f : X \times Y \to \mathbb{R}$ be a separately continuous mapping. If either*

(1) *X is a Baire space and Y is a compact Hausdorff space such that for each point y of a dense subset E of Y, O has a winning strategy in the game $G(Y, \{y\})$, or*
(2) *X is an α-favorable space and Y is a compact Hausdorff space such that for each point y of a dense subset E of Y, P does not have a winning strategy for the game $G(Y, \{y\})$,*

then there exists a dense subset D of $X \times Y$ such that f is jointly continuous at each point of D.

Proof. Let

$$D = \{(x, y) \in X \times Y : f \text{ is jointly continouos at } (x, y)\}.$$

Let V and W be open subsets X and Y respectively. Thanks to Theorem 2.2, for each point $y \in E$, there exists a dense subset A_y of x such that f is jointly continuous on $A_y \times \{y\}$. Hence $A_y \times \{y\} \subset (V \times W) \cap D$. This means that D is dense in $X \times Y$. □

The following question naturally arises:

Problem. Under either conditions of Proposition 2.3 with $E = Y$, can one find a dense subset of A of X such that f is jointly continuous on $A \times Y$?

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Received 07/10/2008; Revised 19/11/2008