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INVERSE SPECTRAL PROBLEM FOR SOME GENERALIZED JACOBI HERMITIAN MATRICES

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Dedicated to my dear father on the occasion of his 50th birthday

Abstract. In this article we will investigate an inverse spectral problem for three-diagonal block Jacobi type Hermitian real-valued matrices with "almost" semidiagonal matrices on the side diagonals.

1. Introduction

This article is a logical continuation of article [5]. Here we investigate an inverse spectral problem for some generalized Jacobi Hermitian matrices. As in [5] under the generalized Jacobi Hermitian matrix we mean a three-diagonal block Jacobi type Hermitian matrix. Using word "some" in the title we mean that the matrix has following special form: matrices on side diagonals have elements which are equal to zero. Later we will give an accurate definition.

In Section 2 we describe an inverse spectral problem for the classical Jacobi matrix. It helps to understand the problem in the general case.

In Section 3 we will recall all necessary results about the direct spectral problem for generalized Jacobi Hermitian matrices from [5]. Also, we will give there a new result, a recurrence relation for calculating the whole vector which consists of generalized eigenvalues of the considered matrix (the so-called, polynomials of the first order) and two examples which show that this vector has a complex structure in the general case.

Section 4 contains the main results of this article. At first, we define the matrix J for which the inverse spectral problem will be considered. It is necessary to admit that in this section we will use the same notations as in Section 3 but they will describe objects for the new matrix under investigation. Then we consider an orthogonalization procedure for some system of vectors for a given measure. The obtained new vectors give a possibility to correctly recover elements of the matrix, i.e., to solve the inverse spectral problem.

Now we give a short list of notations used in this article.

Let $X = (x_{ij})_{i=0}^{N}{}_{j=0}^{M}, N = 0, 1, \dots, \infty, M = 0, 1, \dots, \infty$, be some matrix, X^T and X^* denote, respectively, the transposed and the adjoint matrix of X. It is necessary to admit that in what follows we will understand a vector as a column-vector, i.e., the matrix with single column; and often we will write the vector as a transposed one to row-vector, i.e., as a matrix with one row (line matrix). Also, $X_{\cdot,i}$ and $X_{i,\cdot}$ denotes respectively the i-th column and i-th row of X; $X_{j;\cdot,i}$ ($X_{j;i,\cdot}$) denotes the i-th column (row) of the matrix X_j; $X_{j;i,k}$ denotes the element of the matrix X_j which belongs to i-th row and k-th column,

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i.e., $X_{j;i,k} = (X_j)_{i,k}$; by I_j we denote the identity matrix in $\mathbb{C}^j$;

$$P_{\alpha;(j,\cdot)}(\lambda) = \begin{pmatrix} P_{\alpha;(j,0)}(\lambda) \\ P_{\alpha;(j,1)}(\lambda) \\ \dots \\ P_{\alpha;(j,j)}(\lambda) \\ 0 \\ \dots \end{pmatrix};$$

$\delta_j = (0, \dots, 0, \underbrace{1}_{j\text{th place}}, 0, \dots)^T$ (the indexing starts with 0); $\delta_{j,k} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$

2. The inverse spectral problem for the classical Jacobi matrix

In the classical theory one investigates, on the space ℓ_2 of sequences $f = (f_n)_{n=0}^{\infty}, f_n \in \mathbb{C}$, the Hermitian Jacobi matrix:

$$(1) \qquad J = \begin{bmatrix} b_0 & a_0 & 0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \quad b_n \in \mathbb{R}, \quad a_n > 0, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

This matrix generates, on finite sequences $f \in \ell_{\text{fin}}$, an operator on ℓ_2, which is Hermitian with equal deficiency indices and, therefore, has a selfadjoint extension on ℓ_2. Under some conditions on J (for example, $\sum\limits_{n=0}^{\infty} \frac{1}{a_n} = \infty$) the closure $\widetilde{J}$ of J is selfadjoint.

The inverse spectral problem in this classical case is the following. Consider a Borel probability measure $d\rho(\lambda)$ on $\mathbb{R}$ for which all the moments s_n exist:

$$(2) \qquad s_n = \int\limits_{\mathbb{R}} \lambda^n d\rho(\lambda), \quad n \in \mathbb{N}_0$$

(and support of $d\rho(\lambda)$ contains an infinite set on finite interval).

The problem is following: it is necessary to recover the corresponding Jacobi matrix J in such a way that the initial measure $d\rho(\lambda)$ would be equal to the spectral measure of $\widetilde{J}$.

A method for such a reconstruction is simple: it is necessary to take the sequence of functions

$$(3) \qquad 1, \lambda, \lambda^2, \dots \in L^2(\mathbb{R}, d\rho(\lambda))$$

(which is linearly independent) and apply to it the classical Gram-Schmidt orthogonalization procedure. As a result, we get a sequence of orthonormal polynomials,

$$(4) \qquad P_0(\lambda) = 1, P_1(\lambda), P_2(\lambda), \dots.$$

Then the matrix J is reconstructed by the formulas: $\forall n \in \mathbb{N}_0$

$$(5) \qquad a_n = \int\limits_{\mathbb{R}} \lambda P_n(\lambda) P_{n+1}(\lambda) d\rho(\lambda), \quad b_n = \int\limits_{\mathbb{R}} \lambda \big(P_n(\lambda)\big)^2 d\rho(\lambda).$$

3. About direct spectral problem for generalized Jacobi Hermitian matrices

At first we recall necessary results from [5].

Let us consider the complex Hilbert space

$$(6) \qquad \mathbf{l}_2 = H_0 \oplus H_1 \oplus H_2 \oplus \dots, H_i = \mathbb{C}^{i+1}, \quad i \in \mathbb{N}_0,$$

of vectors $\mathbf{l}_2 \ni f = (f_n)_{n=0}^{\infty}$, where $f_n = (f_{n;j})_{j=0}^{n} \in H_n$; $f = \sum_{n=0}^{\infty}\sum_{j=0}^{n} f_{n;j}e_{n;j}$, (here $e_{n;j}, n = 0,1,\dots, j = 0,1,\dots,n$, are elements of the standard basis in $\mathbf{l}_2$) with the scalar product

$$(f,g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty}(f_n,g_n)_{H_n}, \qquad f,g \in \mathbf{l}_2.$$

Consider the Hilbert space rigging

$$\mathbf{l} = (\mathbf{l}_{\mathrm{fin}})' \supset \mathbf{l}_2(p^{-1}) \supset \mathbf{l}_2 \supset \mathbf{l}_2(p) \supset \mathbf{l}_{\mathrm{fin}}, \tag{7}$$

where $\mathbf{l}_{\mathrm{fin}}$ is the space of finite vectors, $\mathbf{l}$ is the space of arbitrary vectors, $\mathbf{l}_2(p)$ is the space of infinite vectors with the scalar product

$$(f,g)_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty}(f_n,g_n)_{H_n}p_n; f,g \in \mathbf{l}_2(p)$$

(here $p = (p_n)_{n=0}^{\infty}, p_n \geq 1$, is a given weight). Let us suppose that the embedding of the positive space $\mathbf{l}_2(p) \subset \mathbf{l}_2$ is quasinuclear. This is true if $\sum_{n=0}^{\infty}\frac{n+1}{p_n} < \infty$.

Let us consider, in space (6), the Hermitian matrix $J = (J_{j,k})_{j,k=0}^{\infty}$ with operator-valued complex elements $J_{j,k}\colon H_k \to H_j$, $J_{j,k} = (J_{j,k;\alpha,\beta})_{\alpha=0}^{j}{}_{\beta=0}^{k}$, of the following block Jacobi structure:

$$J = \begin{bmatrix} b_0 & c_0 & 0 & 0 & \dots \\ a_0 & b_1 & c_1 & 0 & \dots \\ 0 & a_1 & b_2 & c_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \text{where} \quad \begin{aligned} a_i &= J_{i+1,i} : H_i \to H_{i+1}, \\ b_i &= J_{i,i} : H_i \to H_i, \\ c_i &= J_{i,i+1} : H_{i+1} \to H_i. \end{aligned} \tag{8}$$

It is necessary to assume that $b_j = (b_j)^*, c_j = (a_j)^*$.

Let $u \in \mathbf{l}_2$. Then the matrix J acts in following way:

$$(Ju)_j = a_{j-1}u_{j-1} + b_ju_j + c_ju_{j+1}, \quad \text{where} \quad u_{-1} = 0. \tag{9}$$

The following analogue of Green's formula takes place $\forall k,l \in \mathbb{N}_0, k \leq l$:

$$\begin{aligned} \sum_{j=k}^{l}\left[\big((Ju)_j,v_j\big)_{H_j} - \big(u_j,(Jv)_j\big)_{H_j}\right] &= \left[(c_lu_{l+1},v_l)_{H_l} - (a_lu_l,v_{l+1})_{H_{l+1}}\right] \\ &\quad - \left[(c_{k-1}u_k,v_{k-1})_{H_{k-1}} - (a_{k-1}u_{k-1},v_k)_{H_k}\right], \quad \forall u,v \in \mathbf{l}_2. \end{aligned} \tag{10}$$

Then we construct an operator $\mathbf{J}$ in $\mathbf{l}_2$ (an analog of $\widetilde{J}$ in the classical case) from the matrix J. Let the operator $\mathbf{J}$ be selfadjoint (necessary and sufficient conditions for selfadjointness were given in [5]). Consider an equation which gives a possibility to find eigenvectors for the operator $\mathbf{J}$,

$$(J\varphi(\lambda))_j = a_{j-1}\varphi_{j-1}(\lambda) + b_j\varphi_j(\lambda) + c_j\varphi_{j+1}(\lambda) = \lambda\varphi_j(\lambda), \quad \lambda \in \mathbb{R}, \quad j \in \mathbb{N}_0, \tag{11}$$

where $\varphi \in \mathbf{l}, \varphi_{-1}(\lambda) = 0$.

As we can see, none of equations in system (11) defines $\varphi_{j+1}(\lambda)$ in a unique way by means of $\varphi_j(\lambda)$ and $\varphi_{j-1}(\lambda)$. Indeed, first of all, from (11) we need to obtain $n+2$ variables $\varphi_{j+1;0},\varphi_{j+1;1},\dots,\varphi_{j+1;j+1}$ but there are only $n+1$ equalities. And secondly, c_j is a $(j+1)\times(j+2)$-matrix and, therefore, its inverse matrix is not defined.

So, we assume that

1) $\operatorname{rank} c_j = j+1$;
2) the matrix $c_j = \left\{c_{j;\alpha,\beta}\right\}_{\alpha=0}^{j}{}_{\beta=0}^{j+1}$ is as follows: for the matrix $\widetilde{c}_j := \left\{c_{j;\alpha,\beta}\right\}_{\alpha=0}^{j}{}_{\beta=1}^{j+1}$ there exists an inverse $\widetilde{c}_j^{\,-1}$;

3) let $\varphi_{j;0}(\lambda) = \varphi_{j;0} \in \mathbb{C}, j = 0, 1, \ldots,$ be some fixed complex constants, i.e., $\varphi_{j;0}$ does not depend on λ, where $\varphi_{0,0} := \varphi_0$. Thus all of the above indicates that $\varphi_{j;0}$ generates a "boundary conditions" vector $\varphi_{\cdot;0} := \begin{pmatrix}\varphi_0, & \varphi_{1;0}, & \varphi_{2;0}, & \ldots\end{pmatrix}^T$.

So, now we have to find a solution of the difference equation (11) with the following Cauchy conditions: $\varphi_{-1}(\lambda) = 0$ and $\varphi_{\cdot;0}$ is some given boundary conditions vector.

Let us denote by $P_{\alpha;(j,\cdot)}(\lambda) := \big(P_{\alpha;(j,k)}(\lambda)\big)_{k=0}^{j}, \alpha = 0, 1, \ldots, j = 0, 1, \ldots,$ a solution of equation (11) with the boundary conditions $P_{\alpha;(j,0)} = \delta_{j,\alpha}, j = 0, 1, \ldots$. Then for any fixed α and $\forall k = 1, 2, \ldots, j$, the following equality takes place:

$$
P_{\alpha;(j,k)}(\lambda) = \begin{cases} 0, & j = 0, \ldots, \alpha - 1 \\ -(\widetilde{c}_{j-1}^{\,-1})_{k-1,\cdot} c_{j-1;\cdot,0}, & j = \alpha, \\ \big(\widetilde{c}_{j-1}^{\,-1}(\lambda I_{j-1} - b_{j-1})\big)_{k-1,\cdot} P_{\alpha;(j-1,\cdot)}(\lambda), & j = \alpha + 1, \\ \big(\widetilde{c}_{j-1}^{\,-1}(\lambda I_{j-1} - b_{j-1})\big)_{k-1,\cdot} P_{\alpha;(j-1,\cdot)}(\lambda) - \\ \quad - \big(\widetilde{c}_{j-1}^{\,-1} a_{j-2}\big)_{k-1,\cdot} P_{\alpha;(j-2,\cdot)}(\lambda) & j = \alpha + 2, \alpha + 3, \ldots \end{cases} \tag{12}
$$

And $\varphi_j(\lambda)$ satisfy the following formulas in the vector and coordinate forms:

$$
\begin{aligned} \varphi_j(\lambda) &= \sum_{\alpha=0}^{j} \varphi_{\alpha;0} P_{\alpha;(j,\cdot)}(\lambda), \\ \varphi_{j;k}(\lambda) &= \sum_{\alpha=0}^{j} \varphi_{\alpha;0} P_{\alpha;(j,k)}(\lambda), \quad j = 0, 1, \ldots, \quad k = 0, 1, \ldots, j. \end{aligned} \tag{13}
$$

It follows from [2], Ch. 5, that for our operator $\mathbf{J}$ we have the representation

$$
\mathbf{J} f = \int_{\mathbb{R}} \lambda \Phi(\lambda) d\sigma(\lambda) f, \quad f \in \mathbf{l}_2(p) \tag{14}
$$

and a resolution of identity, which corresponds to $\mathbf{J}$, can be represented in the form

$$
E(\triangle) = \int_{\triangle} \Phi(\lambda) d\sigma(\lambda), \quad \triangle \in \mathcal{B}(\mathbb{R}), \tag{15}
$$

where $\mathcal{B}(\mathbb{R})$ is a Borel sigma algebra on $\mathbb{R}$, $d\sigma(\lambda)$ is a nonnegative finite measure with infinite support, $\Phi(\lambda)\colon \mathbf{l}_2(p) \to \mathbf{l}_2(p^{-1})$ is a generalized projection operator and $\Phi(\lambda)$ is a positive-defined kernel, i.e. $\forall f \in \mathbf{l}_2(p)\, (\Phi(\lambda) f, f)_{\mathbf{l}_2} \geq 0$. For all $f, g \in \mathbf{l}_2(p)$ we have the Parseval equality

$$
(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} (\Phi(\lambda) f, g)_{\mathbf{l}_2} d\sigma(\lambda). \tag{16}
$$

Let us denote by π_n the operator in $\mathbf{l}_2$ of orthogonal projection on $H_n, n \in \mathbb{N}_0$. Hence $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$ we have $f_n = \pi_n f$. This operator acts analogously in the space $\mathbf{l}_2(p)$ and $\mathbf{l}_2(p^{-1})$. Let us consider the operator matrix $(\Phi_{j,k}(\lambda))_{j,k=0}^{\infty}$ where

$$
\Phi_{j,k}(\lambda) = \pi_j \Phi(\lambda) \pi_k \colon \mathbf{l}_2 \to H_j \quad (\text{or, in fact,} \quad H_k \to H_j). \tag{17}
$$

The following representation takes place:

$$
\Phi_{j,k;l,m}(\lambda) = \sum_{\alpha=0}^{j} \sum_{\beta=0}^{k} \Phi_{\alpha,\beta;0,0}(\lambda) \overline{P_{\beta;(k,m)}(\lambda)} P_{\alpha;(j,k)}(\lambda),
$$

$j, k \in \mathbb{N}_0, l = 0, \ldots, j, m = 0, \ldots, k$, where $\Phi_{j,k;l,m}(\lambda)$ is the element of the $(j+1)\times(k+1)$-matrix (17) and it can be understand also as

$$
\Phi_{j,k;l,m}(\lambda) = (\Phi(\lambda) e_{k;m}, e_{j;l})_{\mathbf{l}_2}, \quad j, k \in \mathbb{N}_0, \quad l = 0, \ldots, j, \quad m = 0, \ldots, k. \tag{18}
$$

Let us construct the matrix spectral measure $\Sigma(\cdot)$ by the formula

$$d\Sigma(\lambda)=\begin{pmatrix}\Phi_{0,0;0,0}(\lambda) & \Phi_{0,1;0,0}(\lambda) & \dots\\ \Phi_{1,0;0,0}(\lambda) & \Phi_{1,1;0,0}(\lambda) & \dots\\ \vdots & \vdots & \ddots\end{pmatrix}d\sigma(\lambda)=\left(\Phi_{\alpha,\beta;0,0}(\lambda)d\sigma(\lambda)\right)_{\alpha,\beta=0}^{\infty}. \tag{19}$$

Denote $\sigma_{\alpha\beta}(\cdot):=(\Sigma(\cdot)\delta_\beta,\delta_\alpha)_{\ell_2}$. From (15), (18) and (19), it follow that

$$(E(\triangle)e_{\beta;0},e_{\alpha;0})_{\mathbf{l}_2}=(\Sigma(\triangle)\delta_\beta,\delta_\alpha)_{\ell_2}=\sigma_{\alpha\beta}(\triangle),\quad \triangle\in\mathcal{B}(\mathbb{R}). \tag{20}$$

Consider the space of finite vectors $u(\lambda)=\big(u_0(\lambda),u_1(\lambda),\dots\big)$, $\lambda\in\mathbb{R}$, ($u_i(\cdot)$, $i=0,1,\dots$, are complex-valued functions of the real variable) with the scalar product

$$(u(\lambda),v(\lambda))_{L^2(\mathbb{R},d\Sigma(\lambda))}=\int\limits_{\mathbb{R}}(d\Sigma(\lambda)u(\lambda),v(\lambda))_{\ell_2}.$$

Since $\Phi(\lambda)$ is a positive-defined kernel, the scalar product $(\cdot,\cdot)_{L^2}$ is well defined. Let us introduce the Hilbert space $L^2(\mathbb{R},d\Sigma(\lambda))$ as a completion of the given space of finite vectors with respect to the scalar product $(\cdot,\cdot)_{L^2(\mathbb{R},d\Sigma(\lambda))}$. Also, introduce for $f\in\mathbf{l}_{\mathrm{fin}}$, the Fourier transform $\widehat{f}(\lambda)=\begin{pmatrix}\widehat{f}_0(\lambda)\\ \widehat{f}_1(\lambda)\\ \dots\end{pmatrix}\in L^2(\mathbb{R},d\Sigma(\lambda))$ by the formula

$$\widehat{f}_\alpha(\lambda)=\sum_{j=\alpha}^{\infty}\left(f_j,P_{\alpha;(j,\cdot)}(\lambda)\right)_{H_j},\quad \alpha\in\mathbb{N}_0. \tag{21}$$

According to these notations, the following Parseval equality takes place: $\forall f,g\in\mathbf{l}_{\mathrm{fin}}$

$$(f,g)_{\mathbf{l}_2}=(\widehat{f}(\lambda),\widehat{g}(\lambda))_{L^2(\mathbb{R},d\Sigma(\lambda))}. \tag{22}$$

Let us denote

$$P_{\cdot;(j,k)}(\lambda):=\begin{pmatrix}P_{0;(j,k)}(\lambda)\\ \dots\\ P_{j;(j,k)}(\lambda)\\ 0\\ \dots\end{pmatrix}=\left(P_{\alpha;(j,k)}(\lambda)\right)_{\alpha=0}^{\infty},\quad j=0,1,\dots,\quad k=0,\dots,j.$$

Vectors elements of which are polynomials will be called "vectors of polynomials". These vectors of polynomials are very important for the inverse spectral problem. We will represent in order the infinite collection of all these vectors,

$$P_{\cdot;(0,0)}(\lambda);P_{\cdot;(1,0)}(\lambda),P_{\cdot;(1,1)}(\lambda);P_{\cdot;(2,0)}(\lambda),P_{\cdot;(2,1)}(\lambda),P_{\cdot;(2,2)}(\lambda);\dots;$$
$$P_{\cdot;(j,0)}(\lambda),\dots,P_{\cdot;(j,k)}(\lambda),\dots,P_{\cdot;(j,j)}(\lambda);\dots,\quad j=0,1,\dots,\quad k=0,\dots,j.$$

So, it is convenient to write this set of vectors of polynomials $P_{\cdot;(j,k)}(\lambda)$ as

$$\begin{array}{cccccc} & & & P_{\cdot;(j,j)}(\lambda) & & \\ & & & \dots & & \\ k & & P_{\cdot;(2,2)}(\lambda) & P_{\cdot;(j,k)}(\lambda) & & \\ \uparrow & P_{\cdot;(1,1)}(\lambda) & P_{\cdot;(2,1)}(\lambda) & \dots & & \\ P_{\cdot;(0,0)}(\lambda) & P_{\cdot;(1,0)}(\lambda) & P_{\cdot;(2,0)}(\lambda)\quad\dots & P_{\cdot;(j,0)}(\lambda) & \dots & \to j\end{array} \tag{23}$$

From (22) we obtain the orthogonality relations: $\forall N,M\in\mathbb{N}_0,\xi=0,\dots,N,\zeta=0,\dots,M$,

$$\delta_{N,M}\delta_{\xi,\zeta}=\int\limits_{\mathbb{R}}\left(d\Sigma(\lambda)\overline{P_{\cdot;(N,\xi)}(\lambda)},\overline{P_{\cdot;(M,\zeta)}(\lambda)}\right)_{\ell_2}. \tag{24}$$

The operator $\mathbf{J}$ acts as follows: $\forall f,g\in\mathbf{l}_{\mathrm{fin}}$

$$(\mathbf{J}f,g)_{\mathbf{l}_2}=(Jf,g)_{\mathbf{l}_2}=\left(\lambda\widehat{f}(\lambda),\widehat{g}(\lambda)\right)_{L^2(\mathbb{R},d\Sigma(\lambda))}. \tag{25}$$

Elements of the matrix J can be recovered by the formulas: $\forall j,k\in\mathbb{N}_0$, $l=0,\dots,j, m=0,\dots,k$,

$$J_{j,k;l,m}=\int\limits_{\mathbb{R}}\lambda\left(d\Sigma(\lambda)\overline{P_{\cdot;(k,m)}(\lambda)},\overline{P_{\cdot;(j,l)}(\lambda)}\right)_{\ell_2}. \tag{26}$$

Now we will give some new results which were not published in [5]. They are very useful for a better understanding of the inverse spectral problem in the general case.

The equalities (12) give a possibility to calculate $P_{\alpha;(j,k)}(\lambda)$ step by step for all permitted indexes. In following statement we prove a recurrence relation which allows to obtain the whole $P_{\cdot;(j,k)}(\lambda)$ at once.

Proposition 1. *Let us denote* $\forall j\in\mathbb{N}_0$

$$B_j(\lambda):=\widetilde{c_j}^{-1}(\lambda I_j-b_j),\quad A_j:=-\widetilde{c_j}^{-1}a_{j-1},\quad C_j:=-\widetilde{c_j}^{-1}c_{j;\cdot,0}. \tag{27}$$

Here $B_j(\lambda)$ is a $(j+1)\times(j+1)$-matrix elements of which are polynomials of the first degree, A_j is a $(j+1)\times j$-matrix and C_j is a vector from H_j. Then $\forall j=1,2,\dots,i=1,\dots,j$, the following recurrence formula takes place:

$$P_{\cdot;(j,i)}(\lambda)=\sum_{k=0}^{j-1}B_{j-1;i-1,k}(\lambda)P_{\cdot;(j-1,k)}(\lambda)+\sum_{l=0}^{j-2}A_{j-1;i-1,l}P_{\cdot;(j-2,l)}(\lambda)+C_{j-1;i-1}\delta_j. \tag{28}$$

Proof. Let us prove (28) for $P_{\cdot;(j,i)}(\lambda), j=0,1,2, i=0,\dots,j$, at first. It helps to understand the scheme of the proof in general. 1) $P_{\cdot;(0,0)}(\lambda)=\delta_0$.
2) $P_{\cdot;(1,0)}(\lambda)=\delta_1$;

$$P_{\cdot;(1,1)}(\lambda)=\begin{pmatrix}P_{0;(1,1)}(\lambda)\\ P_{1;(1,1)}(\lambda)\\ 0\\ \dots\end{pmatrix}=\begin{pmatrix}\widetilde{c_0}^{-1}(\lambda-b_0)P_{0;(0,\cdot)}(\lambda)\\ \widetilde{c_0}^{-1}c_{0;0,0}\\ 0\\ \dots\end{pmatrix}=\widetilde{c_0}^{-1}(\lambda-b_0)\delta_0-\widetilde{c_0}^{-1}c_{0;0,0}\delta_1.$$

3) $P_{\cdot;(2,0)}(\lambda)=\delta_2$;

$$\begin{aligned}P_{\cdot;(2,1)}(\lambda)&=\begin{pmatrix}P_{0;(2,1)}(\lambda)\\ P_{1;(2,1)}(\lambda)\\ P_{2;(2,1)}(\lambda)\\ 0\\ \dots\end{pmatrix}\\
&=\begin{pmatrix}\left(\widetilde{c_1}^{-1}(\lambda I_1-b_1)\right)_{0,\cdot}P_{0;(1,\cdot)}(\lambda)-\left(\widetilde{c_1}^{-1}a_0\right)_{0,\cdot}P_{0;(0,\cdot)}(\lambda)\\ \left(\widetilde{c_1}^{-1}(\lambda I_1-b_1)\right)_{0,\cdot}P_{1;(1,\cdot)}(\lambda)\\ \left(\widetilde{c_1}^{-1}\right)_{0,\cdot}c_{1;\cdot,0}\\ 0\\ \dots\end{pmatrix}=\left(\widetilde{c_1}^{-1}(\lambda I_1-b_1)\right)_{0,\cdot}\\
&\quad\times\left(P_{0;(1,\cdot)}(\lambda)\delta_0+P_{1;(1,\cdot)}(\lambda)\delta_1\right)-\left(\widetilde{c_1}^{-1}a_0\right)_{0,\cdot}P_{0;(0,\cdot)}(\lambda)\delta_0-\left(\widetilde{c_1}^{-1}\right)_{0,\cdot}c_{1;\cdot,0}\delta_2\\
&=\sum_{k=0}^{1}B_{1;0,k}(\lambda)P_{\cdot;(1,k)}(\lambda)+A_{1;0,0}P_{\cdot;(0,0)}(\lambda)+C_{1;0}\delta_2;\end{aligned}$$

$$P_{\cdot;(2,2)}(\lambda) = \begin{pmatrix} B_{1;1,\cdot}(\lambda)P_{0;(1,\cdot)}(\lambda) + A_{1;1,\cdot}P_{0;(0,\cdot)}(\lambda) \\ B_{1;1,\cdot}(\lambda)P_{1;(1,\cdot)}(\lambda) \\ C_{1;1} \\ 0 \\ \dots \end{pmatrix} = \sum_{k=0}^{1} B_{1;1,k}(\lambda)P_{\cdot;(1,k)}(\lambda)$$

$$+ A_{1;1,0}\delta_0 + C_{1;1}\delta_2.$$

4) Let us prove that equality (28) takes place $\forall j \in \mathbb{N}_0, i = 0, \dots, j$. Indeed,

$$P_{\cdot;(j,i)}(\lambda) = \begin{pmatrix} B_{j-1;i-1,\cdot}(\lambda)P_{0;(j-1,\cdot)}(\lambda) + A_{j-1;i-1,\cdot}P_{0;(j-2,\cdot)}(\lambda) \\ \dots \\ B_{j-1;i-1,\cdot}(\lambda)P_{j-2;(j-1,\cdot)}(\lambda) + A_{j-1;i-1,\cdot}P_{j-2;(j-2,\cdot)}(\lambda) \\ B_{j-1;i-1,\cdot}(\lambda)P_{j-1;(j-1,\cdot)}(\lambda) \\ C_{j-1;i-1} \\ 0 \\ \dots \end{pmatrix}$$

$$= \sum_{k=0}^{j-1} B_{j-1;i-1,k}(\lambda)(P_{0;(j-1,k)}(\lambda)\delta_0 + P_{1;(j-1,k)}(\lambda))\delta_1 + \dots + P_{j-1;(j-1,k)}(\lambda))\delta_{j-1})$$

$$+ \sum_{l=0}^{j-2} A_{j-1;i-1,l}(P_{0;(j-2,l)}(\lambda)\delta_0 + \dots + P_{j-2;(j-2,l)}(\lambda)\delta_{j-2}) + C_{j-1;i-1}\delta_j$$

$$= \sum_{k=0}^{j-1} B_{j-1;i-1,k}(\lambda)P_{\cdot;(j-1,k)}(\lambda) + \sum_{l=0}^{j-2} A_{j-1;i-1,l}P_{\cdot;(j-2,l)}(\lambda) + C_{j-1;i-1}\delta_j.$$

□

Now we will consider two examples of the matrix J and calculate the vector of the polynomials $P_{\cdot;(j,k)}(\lambda)$ for $j = 0, 1, 2, 3, k = 0, \dots, j$. These calculations will help us to understand the construction and the form of $P_{\cdot;(j,k)}(\lambda)$ in the general case. This information is very important for the corresponding inverse spectral problem.

Example 1. Consider a matrix (8) of the form: $\forall j = 0, 1, \dots$

$$b_j = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \quad c_j = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad a_j = (c_j)^*.$$

So, $\widetilde{c_j}^{-1} = I_j$, $c_{j;\cdot,0} = (1, 0, \dots, 0)^T \in H_j, j = 0, 1, \dots$ Using (28) we obtain
1) $P_{\cdot;(0,0)}(\lambda) = \delta_0 = (1, 0, 0, \dots)^T$.
2) $P_{\cdot;(1,0)}(\lambda) = \delta_1 = (0, 1, 0, \dots)^T$;
$P_{\cdot;(1,1)}(\lambda) = \lambda\delta_0 - \delta_1 = (\lambda, -1, 0, \dots)^T$.
3) $P_{\cdot;(2,0)}(\lambda) = \delta_2$;
$P_{\cdot;(2,1)}(\lambda) = \lambda P_{\cdot;(1,0)}(\lambda) - P_{\cdot;(0,0)}(\lambda) - \delta_2 = (-1, \lambda, -1, 0, \dots)^T$;
$P_{\cdot;(2,2)}(\lambda) = \lambda P_{\cdot;(1,1)}(\lambda) - \delta_0 - \delta_2 = (\lambda^2 - 1, -\lambda, -1, 0, \dots)^T$.
4) $P_{\cdot;(3,0)}(\lambda) = \delta_3$;
$B_2(\lambda) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, A_2 = -\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, C_2 = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};$
$P_{\cdot;(3,1)}(\lambda) = \lambda P_{\cdot;(2,0)}(\lambda) - P_{\cdot;(1,0)}(\lambda) - \delta_3 = \lambda\delta_2 - \delta_1 - \delta_3 = (0, -1, \lambda, -1, 0, \dots)^T$;

$P_{\cdot;(3,2)}(\lambda) = \lambda P_{\cdot;(2,1)}(\lambda) - P_{\cdot;(1,0)}(\lambda) = \lambda^2\delta_1 - \lambda\delta_0 - \lambda\delta_2 - \delta_1 = (-\lambda, \lambda^2 - 1, -\lambda, 0, \dots)^T$;
$P_{\cdot;(3,3)}(\lambda) = \lambda P_{\cdot;(2,2)}(\lambda) - P_{\cdot;(1,1)}(\lambda) = (\lambda^3 - 2\lambda, -\lambda^2 + 1, -\lambda, 0, \dots)^T$.

Example 2. Let us consider a matrix (8) of the form: $\forall j = 0, 1, \dots$

$$b_j = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \quad c_j = \begin{pmatrix} 0 & 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad a_j = (c_j)^*.$$

Then $\widetilde{c_j}^{-1} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$. The vectors of polynomials $P_{\cdot;(j,k)}(\lambda)$ for this matrix are following:
1) $P_{\cdot;(0,0)}(\lambda) = \delta_0$.
2) $P_{\cdot;(1,0)}(\lambda) = \delta_1$;
$P_{\cdot;(1,1)}(\lambda) = \lambda\delta_0$.
3) $B_1 = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}, A_1 = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}, C_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;
$P_{\cdot;(2,0)}(\lambda) = \delta_2$;
$P_{\cdot;(2,1)}(\lambda) = \lambda P_{\cdot;(1,0)}(\lambda) + \lambda P_{\cdot;(1,1)}(\lambda) - \delta_0 = \lambda\delta_1 + \lambda^2\delta_0 - \delta_0 = (\lambda^2 - 1, \lambda, 0, \dots)^T$;
$P_{\cdot;(2,2)}(\lambda) = \lambda P_{\cdot;(1,1)}(\lambda) - \delta_0 = (\lambda^2 - 1, 0, \dots)^T$.
4) $B_2 = \begin{pmatrix} \lambda & \lambda & \lambda \\ 0 & \lambda & \lambda \\ 0 & 0 & \lambda \end{pmatrix}, A_2 = -\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$;
$P_{\cdot;(3,0)}(\lambda) = \delta_3$;
$P_{\cdot;(3,1)}(\lambda) = \lambda P_{\cdot;(2,0)}(\lambda) + \lambda P_{\cdot;(2,1)}(\lambda) + \lambda P_{\cdot;(2,2)}(\lambda) - P_{\cdot;(1,1)}(\lambda)$
$= \lambda\delta_2 + \lambda(\lambda^2 - 1, \lambda, 0, \dots)^T + \lambda(\lambda^2 - 1, 0, \dots)^T - \lambda\delta_0 = (2\lambda^3 - 3\lambda, \lambda^2, \lambda, 0, \dots)^T$;
$P_{\cdot;(3,2)}(\lambda) = \lambda P_{\cdot;(2,1)}(\lambda) + \lambda P_{\cdot;(2,2)}(\lambda) - P_{\cdot;(1,1)}(\lambda) = (2\lambda^3 - 3\lambda, \lambda^2, 0, \dots)^T$;
$P_{\cdot;(3,3)}(\lambda) = \lambda P_{\cdot;(2,2)}(\lambda) + P_{\cdot;(1,0)}(\lambda) - P_{\cdot;(1,1)}(\lambda) = (\lambda^3 - 2\lambda, 1, 0, \dots)^T$.

Lemma 1. *The defined above polynomials of the first order, $P_{\alpha;(j,k)}(\lambda)$, for the matrix J (8) have the following form:*
1) *if $k = 0$, then $\forall j \in \mathbb{N}_0$ $P_{\cdot;(j,k)}(\lambda) = \delta_j$;*
2) *if $k = 1, 2, \dots, j$, then $\forall j, \alpha \in \mathbb{N}_0$ the polynomial $P_{\alpha;(j,k)}(\lambda)$ is as follows:*
2.1) *the degree of $P_{\alpha;(j,k)}(\lambda)$ is less or equal than $j - \alpha$ or $P_{\alpha;(j,k)}(\lambda) = 0$, where $\alpha = 0, \dots, j - 1$;*
2.2) *$P_{j;(j,k)}(\lambda)$ is some complex constant;*
2.3) *$P_{\alpha;(j,k)}(\lambda) = 0$, where $\alpha = j + 1, j + 2, \dots$.*
Consider, for some fixed $j \in \mathbb{N}_0$, the system of vectors,

$$(1, 0, \dots)^T; (0, 1, 0, \dots)^T, (\lambda, 0, \dots)^T; (0, 0, 1, 0, \dots)^T, (0, \lambda, 0, \dots)^T, (\lambda^2, 0, \dots)^T;$$
$$(\underbrace{0, \dots, 0}_{j}, 1, 0, \dots)^T, \dots, (\underbrace{0, \dots, 0}_{j-k}, \lambda^k, 0, \dots)^T, \dots (\lambda^j, 0, \dots)^T; \quad k = 0, \dots, j, \quad \lambda \in \mathbb{R},$$

or, what is the same but in another words and a more convenient way, the set of vectors of polynomials

$$
\begin{array}{ccccc}
 & & & & \lambda^j\delta_0 \\
 & & & & \ldots \\
 & & \lambda^2\delta_0 & & \lambda^k\delta_{j-k} \\
 & \lambda\delta_0 & \lambda\delta_1 & & \ldots \\
\delta_0 & \delta_1 & \delta_2 & \ldots & \delta_j
\end{array} \tag{29}
$$

Then, for any fixed $j \in \mathbb{N}_0$ and for all $k = 1, \ldots, j$, the vector of polynomials $P_{\cdot;(j,k)}(\lambda)$ is a linear combination of elements of system (29)*.*

Proof. This Lemma follows from identities (12) and (28). □

Remark 1. It is necessary to admit that in some cases, i.e. for some matrices J, the vector of polynomials $P_{\cdot;(j,k)}(\lambda), j = 0, 1, \ldots, k = 0, \ldots, j$, is not a linear combination of all elements of set (29) (see, Example 1). But in the general case we need all "prime" vectors of polynomials from (29) for the construction of $P_{\cdot;(j,k)}(\lambda)$ (see, Example 2).

Remark 2. This Section contains results which was published in [5]. In that article we consider the direct spectral problem for some selfadjoint operator $\mathbf{J}$ generated by a generalized Jacobi matrix, i.e., some eigenvector expansion of $\mathbf{J}$ was constructed there.

Let us remark (and it will be used later) that all of the results described above hold true if $\mathbf{J}$ is not selfadjoint. Now we will give some respective explanations.

First of all, we assume that it follows from the construction that $\mathbf{J}$ is a Hermitian operator (because $\mathbf{J}$ is a Hermitian matrix). If $\mathbf{J}$ is not selfadjoint, it can be extended to a selfadjoint operator with or without leaving $\mathbf{l}_2$. If the operator $\mathbf{J}$ is selfadjoint, it is well known that there is exist a resolution of identity which corresponds to it and such a resolution will be called ordinary. If $\mathbf{J}$ is Hermitian, there is exists some ordinary resolution of identity for its selfadjoint extension and we will call such a resolution a generalized resolution of identity for the operator $\mathbf{J}$.

Since the embedding $\mathcal{H}_0 \supset \mathcal{H}_+$ is quasinuclear, from [2], Ch. V (see, [2], Ch. V, Theorems 2.1 and 2.3) it follows that for any $\triangle \in \mathcal{B}(\mathbb{R})$ for the ordinary (the generalized) resolution of identity $E(\triangle)$, which corresponds to a selfadjoint (Hermitian) operator $\mathbf{J}$, (15) takes place. So, if $\mathbf{J}$ is Hermitian, the results of paper [5], i.e., the construction of the spectral measure, the generalized eigenvector expansion and so on, are valid.

4. The inverse spectral problem for the generalized Jacobi Hermitian matrix with an "almost" semidiagonal matrix on the side diagonals

In this section we will consider the inverse spectral problem for some kind of the generalized Jacobi Hermitian matrix. In the next section we will give some explanations about the inverse spectral problem for the general situation, i.e., for the matrix (8).

It is necessary to remark that in this section, we will use the same notations as in Section 3. They will denote the same objects as above but for a more concrete matrix.

Let us consider the space $\mathbf{l}_2$ of form (6). Consider a matrix J on the space $\mathbf{l}_2$ of the form (8) with elements of following type:

$$
c_j = \begin{pmatrix} c_{j;0,0} & c_{j;0,1} & 0 & \ldots & 0 \\ c_{j;1,0} & c_{j;1,1} & c_{j;1,2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{j;j,0} & c_{j;j,1} & c_{j;j,2} & \ldots & c_{j;j,j+1} \end{pmatrix}, \quad c_{j;i.i+1} > 0, \quad i = 0, \ldots, j; \tag{30}
$$
$$
b_j = (b_j)^T, \quad a_j = (c_j)^T, \quad j = 0, 1, \ldots,
$$

where $a_j, b_j, c_j, j = 0, 1, \dots,$ are *real-valued* matrices. The matrix J generates an operator $\mathbf{J}$ on $\mathbf{l}_2$ as the closure in $\mathbf{l}_2$ of the operator $\mathbf{l}_{\text{fin}} \ni f \mapsto Jf \in \mathbf{l}_{\text{fin}}$.

The direct spectral problem for such a matrix was solved in [5] and we will partially describe its solution in Section 3 (in fact, it was solved for a more general matrix). First of all, it is necessary to assume that, since the matrix J is real-valued, that the respective vectors of polynomials are real. This means that $P_{\alpha;(j,k)}(\lambda) \in \mathbb{R}$ for any permitted α, j, k and a fixed $\lambda \in \mathbb{R}$. Since, in comparison with article [5], we have a more simple matrix, we can calculate $P_{\cdot;(j,k)}(\lambda), j = 0, 1, \dots, k = 0, \dots, j,$ more accurately. Let us do this.

The matrix $\widetilde{c_j}$ is lower subdiagonal matrix with positive elements on the main diagonal. Then $\widetilde{c_j}^{-1}$ is as follows:

(31)
$$\widetilde{c_j}^{-1} = \begin{pmatrix} \frac{1}{c_{j;0,1}} & 0 & 0 & \dots & 0 \\ * & \frac{1}{c_{j;1,2}} & 0 & \dots & 0 \\ * & * & \frac{1}{c_{j;2,3}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & \frac{1}{c_{j;j,j+1}} \end{pmatrix}; \quad \text{here } * \text{ denotes some real constants.}$$

Using (28) and (31) we obtain the following:
1) $P_{\cdot;(0,0)}(\lambda) = \delta_0$.
2) $P_{\cdot;(1,0)}(\lambda) = \delta_1$;
$P_{\cdot;(1,1)}(\lambda) = B_{0;0,0}(\lambda) P_{\cdot;(0,0)}(\lambda) + C_{0;0}\delta_1 = (\frac{1}{c_{0;0,1}}\lambda - \frac{b_0}{c_{0;0,1}}, \frac{c_{0;0,0}}{c_{0;0,1}}, 0, \dots)^T$.
3) $P_{\cdot;(2,0)}(\lambda) = \delta_2$;

$$\begin{aligned} P_{\cdot;(2,1)}(\lambda) &= \sum_{k=0}^{1} B_{1;0,k}(\lambda) P_{\cdot;(1,k)}(\lambda) + A_{1;0,0} P_{\cdot;(0,0)}(\lambda) + C_{1;0}\delta_2 \\ &= \lambda \left(\widetilde{c_1}^{-1}\right)_{0,0} P_{\cdot;(1,0)}(\lambda) - \sum_{k=0}^{1} \left(\widetilde{c_1}^{-1} b_1\right)_{0,k} P_{\cdot;(1,k)}(\lambda) + A_{1;0,0} P_{\cdot;(0,0)}(\lambda) + C_{1;0}\delta_2 \\ &= (*\lambda + *; \frac{1}{c_{1;0,1}}\lambda + *; *; 0; \dots)^T; \end{aligned}$$

(here, and in what follows, $*$ mean some real constants in the respective place);

$$\begin{aligned} P_{\cdot;(2,2)}(\lambda) &= \sum_{k=0}^{1} B_{1;1,k}(\lambda) P_{\cdot;(1,k)}(\lambda) + A_{1;1,0} P_{\cdot;(0,0)}(\lambda) + C_{1;1}\delta_2 \\ &= \lambda \left(\widetilde{c_1}^{-1}\right)_{1,0} P_{\cdot;(1,0)}(\lambda) + \lambda \left(\widetilde{c_1}^{-1}\right)_{1,1} P_{\cdot;(1,1)}(\lambda) \\ &\quad - \sum_{k=0}^{1} \left(\widetilde{c_1}^{-1} b_1\right)_{1,k} P_{\cdot;(1,k)}(\lambda) + A_{1;1,0} P_{\cdot;(0,0)}(\lambda) + C_{1;1}\delta_2 \\ &= (\frac{1}{c_{1;1,2}} \frac{1}{c_{0;0,1}}\lambda^2 + *\lambda + *, *\lambda + *, *, 0, \dots)^T. \end{aligned}$$

4) $B_2(\lambda)=\widetilde{c_2}^{\,-1}(\lambda I_2-b_2)=\begin{pmatrix}\frac{\lambda}{c_{2;0,1}} & 0 & 0\\ *\lambda & \frac{\lambda}{c_{2;1,2}} & 0\\ *\lambda & *\lambda & \frac{\lambda}{c_{2;2,3}}\end{pmatrix}-\widetilde{c_2}^{\,-1}b_2;$

$P_{\cdot;(3,0)}(\lambda)=\delta_3;$

$$P_{\cdot;(3,1)}(\lambda)=\sum_{k=0}^{2}B_{2;0,k}(\lambda)P_{\cdot;(2,k)}(\lambda)+\sum_{l=0}^{1}A_{2;0,l}P_{\cdot;(1,l)}(\lambda)+C_{2;0}\delta_3$$
$$=(*\lambda^2+*\lambda+*,*\lambda+*,\frac{1}{c_{2;0,1}}\lambda+*,*,0,\ldots)^T;$$

$$P_{\cdot;(3,2)}(\lambda)=\sum_{k=0}^{2}B_{2;1,k}(\lambda)P_{\cdot;(2,k)}(\lambda)+\sum_{l=0}^{1}A_{2;1,l}P_{\cdot;(1,l)}(\lambda)+C_{2;1}\delta_3$$
$$=(*\lambda^2+*\lambda+*,\frac{1}{c_{2;1,2}}\frac{1}{c_{1;0,1}}\lambda^2+*\lambda+*,*\lambda+*,*,0,\ldots)^T;$$

$$P_{\cdot;(3,3)}(\lambda)=\sum_{k=0}^{2}B_{2;2,k}(\lambda)P_{\cdot;(2,k)}(\lambda)+\sum_{l=0}^{1}A_{2;2,l}P_{\cdot;(1,l)}(\lambda)+C_{2;2}\delta_3$$
$$=(\frac{1}{c_{2;2,3}}\frac{1}{c_{1;1,2}}\frac{1}{c_{0;0,1}}\lambda^3+*\lambda^2+*\lambda+*,*\lambda^2+*\lambda+*,*\lambda+*,*,0,\ldots)^T.$$

5) Let us suppose, by induction, that for any $j=n-1,n,\ n\in\mathbb{N}$, following equalities take place:

$$\begin{aligned}&P_{\cdot;(j,0)}(\lambda)=\delta_j;\\ &P_{\cdot:(j,k)}(\lambda)=\sum_{r=0}^{k-1}\sum_{s=0}^{r}*\lambda^s\delta_{j-r}+\sum_{s=0}^{k-1}*\lambda^s\delta_{j-k}+\prod_{i=0}^{k-1}\frac{1}{c_{j-k+i;i,i+1}}\lambda^k\delta_{j-k}\\ &\qquad+\sum_{r=k+1}^{j}\sum_{s=0}^{r-1}*\lambda^s\delta_{j-r},\quad k=1,2,\ldots,j.\end{aligned}\tag{32}$$

For a better understanding we give the following picture:

(33)

	1	λ	λ^2	$\ldots$	λ^{k-1}	λ^k	λ^{k+1}	$\ldots$	λ^{j-2}	λ^{j-1}	λ^j
δ_0	$*$	$*$	$*$	$\ldots$	$*$	$*$	$*$	$\ldots$	$*$	$*$	0
δ_1	$*$	$*$	$*$	$\ldots$	$*$	$*$	$*$	$\ldots$	$*$	0	0
δ_2	$*$	$*$	$*$	$\ldots$	$*$	$*$	$*$	$\ldots$	0	0	0
$\vdots$											
$\delta_{j-(k+1)}$	$*$	$*$	$*$	$\ldots$	$*$	$*$	0	$\ldots$	0	0	0
δ_{j-k}	$*$	$*$	$*$	$\ldots$	$*$	$+$	0	$\ldots$	0	0	0
$\delta_{j-(k-1)}$	$*$	$*$	$*$	$\ldots$	$*$	0	0	$\ldots$	0	0	0
$\vdots$											
δ_{j-2}	$*$	$*$	$*$	$\ldots$	0	0	0	$\ldots$	0	0	0
δ_{j-1}	$*$	$*$	0	$\ldots$	0	0	0	$\ldots$	0	0	0
δ_j	$*$	0	0	$\ldots$	0	0	0	$\ldots$	0	0	0

Table in (33) should be understood in the following way: if at the intersection of the column indicated with λ^s and a row marked with δ_r there is " $*$ ", then the vector of the polynomials $P_{\cdot;(j,k)}(\lambda)$ contains the respective element $\lambda^s\delta_r$ with some real coefficient.

The sign " + " means that the element $\lambda^k\delta_{j-k}$ has a positive coefficient $\prod_{i=0}^{k-1}\frac{1}{c_{j-k+i;i,i+1}}$ and "0" stands for an element that does not enter $P_{\cdot;(j,k)}(\lambda)$.

6) Now we will show that (32) holds true for $j=n+1$.

From the definition of vectors of the polynomials, it follows that $P_{\cdot;(n+1,0)}(\lambda)=\delta_{n+1}$. Using (27), (28), (31), (32) and (33) we obtain

$$
\begin{aligned}
&P_{\cdot;(n+1,i)}(\lambda)=\sum_{k=0}^{n}B_{n;i-1,k}(\lambda)P_{\cdot;(n,k)}(\lambda)+\sum_{l=0}^{n-1}A_{n-1;i-1,l}P_{\cdot;(n-1,l)}(\lambda)+C_{n;i-1}\delta_{n+1}\\
&=\sum_{k=0}^{i-2}\lambda(\widetilde{c}_n^{-1})_{i-1,k}P_{\cdot;(n;k)}(\lambda)+\lambda\frac{1}{c_{n;i-1,i}}P_{\cdot;(n,i-1)}(\lambda)-\sum_{k=0}^{n}(\widetilde{c}_n^{-1}b_n)_{i-1,k}P_{\cdot;(n,k)}(\lambda)\\
&+\sum_{l=0}^{n-1}A_{n-1;i-1,l}P_{\cdot;(n-1,l)}(\lambda)+C_{n;i-1}\delta_{n+1}=\lambda\left[\sum_{r=0}^{i-2}\sum_{s=0}^{r}*\lambda^s\delta_{n-r}+\sum_{r=i-1}^{n}\sum_{s=0}^{r-1}*\lambda^s\delta_{n-r}\right]\\
&+\lambda\left[\frac{1}{c_{n;i-1,i}}\prod_{l=0}^{i-2}\frac{1}{c_{n-i+1+l;l,l+1}}\lambda^{i-1}\delta_{n-i+1}+\sum_{r=0}^{i-2}\sum_{s=0}^{r}*\lambda^s\delta_{n-r}+\sum_{r=i-1}^{n}\sum_{s=0}^{r-1}*\lambda^s\delta_{n-r}\right]\\
&+\left[\sum_{r=0}^{n}\sum_{s=0}^{r}*\lambda^s\delta_{n-r}\right]+\left[\sum_{r=0}^{n-1}\sum_{s=0}^{r}*\lambda^s\delta_{n-1-r}\right]+[*\delta_{n+1}]=\sum_{r=0}^{i-2}\sum_{s=1}^{r+1}*\lambda^s\delta_{n+1-(r+1)}\\
&+\sum_{r=i-1}^{n}\sum_{s=1}^{r}*\lambda^s\delta_{n+1-(r+1)}+\prod_{l=0}^{i-1}\frac{1}{c_{n+1-i+l;l,l+1}}\lambda^i\delta_{n+1-i}+\sum_{r=0}^{n}\sum_{s=0}^{r}*\lambda^s\delta_{n+1-(r+1)}\\
&+*\delta_{n+1}=\sum_{r=1}^{i-1}\sum_{s=1}^{r}*\lambda^s\delta_{n+1-r}+\sum_{r=i}^{n+1}\sum_{s=1}^{r-1}*\lambda^s\delta_{n+1-r}+\prod_{l=0}^{i-1}\frac{1}{c_{n+1-i+l;l,l+1}}\lambda^i\delta_{n+1-i}\\
&+\sum_{r=1}^{n+1}\sum_{s=0}^{r-1}*\lambda^s\delta_{n+1-r}+*\delta_{n+1}=\sum_{r=0}^{i-1}\sum_{s=0}^{r}*\lambda^s\delta_{n+1-r}+\prod_{l=0}^{i-1}\frac{1}{c_{n+1-i+l;l,l+1}}\lambda^i\delta_{n+1-i}\\
&+\sum_{r=i}^{n+1}\sum_{s=0}^{r-1}*\lambda^s\delta_{n+1-r},\quad i=1,\dots,n+1.
\end{aligned}
$$

By induction from these calculations we obtain the following Lemma.

Lemma 2. *For any fixed $j\in\mathbb{N}_0$ and $k=0,\dots,j$, the vector of polynomials $P_{\cdot;(j,k)}(\lambda)$ for the matrix J of type* (8) *with elements* (30) *is a linear combination of elements of the system*

$$
\begin{array}{cccccc}
 & & & & \lambda^{j-1}\delta_0 & \\
 & & & & & \lambda^k\delta_{j-k} \\
 & & \lambda^2\delta_0 & & \dots & \dots \\
 & \lambda\delta_0 & \lambda\delta_1 & & \lambda\delta_{j-2} & \lambda\delta_{j-1} \\
\delta_0 & \delta_1 & \delta_2 & \dots & \delta_{j-1} & \delta_j
\end{array}
\tag{34}
$$

This linear combination consist of $\lambda^k\delta_{j-k}$ with positive coefficient $\prod_{i=0}^{k-1}\frac{1}{c_{j-k+i;i,i+1}}$ and all other vectors of (34) *with some real coefficients.*

It is necessary to note that in this case, i.e., when the matrix J has elements (30), the vectors of polynomials is a linear combination of not all the vectors of system (29) but just those ones that belong to (34).

Consider the orthogonality relations (24) which corresponds to the matrix J with elements (30).

Since $\|P_{\cdot;(j,k)}(\lambda)\|_{L^2} = 1, j \in \mathbb{N}_0, k = 0, \dots, j$, and $P_{\cdot;(j,0)}(\lambda) = \delta_j, j \in \mathbb{N}_0$, we obtain

$$\int_{\mathbb{R}} d\sigma_{\alpha\alpha}(\lambda) = 1, \quad \alpha = 0, 1, \dots \tag{35}$$

Also, from (24), it follows that

$$(P_{\cdot;(N,0)}(\lambda), P_{\cdot;(j,k)}(\lambda))_{L^2} = 0, \quad N \in \mathbb{N}_0, \quad j = 0, \dots, N-1, \quad k = 0, \dots, j. \tag{36}$$

Let us fix some $N \in \mathbb{N}_0$. Consider (36) step by step w.r.t. (j,k).
1) $0 = (P_{\cdot;(N,0)}(\lambda), P_{\cdot;(0,0)}(\lambda))_{L^2} = (\delta_N, \delta_0)_{L^2} = \int_{\mathbb{R}} d\sigma_{0N}(\lambda)$.
2.1) $0 = (P_{\cdot;(N,0)}(\lambda), P_{\cdot;(1,0)}(\lambda))_{L^2} = (\delta_N, \delta_1)_{L^2} = \int_{\mathbb{R}} d\sigma_{1N}(\lambda)$,
2.2) $0 = (P_{\cdot;(N,0)}(\lambda), P_{\cdot;(1,1)}(\lambda))_{L^2} = (\delta_N, P_{\cdot;(1,1)}(\lambda))_{L^2}$. From Lemma 2 it follows that $P_{\cdot;(1,1)}(\lambda)$ is a linear combination of prime vectors δ_0, δ_1 and $\lambda\delta_0$. Since $\delta_N \perp \delta_0, \delta_N \perp \delta_1$ in L^2, $0 = (P_{\cdot;(N,0)}(\lambda), P_{\cdot;(1,1)}(\lambda))_{L^2} = (\delta_N, \lambda\delta_0)_{L^2} = \int_{\mathbb{R}} \lambda d\sigma_{0N}(\lambda)$.

So, using Lemma 2 and (36) in the same way we obtain for L^2 step by step that

$$\delta_N \perp \lambda^k \delta_{j-k}, \quad j = 0, \dots, N-1, \quad k = 0, \dots, j.$$

Indeed, let us show this by induction. We suppose by induction that it is true for $0 \le j \le j_0 < N-1$ and show that the vectors $\delta_{j_0+1}, \lambda\delta_{j_0}, \dots, \lambda^{j_0+1}\delta_0$ are orthogonal to δ_N. From (36) it follows that $0 = (P_{\cdot;(N,0)}(\lambda), P_{\cdot;(j_0+1,0)}(\lambda))_{L^2} = (\delta_N, \delta_{j_0+1})_{L^2}$. So, $\delta_N \perp \delta_{j_0+1}$. Let us suppose by induction one more time that, for $0 \le m < j_0 + 1$, the vectors $\delta_{j_0+1}, \lambda\delta_{j_0}, \dots, \lambda^m \delta_{j_0+1-m}$ are orthogonal to δ_N and prove that $\lambda^{m+1}\delta_{j_0-m}$ is orthogonal to δ_N. From Lemma 2 it follows that $P_{\cdot;(j_0+1,m+1)}(\lambda)$ is a linear combination of prime vectors $\delta_0; \delta_1, \lambda\delta_0; \dots; \delta_{j_0}, \dots, \lambda^{j_0}\delta_0; \delta_{j_0+1}, \dots, \lambda^m \delta_{j_0+1-m}, \lambda^{m+1}\delta_{j_0-m}$. According to the assumption, $P_{\cdot;(N,0)}(\lambda)$ is orthogonal to all elements of this linear combination except the last one, i.e., $\lambda^{m+1}\delta_{j_0-m}$. So, using (36) we obtain

$$0 = (P_{\cdot;(N,0)}(\lambda), P_{\cdot;(j_0+1,m+1)}(\lambda))_{L^2} = (P_{\cdot;(N,0)}(\lambda), \lambda^{m+1}\delta_{j_0-m})_{L^2} = (\delta_N, \lambda^{m+1}\delta_{j_0-m})_{L^2}.$$

Thus, $\lambda^{m+1}\delta_{j_0-m} \perp \delta_N$ in L^2 and this ends the proof.

In other words, we obtain following:

$$\int_{\mathbb{R}} \lambda^i d\sigma_{MN}(\lambda) = 0, \quad i + M < N, \quad i, M, N \in \mathbb{N}_0. \tag{37}$$

Also, the following condition takes place:

$$\int_{\mathbb{R}} \lambda^i d\sigma_{MN}(\lambda) \in \mathbb{R}, \quad i, M, N \in \mathbb{N}_0. \tag{38}$$

Let us prove this fact. First of all, $\int_{\mathbb{R}} \lambda^i d\sigma_{MN}(\lambda) = (\lambda^i \delta_N, \delta_M)_{L^2},\ i, M, N \in \mathbb{N}_0$. It is necessary to admit that $\delta_M = P_{\cdot;(M,0)}(\lambda),\ M \in \mathbb{N}_0$. Since the vectors of polynomials are real and Lemma 2 is valid, making a simple calculations it is easy to obtain that $\lambda^i \delta_N$ is a linear combination of the form

$$\lambda^i \delta_N = \sum_{j=0}^{i+N-1} \sum_{k=0}^{j} r_{j,k} P_{\cdot;(j,k)}(\lambda) + \sum_{k=0}^{i} r_{i+N,k} P_{\cdot;(j,k)}(\lambda), \quad i, N \in \mathbb{N}_0,$$

where $r_{j,k}$ is some real coefficient. Also this formula holds true because the vectors of the polynomials $P_{\cdot;(j,k)}(\lambda)$ form a basis in the real span of elements (34) (this will be shown later). Note that $r_{i+N,i} = \prod_{k=0}^{i-1} c_{N+k;k,k+1},\ i, N \in \mathbb{N}_0$. Since the vectors of the polynomials are orthonormal,

$$\int_{\mathbb{R}} \lambda^i d\sigma_{MN}(\lambda) = (\lambda^i \delta_N, \delta_M)_{L^2}$$
$$= \Big(\sum_{j=0}^{i+N-1} \sum_{k=0}^{j} r_{j,k} P_{\cdot;(j,k)}(\lambda) + \sum_{k=0}^{i} r_{i+N,k} P_{\cdot;(j,k)}(\lambda), P_{\cdot;(M,0)}(\lambda) \Big)_{L^2} = r_{M,0} \in \mathbb{R}.$$

So, the spectral measure $\Sigma(\cdot)$ of the matrix J with elements (30) satisfies conditions (35), (37), (38).

Now we proceed to consider the inverse spectral problem for the matrix J.

That is, we have an arbitrary operator-valued measure $\Sigma(\cdot):\mathcal{B}(\mathbb{R})\to\mathcal{L}(\ell_2)$, i.e., $\Sigma(\triangle)$ is a linear bounded operator on ℓ_2. This measure is such that:

i) $\Sigma(\cdot)$ is a nonnegative, i.e., $\forall\ \triangle\in\mathcal{B}(\mathbb{R})\quad\Sigma(\triangle)$ is a nonnegative operator on ℓ_2;

ii) equations (35), (37), (38) are valid, where $\sigma_{\alpha\beta}(\triangle):=(\Sigma(\triangle)\delta_\beta,\delta_\alpha)_{\ell_2},\ \ \alpha,\beta\in\mathbb{N}_0$, $\triangle\in\mathcal{B}(\mathbb{R})$;

iii) the following inequality takes place:

$$\int_{\mathbb{R}}|\lambda|^m d\sigma_{\alpha\alpha}(\lambda)<\infty,\quad m=0,1,\ldots,\quad \alpha=0,1,\ldots. \tag{39}$$

Since $\Sigma(\cdot)$ is not a usual measure, we will give some necessary definitions.

Let Θ is set of elements of full measure from $\mathcal{B}(\mathbb{R})$, i.e., $\Theta:=\{\theta\in\mathcal{B}(\mathbb{R})|\Sigma(\theta)=\Sigma(\mathbb{R})\}$. The support of the measure $\Sigma(\cdot)$ can be defined in a usual way as the intersection of closed sets of full measure, i.e., $\operatorname{supp}\Sigma=\bigcap_\tau\theta_\tau$, where $\theta_\tau\in\Theta$ and $\theta_\tau=\overline{\theta_\tau}$.

Remark 3. Since $\Sigma(\cdot)$ has to be a spectral measure of the operator $\mathbf{J}$, which is constructed from a real-valued matrix J with elements (30), the above-mentioned assumptions are natural and, what is more, necessary for $\Sigma(\cdot)$, and this will be shown later.

Also, it is necessary to assume that here and later on, the words a "spectral matrix (measure)" mean that the matrix (the measure) $\Sigma(\cdot)$ is generated by some J in accordance with the direct spectral problem, i.e., there exists a generalized Hermitian Jacobi matrix J that generates a Hermitian or a selfadjoint operator $\mathbf{J}$ and such that we can construct $\Sigma(\cdot)$ from J by the procedure described in Section 3.

From nonnegativeness of the measure $\Sigma(\cdot):\mathcal{B}(\mathbb{R})\to\mathcal{L}(\ell_2)$, it follows that $\Sigma(\triangle)$ is a selfadjoint operator for any fixed $\triangle\in\mathcal{B}(\mathbb{R})$ (see [4], Ch.8, Theorem 5.3), i.e., $\forall f,g\in\ell_2$

$$(\Sigma(\triangle)f,g)_{\ell_2}=(f,\Sigma(\triangle)g)_{\ell_2}=\overline{(\Sigma(\triangle)g,f)}_{\ell_2}.$$

Since $\Sigma(\cdot)$ is a selfadjoint measure, we obtain that

$$\sigma_{\alpha\beta}(\cdot)=\overline{\sigma_{\beta\alpha}(\cdot)},\quad \alpha,\beta=0,1,\ldots. \tag{40}$$

The measure $\Sigma(\cdot)$ we will be also understood as a matrix $(\sigma_{\alpha\beta}(\cdot))_{\alpha,\beta=0}^{\infty}$. Similarly to Section 3, using measure $\Sigma(\cdot)$, we construct the space $L^2:=L^2(\mathbb{R},d\Sigma(\cdot))$ of infinite vectors the elements of which are complex-valued functions of real variable, with the scalar product $(\cdot,\cdot)_{L^2}$.

Lemma 3. *For any finite vector $\xi(\lambda)=(\xi_0(\lambda),\xi_1(\lambda),\ldots)^T$, where $\xi_\alpha(\lambda),\alpha=0,1,\ldots$, are some polynomials, we have $\int_{\mathbb{R}}(d\Sigma(\lambda)\xi(\lambda),\xi(\lambda))_{\ell_2}<\infty$, i.e. $\xi(\lambda)\in L^2$.*

Proof. This statement follows from condition (39). Indeed, let us denote $\xi(\lambda,\alpha):=(0,\ldots,0,\xi_\alpha(\lambda),0,\ldots)^T=\xi_\alpha(\lambda)\delta_\alpha$, where $\xi_\alpha(\lambda)$ is some polynomial. Then

$$\begin{aligned}(\xi(\lambda,\alpha),\xi(\lambda,\alpha))_{L^2}&=\int_{\mathbb{R}}(d\Sigma(\lambda)\xi(\lambda,\alpha),\xi(\lambda,\alpha))_{\ell_2}=\int_{\mathbb{R}}|\xi_\alpha(\lambda)|^2d\sigma_{\alpha\alpha}(\lambda)\\&\le\int_{\mathbb{R}}\sum_{i=0}^{N_0}|a_i|\,|\lambda|^j d\sigma_{\alpha\alpha}(\lambda)<\infty\end{aligned}$$

(here, $\mathbb{N}\ni N_0$ is the degree of the polynomial $\xi_\alpha(\lambda),a_i\in\mathbb{R}$). So, $\xi(\lambda,\alpha)\in L^2$.

From nonnegativeness of $\Sigma(\cdot)$ and (40) it follows that

$$\begin{aligned}0&\le(\Sigma(\triangle)(\delta_\alpha-\delta_\beta),(\delta_\alpha-\delta_\beta))_{\ell_2}=\sigma_{\alpha\alpha}(\triangle)-\sigma_{\beta\alpha}(\triangle)-\sigma_{\alpha\beta}(\triangle)+\sigma_{\beta\beta}(\triangle)\\&=\sigma_{\alpha\alpha}(\triangle)+\sigma_{\beta\beta}(\triangle)-(\sigma_{\alpha\beta}(\triangle)+\overline{\sigma_{\alpha\beta}(\triangle)}),\quad \alpha,\beta\in\mathbb{N}_0,\quad \triangle\in\mathcal{B}(\mathbb{R}).\end{aligned}$$

From this inequality we obtain that $\operatorname{Re}\sigma_{\alpha\beta}(\triangle) \leq \frac{1}{2}(\sigma_{\alpha\alpha}(\triangle)+\sigma_{\beta\beta}(\triangle)),\ \alpha,\beta\in\mathbb{N}_0, \triangle\in \mathcal{B}(\mathbb{R})$. On the other hand, for all $\alpha,\beta\in\mathbb{N}_0$ and $\triangle\in\mathcal{B}(\mathbb{R})$,

$$0 \leq (\Sigma(\triangle)(i\delta_\alpha-\delta_\beta),(i\delta_\alpha-\delta_\beta))_{\ell_2} = \sigma_{\alpha\alpha}(\triangle)+\sigma_{\beta\beta}(\triangle)+(i\sigma_{\alpha\beta}(\triangle)+\overline{i\sigma_{\alpha\beta}(\triangle)}).$$

Therefore, $\operatorname{Im}\sigma_{\alpha\beta}(\triangle) \leq \frac{1}{2}(\sigma_{\alpha\alpha}(\triangle)+\sigma_{\beta\beta}(\triangle)),\ \alpha,\beta\in\mathbb{N}_0, \triangle\in\mathcal{B}(\mathbb{R})$.

Thus, $\xi(\lambda,\alpha)\in L^2$ and

$$|\sigma_{\alpha\beta}(\triangle)| \leq \frac{1}{\sqrt{2}}(\sigma_{\alpha\alpha}(\triangle)+\sigma_{\beta\beta}(\triangle)), \quad \alpha,\beta\in\mathbb{N}_0, \quad \triangle\in\mathcal{B}(\mathbb{R}). \tag{41}$$

From these facts it follows that $\xi(\lambda)\in L^2$, because $\xi(\lambda)$ is a finite linear combination of the corresponding $\xi(\lambda,\alpha),\alpha=0,1,\dots$ □

Let us consider a system of vectors (34) in the space L^2 and we will suppose that system (34) $\forall\,(j,k): j=0,1,\dots,k=0,\dots,j$, consists of linearly independent elements.

Here we assume that system (34) consists of linearly independent vectors. The following statement gives a necessary and sufficient condition for this assumption.

Theorem 1. *Let $\Sigma(\cdot)=(\sigma_{\alpha\beta}(\cdot))_{\alpha,\beta=0}^{\infty}$ be a nonnegative operator-valued measure. Vectors of system* (34) *are linearly independent in the space L^2 for all permitted j,k, i.e. $j,k: j=0,1,\dots,k=0,\dots,j$, if and only if the support of the measure $\Sigma(\cdot)$ has an infinite set of points.*

Proof. **Necessity.** Let us suppose the opposite, i.e., let $\Sigma(\cdot)$ have a finite support. Then for some fixed $\alpha_0\in\mathbb{N}_0$, $\sigma_{\alpha_0\alpha_0}(\cdot)$ also has a support with a finite set of points. Let us construct a polynomial $T(\lambda):=T_n\lambda^n+\ldots+T_0, T_i\in\mathbb{R}, i\in\mathbb{N}_0,\lambda\in\mathbb{R}$, such that $\exists\lambda_0\in\mathbb{R}: T(\lambda_0)\neq 0$ and

$$I := \int_{\mathbb{R}} |T(\lambda)|^2 d\sigma_{\alpha_0\alpha_0}(\lambda) = 0.$$

Consider $\xi(\lambda):=T(\lambda)\delta_{\alpha_0}$. Then

$$(\xi(\lambda),\xi(\lambda))_{L^2} = I = 0,$$

i.e., the elements of system (34) are linearly dependent. We obtain a contradiction.

Sufficiency. Let us consider some monotone increasing sequence $(q_\alpha)_{\alpha=0}^{\infty}$ such that $\forall\alpha\in\mathbb{N}_0$, $q_\alpha>0$ and $\sum_{\alpha=0}^{\infty}\frac{1}{q_\alpha}<\infty$. Let us construct the measure $\mu(\cdot)=\sum_{\alpha=0}^{\infty}\frac{\sigma_{\alpha\alpha}(\cdot)}{q_\alpha}$ on $\mathbb{R}$. This measure has also an infinite support.

From (35) it follows that the measure $\mu(\cdot)$ is finite. Since (41) holds, $\forall\alpha,\beta\in\mathbb{N}_0$, $|\sigma_{\alpha\beta}(\cdot)|\leq q_{\max\{\alpha,\beta\}}\mu(\cdot)$. Therefore, the measure $\Sigma(\cdot)$ is absolutely continuous w.r.t. $\mu(\cdot)$. Thus, Radon-Nikodym theorem takes place, i.e., there is a nonnegative matrix $M(\lambda)=(M_{\alpha,\beta}(\lambda))_{\alpha,\beta=0}^{\infty}$ such that $d\Sigma(\lambda)=M(\lambda)d\mu(\lambda)$, where $M_{\alpha,\beta}(\lambda)$ is a finite integrable function for all $\alpha,\beta\in\mathbb{N}_0$. This matrix is positive on the set of full measure for $d\mu(\lambda)$.

Assume that some linear combination $\xi(\lambda),\lambda\in\mathbb{R}$, of vectors (34) is equal to zero in the space L^2, i.e.,

$$0=\|\xi(\lambda)\|_{L^2}^2=\int_{\mathbb{R}}(d\Sigma(\lambda)\xi(\lambda),\xi(\lambda))_{\ell_2}=\int_{\mathbb{R}}(M(\lambda)\xi(\lambda),\xi(\lambda))_{\ell_2}d\mu(\lambda).$$

Then $(M(\lambda)\xi(\lambda),\xi(\lambda))_{\ell_2}=0$ for μ-almost all $\lambda\in\mathbb{R}$. But $M(\lambda)$ is positive on the set of full measure for $d\mu(\lambda)$; therefore $\xi(\lambda)=(\xi_0(\lambda),\xi_1(\lambda),\dots)=\bar{0}$ for μ-almost all λ. The functions $\xi_i(\lambda), i\in\mathbb{N}_0$, are some ordinary polynomials. Since $d\mu(\lambda)$ has an infinite support, equality to zero of functions $\xi_i(\lambda), i\in\mathbb{N}_0$, means that all their coefficients are also equal to zero. So, the vectors of system (34) are linearly independent in L^2 for all $j,k: j=0,1,\dots,k=0,\dots,j$. □

Now we will make (34) orthonormal in L^2. We will apply the orthonormalization procedure to (34) in following order:

$$\begin{array}{ccccccc} & & & & \lambda^2\delta_0 & & \\ & & & & \uparrow & \searrow & \\ & & \lambda\delta_0 & & \lambda\delta_1 & & \\ & & \uparrow & \searrow & \uparrow & & \\ \delta_0 & \rightarrow & \delta_1 & & \delta_2 & & \dots \end{array}$$

and in a classical way.

According to this procedure we obtain some vectors of polynomials $Q_{j;k}(\lambda)$ which are defined in a unique way and they form the following system:

$$\begin{array}{ccccc} & & & & Q_{j;k}(\lambda) \\ & & Q_{2;2}(\lambda) & & \vdots \\ & Q_{1;1}(\lambda) & Q_{2;1}(\lambda) & & Q_{j;1}(\lambda) \\ Q_{0;0}(\lambda) & Q_{1;0}(\lambda) & Q_{2;0}(\lambda) & \dots & Q_{j;0}(\lambda) \end{array} \tag{42}$$

From the construction and (37) it follows that the following orthogonality relations hold:

$$(Q_{j;k}(\lambda), Q_{m;l}(\lambda))_{L^2} = \delta_{j,m}\delta_{k,l}, \quad j,m = 0,1,\dots, \quad k = 0,\dots,j, \quad l = 0,\dots,m. \tag{43}$$

From the procedure of constructing $Q_{j;k}(\lambda)$, it is easy to see that $\forall j \in \mathbb{N}_0, k = 0,\dots,j$,

$$Q_{j;k}(\lambda) = q_{j;k}\lambda^k\delta_{j-k} + S_{j;k}(\lambda), \tag{44}$$

where $q_{j;k}$ is some positive constant and $S_{j;k}(\lambda)$ denotes the linear combination with real coefficients of vectors (34) without $\lambda^k\delta_{j-k}$.

Let us show this by using induction. From (35) it follows that $Q_{0;0}(\lambda) = \delta_0$. From (35) and (37) we have that $Q_{1;0}(\lambda) = \delta_1$. The construction of $Q_{1;1}(\lambda)$ is the following. Consider the linear combination $Q'_{1;1}(\lambda) = \lambda\delta_0 + k_{1;0}Q_{1;0}(\lambda) + k_{0;0}Q_{0;0}(\lambda)$, where $k_{1;0}, k_{0;0}$ are some complex constants. According to the construction, $Q'_{1;1}(\lambda) \perp Q_{1;0}(\lambda)$ in L^2. So, from (35) and (37) it follows that

$$0 = k_{1;0} + (\lambda\delta_0, \delta_1)_{L^2} = k_{1;0} + \int_{\mathbb{R}} \lambda d\sigma_{10}(\lambda).$$

Therefore, from (38) we obtain that $k_{1;0} = -\int_{\mathbb{R}} \lambda d\sigma_{10}(\lambda) \in \mathbb{R}$. In the same way it is easy to show that $k_{0;0} = -\int_{\mathbb{R}} \lambda d\sigma_{00}(\lambda) \in \mathbb{R}$. Since $Q_{1;1}(\lambda) = \frac{Q'_{1;1}(\lambda)}{\left\|Q'_{1;1}(\lambda)\right\|_{L^2}}$, from the above it follows that $Q_{1;1}(\lambda)$ is a linear combination with real coefficient of the vectors $\delta_0; \delta_1, \lambda\delta_0$. Let us suppose by induction that for some fixed j,k , $Q_{m;l}(\lambda)$ is a linear combination with real coefficient of the respective vectors in the set (34) for all $(m,l) \in \Delta_{j,k}$, where

$$\Delta_{j,k} = \{(m,l) | \text{if } m = 0,\dots,j-1 \text{ then } l = 0,\dots,m, \text{and if } m = j \text{ then } l = 0,\dots,k\}.$$

Now we will show that the next vector of polynomials, in sense of the order of orthonormalization, is a respective linear combination of elements of system (34) with real coefficients. If $k = j$ then from (37) it follows that $Q_{j+1;0}(\lambda) = \delta_j$. If $k = 0,\dots,j-1$ then we have to prove that $Q_{j;k+1}(\lambda)$ is a linear combination of respective prime vectors with real coefficients. Consider the linear combination of the form $Q'_{j;k+1}(\lambda) = \lambda^{k+1}\delta_{j-(k+1)} + \sum_{(m,l)\in\Delta_{j,k}} k_{m;l}Q_{m;l}(\lambda)$. According to the construction, $Q_{j;k+1}(\lambda) \perp Q_{m;l}(\lambda)$, $(m,l) \in \Delta_{j,k}$. So, from (35) it follows that

$$0 = (Q'_{j;k+1}(\lambda), Q_{m;l}(\lambda))_{L^2} = (\lambda^{k+1}\delta_{j-(k+1)}, Q_{m;l}(\lambda))_{L^2} + k_{m;l}, \quad (m,l) \in \Delta_{j,k}.$$

According to the assumption, $Q_{m;l}(\lambda)$ is a linear combination with real coefficients of respective vectors from system (34). Since condition (38) is satisfied,

$$(\lambda^{k+1}\delta_{j-(k+1)}, Q_{m;l}(\lambda))_{L^2} \in \mathbb{R}.$$

Thus, $k_{m;l} \in \mathbb{R}$ for all $(m,l) \in \Delta_{j,k}$, i.e., $Q'_{j;k+1}(\lambda)$ is a linear combination of respective vectors with real coefficients. Since $Q_{m;l}(\lambda) = \frac{Q'_{m;l}(\lambda)}{\left\|Q'_{m;l}(\lambda)\right\|_{L^2}}$, $Q_{m;l}(\lambda)$ is also a linear combination with real coefficients of elements of system (34).

In particular, from (35) and (37) it also follows that

$$Q_{j;0}(\lambda) = \delta_j, \quad j \in \mathbb{N}_0.$$

So, all the elements $Q_{j;k}(\lambda)$ of (42) are linear combinations of the form (44) of simple vectors from (34). And vise versa, any element of (34) is a linear combination of vectors of polynomials $Q_{j;k}(\lambda)$ of (42). This follows from representation (44).

Theorem 2. *Let $\Sigma(\lambda)$ be a spectral matrix of some generalized Hermitian Jacobi matrix J with elements* (30).

Then (39) *holds true and the vectors* (34) *are linearly independent. Thus, $Q_{j;k}(\cdot) = P_{\cdot;(j,k)}(\cdot), j \in \mathbb{N}_0, k = 0,\dots,j$, and the real linear span of $P_{\cdot;(m,l)}(\lambda)$, $(m,l) \in \Delta_{j,k}$, coincides with the real linear span of the vectors* (34).

Proof. Let us show that for all $u \in \mathbf{l}_{\text{fin}}$, the identity $\widehat{(J^m u)}(\lambda) = \lambda^m \widehat{u}(\lambda), m \in \mathbb{N}_0$, takes place, where $\widehat{\ }$ is the Fourier transform defined by (21). To prove this, it is sufficient to show that $\widehat{(Ju)}(\lambda) = \lambda \widehat{u}(\lambda)$.

Let $u \in \mathbf{l}_{\text{fin}}$ and $n_0 \in \mathbb{N}$ be such that $u = (u_0, \dots, u_{n_0}, 0, \dots)$. Then according to (21), we have $\widehat{(Ju)}_0(\lambda) = \sum_{k=0}^{\infty} \big((Ju)_k, P_{0;(k,\cdot)}(\lambda)\big)_{H_k}$. Using Green's formula (10) we obtain

$$\begin{aligned}\widehat{(Ju)}_0(\lambda) &= \sum_{k=0}^{\infty} \big((Ju)_k, P_{0;(k,\cdot)}(\lambda)\big)_{H_k} = \sum_{k=0}^{n_0+1} \big(u_k, (J(P_{0;(0,\cdot)}(\lambda), P_{0;(1,\cdot)}(\lambda), \dots)^T)_k\big)_{H_k} \\ &= \sum_{k=0}^{n_0+1} \big(u_k, \lambda P_{0;(k,\cdot)}(\lambda)\big)_{H_k} = \lambda \widehat{u}_0(\lambda).\end{aligned}$$

In the same way we can prove that $\widehat{(Ju)}_j(\lambda) = \lambda \widehat{u}_j(\lambda), j = 1, 2, \dots$ So, $\widehat{(Ju)}(\lambda) = \lambda \widehat{u}(\lambda)$.

Using (25) for all even $m \in \mathbb{N}$ we get

$$\begin{aligned}\infty > (J^m e_{\alpha;0}, e_{\alpha;0})_{\mathbf{l}_2} &= (\lambda^m \widehat{e_{\alpha;0}}, \widehat{e_{\alpha;0}})_{L^2} = (\lambda^m P_{\cdot;(\alpha,0)}(\lambda), P_{\cdot;(\alpha,0)}(\lambda))_{L^2} \\ &= (\lambda^m \delta_\alpha, \delta_\alpha)_{L^2} = \int_{\mathbb{R}} \lambda^m d\sigma_{\alpha\alpha}(\lambda), \quad \alpha = 0, 1, \dots\end{aligned}$$

Therefore statement (39) holds true.

Now we show that system (34) consists of linearly independent vectors. From formulas (32) it is easy to see that $P_{\cdot;(j,k)}(\lambda), j = 0, 1, \dots, k = 0, \dots, j$, is a linear combination of vectors of system (34). Therefore the real linear span of $\frac{j(j+1)}{2} + k + 1$ vectors of system (34) contains the following orthonormal system of vectors in L^2:

$$\begin{array}{ccccc} & & & P_{\cdot;(j-1,j-1)}(\lambda) & \\ & & & & P_{\cdot;(j,k)}(\lambda) \\ & & & \vdots & \vdots \\ & P_{\cdot;(1,1)}(\lambda) & & P_{\cdot;(j-1,1)}(\lambda) & P_{\cdot;(j,1)}(\lambda) \\ P_{\cdot;(0,0)}(\lambda) & P_{\cdot;(1,0)}(\lambda) & \dots & P_{\cdot;(j-1,0)}(\lambda) & P_{\cdot;(j,0)}(\lambda) \end{array} \tag{45}$$

The number of the vectors in system (45) is also equal to $\frac{j(j+1)}{2} + k + 1$. From all of the above-mentioned we obtain linear independence of vectors of system (34).

At last, the real linear span of all vectors of (34), not including $\lambda^k \delta_{j-k}$, coincides with the real linear span of all vectors of system (45), not including $P_{\cdot;(j,k)}(\lambda)$. This is true, since the latter system can be obtained from the former one by a linear combination, and the dimensions of both systems are equal to $\frac{j(j+1)}{2} + k$. Thus from orthogonality of $P_{\cdot;(j,k)}(\lambda)$ to $P_{\cdot;(m,l)}(\lambda)$ $((j,k)\neq(m,l))$, we obtain the orthogonality of $P_{\cdot;(j,k)}(\lambda)$ to all the vectors in (34), except for $\lambda^k\delta_{j-k}$. In addition, $P_{\cdot;(j,k)}(\lambda)$ has the norm equal to 1 in L^2 and the coefficient at $\lambda^k\delta_{j-k}$ is positive; this follows from (32). Since $Q_{j;k}(\lambda)$ is obtained in a unique way, we can make the following conclusion: $P_{\cdot;(j,k)}(\cdot)=Q_{j;k}(\cdot)$ $\forall j=0,1,\dots,k=0,\dots,j$ in the space $L^2(\mathbb{R},d\Sigma(\lambda))$. Since the measure $\Sigma(\cdot)$ has infinite support,

$$\forall\lambda\in\mathbb{R}\quad P_{\cdot;(j,k)}(\lambda)=Q_{j;k}(\lambda),\quad j=0,1,\dots,\quad k=0,\dots,j.$$ □

Theorem 3. *The aggregate $\widehat{\mathbf{l}_{\mathrm{fin}}}$ of the Fourier transforms of vectors from $\mathbf{l}_{\mathrm{fin}}$ consists of all finite vectors of the form $\xi(\lambda)=(\xi_0(\lambda),\xi_1(\lambda),\dots)^T$, where $\xi_\alpha(\lambda),\alpha=0,1,\dots$ is some polynomial of λ. If $\xi(\lambda)\in\widehat{\mathbf{l}_{\mathrm{fin}}}$ is such that $(\xi(\lambda),\xi(\lambda))_{L^2(\mathbb{R},d\Sigma(\lambda))}=0$, then $\forall\alpha=0,1,\dots\ \xi_\alpha(\lambda)\equiv0$.*

Proof. To prove the first statement, it is sufficient to show that all the vectors $\lambda^j\delta_k\in\widehat{\mathbf{l}_{\mathrm{fin}}}$, $j,k\in\mathbb{N}_0$. The latter follows from the fact that $\lambda^j\delta_k$ is a linear combination of the corresponding vectors $P_{\cdot;(m,l)}(\lambda)$ that are Fourier transforms of the vectors $e_{m;l}\in\mathbf{l}_{\mathrm{fin}}$.

The second statement of the theorem follows from the following consideration. The vector $\xi(\lambda)\in\widehat{\mathbf{l}_{\mathrm{fin}}}$ is a linear combination of linearly independent vectors (34). So, if $(\xi(\lambda),\xi(\lambda))_{L^2}=0$, then all the coefficients in this combination are equal to zero. This means that $\xi_\alpha(\lambda)\equiv0,\alpha=0,1,\dots$ □

Remark 4. It follows from Theorem 3 that $u\to\widehat{u}(\lambda)$ is a one-to-one mapping between $\mathbf{l}_{\mathrm{fin}}$ and finite vectors of polynomials. Using the Parseval equality (22) we obtain that this mapping is an isometry between $\mathbf{l}_{\mathrm{fin}}$ and $\widehat{\mathbf{l}_{\mathrm{fin}}}$. By continuity it can be extended to $\mathbf{l}_2$, and its image $\widehat{\mathbf{l}_2}$ is the closure of $\widehat{\mathbf{l}_{\mathrm{fin}}}$ in L^2. Under the action of the isometry, the operator $\mathbf{J}$ transforms into an operator $\widehat{\mathbf{J}}$ of multiplication by the independent variable λ (i.e., the operator $\widehat{u}(\lambda)\to\lambda\widehat{u}(\lambda)$) that is defined on $\widehat{\mathbf{l}_{\mathrm{fin}}}$. It can be extended by closure to the whole domain $D(\widehat{\mathbf{J}})$.

Theorem 4. *The set of all finite vectors of polynomials, i.e. $\widehat{\mathbf{l}_{\mathrm{fin}}}$, is dense in $L^2(\mathbb{R},d\Sigma(\lambda))$ if and only if the spectral measure $d\Sigma(\lambda)$ is generated by the ordinary resolution of identity, that is, $d\Sigma(\lambda)$ is constructed in accordance with the direct spectral problem (see, Section 3 or* [5]*) from some selfadjoint operator A in $\mathbf{l}_2$ generated by a generalized Hermitian Jacobi matrix and, therefore, identity* (20) *holds, where $E(\cdot)$ is an ordinary resolution of identity for A. Such spectral measures will be called orthogonal.*

Proof. **Sufficiency.** Let $\Sigma(\cdot)=(\sigma_{\alpha\beta}(\cdot))_{\alpha,\beta=0}^{\infty}=((E(\cdot)e_{\beta;0},e_{\alpha;0})_{\mathbf{l}_2})_{\alpha,\beta=0}^{\infty}$, where $E(\cdot)$ is an ordinary resolution of identity. Let us prove that the vectors of polynomials are dense in $L^2(\mathbb{R},d\Sigma(\cdot))$. Let us denote by $A\supseteq\mathbf{J}$ the selfadjoint extension of the operator $\mathbf{J}$ in $\mathbf{l}_2$ responding to $E(\cdot)$. Also, by R_z $(Im\,z\neq0)$ we denote its resolvent. Consider some fixed $u\in\mathbf{l}_{\mathrm{fin}}$. Since $R_zu\in D(A)\subseteq\mathbf{l}_2$, for all fixed z there exists $v\in\mathbf{l}_{\mathrm{fin}}$ such that $\|R_zu-v\|_{\mathbf{l}_2}<\varepsilon$. Passing using the isometry from $\mathbf{l}_2$ to L^2 we obtain the following: $\forall\widehat{u}(\lambda)\,\exists\widehat{v}(\lambda)$

$$\left\|\frac{\widehat{u}(\lambda)}{\lambda-z}-\widehat{v}(\lambda)\right\|_{L^2}<\varepsilon. \tag{46}$$

Indeed, let $(A-z\mathbf{1})^{-1}u=f$, where $\mathbf{1}$ is the identity operator in $\mathbf{l}_2$. Then $(A-z\mathbf{1})f=u$ and, therefore, $zf=Af-u$. Using the Fourier transform we get that $\widehat{f}(\lambda)=\frac{1}{z}\widehat{(zf)}(\lambda)=\frac{1}{z}(\lambda\widehat{f}(\lambda)-\widehat{u}(\lambda))$. Thus $\widehat{f}(\lambda)=\frac{\widehat{u}(\lambda)}{\lambda-z}$. Since ε is an arbitrary positive constant in (46), any function $\frac{\widehat{u}(z)}{\lambda-z}$ can be approximate by a vector of polynomials; in other words, $\frac{\widehat{u}(z)}{\lambda-z}\in\widehat{\mathbf{l}_2}$ for any $\widehat{u}(\lambda)$.

Now let $h(\lambda) \in L^2(\mathbb{R}, d\Sigma(\lambda))$ be orthogonal to $\widehat{\mathbf{l}_2}$. Then for all non-real z, the following equality takes place:

$$\int_{\mathbb{R}} \left(d\Sigma(\lambda) \frac{\widehat{u}(\lambda)}{\lambda - z}, h(\lambda) \right)_{\ell_2} = \int_{\mathbb{R}} \frac{1}{\lambda - z} \left(d\Sigma(\lambda) \widehat{u}(\lambda), h(\lambda) \right)_{\ell_2} = 0.$$

Consider some fixed $j \in \mathbb{N}_0$ and put $u := h_j(\lambda) e_{j;0}$. Then $\widehat{u}(\lambda) = h_j(\lambda) P_{\cdot;(j,0)}(\lambda) =$ $= h_j(\lambda)\delta_j$. So,

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} (d\Sigma(\lambda) h_j(\lambda)\delta_j, h(\lambda))_{\ell_2} = 0.$$

If we consider this equality for all j and sum over all of them, then we obtain

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} (d\Sigma(\lambda) h(\lambda), h(\lambda))_{\ell_2} = 0.$$

Let us consider the measure $\eta(\triangle) = \int_{\triangle} (d\Sigma(\lambda) h(\lambda), h(\lambda))_{\ell_2}, \quad \triangle \in \mathcal{B}(\mathbb{R})$.
Since $\int_{\mathbb{R}} \frac{1}{\lambda - z} d\eta(\lambda) = 0$, we have $\eta(\triangle) \equiv 0$. Therefore, $h(\lambda) = 0$. So, $\widehat{\mathbf{l}_2} = L^2(\mathbb{R}, d\Sigma(\cdot))$.

Necessity. Let $\widehat{\mathbf{l}_2} = L^2(\mathbb{R}, d\Sigma(\cdot))$. Let us show that $d\Sigma(\lambda)$ is generated by some selfadjoint extension A of an operator $\mathbf{J}$ in $\mathbf{l}_2$. If we pass by isometry from $\mathbf{l}_2$ to L^2, then the operator $\mathbf{J}$ is transformed into the operator $\widehat{\mathbf{J}}$ of multiplication by the independent variable λ, which is defined, at first, on vectors of polynomials, and, later, on the whole L^2 by closure. Let us denote by $\widehat{A}$ the ordinary operator of multiplication by λ in $L^2(\mathbb{R}, d\Sigma(\lambda))$. This operator is selfadjoint. Then $\widehat{\mathbf{J}} \subseteq \widehat{A}$ and $\Sigma(\triangle) = (\sigma_{\alpha\beta}(\triangle))_{\alpha,\beta=0}^{\infty} =$ $((\widehat{E(\triangle)}\delta_\beta, \delta_\alpha)_{L^2})_{\alpha,\beta=0}^{\infty}$, where $\widehat{E(\triangle)}, \triangle \in \mathcal{B}(\mathbb{R})$, is the resolution of identity corresponding to $\widehat{A}$ and that coincides with the operator of multiplication by the characteristic function. Since $\widehat{\mathbf{l}_2} = L^2(\mathbb{R}, d\Sigma(\cdot))$, we can pass via the inverse isometry from $\widehat{A}$ to the operator A in $\mathbf{l}_2$. It is easy to see that this operator is the required selfadjoint operator which is an extension of $\mathbf{J}$ and generates $d\Sigma(\cdot)$. $\square$

Remark 5. In the situation described in Section 3, if the measure $\Sigma(\cdot)$ is a spectral measure of a selfadjoint operator $\mathbf{J}$ generated by generalized Hermitian matrix, then $\widehat{\mathbf{l}_{\mathrm{fin}}}$ is dense in L^2.

Indeed, in the direct spectral problem for the matrix J (see, article [5]) the operator $\mathbf{J}$ is constructed as a selfadjoint closure of a Hermitian operator in $\mathbf{l}_2$. So, the corresponding spectral measure is generated by the ordinary resolution of identity. Thus, from Theorem 4, it follows that the set of all finite vectors of polynomials is dense in L^2.

Theorem 5. *Let $d\Sigma(\cdot)$ be a non-negative operator-valued measure on $\mathbb{R}$ and* (39) *hold. If for all $u, v \in \mathbf{l}_{\mathrm{fin}}$ and their Fourier transforms $\widehat{u}(\lambda), \widehat{v}(\lambda)$, defined by* (21)*, the Parseval equality* (22) *holds (or, which is the same, orthogonality relations* (24) *holds), then $d\Sigma(\cdot)$ is a spectral measure (matrix), i.e., in the general case, there exists a generalized resolution of identity constructed from a selfadjoint extension of $\mathbf{J}$, and $\Sigma(\cdot) =$ $((E(\cdot)e_{\beta;0}, e_{\alpha;0})_{\mathbf{l}_2})_{\alpha,\beta=0}^{\infty} = (\sigma_{\alpha\beta}(\cdot))_{\alpha,\beta=0}^{\infty}$.*

Proof. Using the Parseval equality (22) we construct an isometry between $\mathbf{l}_2$ and $\widehat{\mathbf{l}_2} \subseteq$ $L^2(\mathbb{R}, d\Sigma(\lambda))$. This isometry transforms the operator $\mathbf{J}$ into the operator $\widehat{\mathbf{J}}$ of multiplication by λ, which is equal to the closure of the operator of multiplication by λ, defined on vectors of polynomials. Using this isomorphism we can pass from the initial problem to the problem of constructing a selfadjoint extension $\widehat{A}$ of the operator $\widehat{J}$ such that $\Sigma(\cdot) = (\sigma_{\alpha\beta}(\cdot))_{\alpha,\beta=0}^{\infty} =$
$= ((\widehat{E}(\cdot)\delta_\beta, \delta_\alpha)_{L^2})_{\alpha,\beta=0}^{\infty}$, where $\widehat{E}(\cdot)$ is a resolution of identity corresponding to $\widehat{A}$. For the operator $\widehat{A}$, we can consider a selfadjoint extension of the operator of the original multiplication by λ in $L^2(\mathbb{R}, d\Sigma(\lambda))$. If $\widehat{\mathbf{l}_2} = L^2$, then the obtained extension corresponds

to an extension of $\mathbf{J}$ without leaving $\mathbf{l}_2$; and, in case $\widehat{\mathbf{l}_2} \subset L^2$, it is an extension with a larger space. $\square$

Remark 6. If we consider the case described in Remark 5, it follows that $\widehat{\mathbf{l}_{\text{fin}}}$ is dense in L^2, i.e., $\widehat{\mathbf{l}_2} = L^2$. Therefore, if conditions of Theorem 5 are satisfied, then $d\Sigma(\cdot)$ is a spectral measure corresponding to an ordinary resolution of identity.

Now we proceed directly to the solution of the inverse spectral problem.

Let us introduce formulas that express elements of J with in terms of $P_{\cdot;(j,k)}(\lambda)$. According to (11) and the definition of $P_{\alpha;(j,k)}(\lambda), \alpha = 0, 1, \ldots, j = 0, 1, \ldots, k = 0, \ldots, j$, we obtain

$$a_{j-1} P_{\alpha;(j-1,\cdot)}(\lambda) + b_j P_{\alpha;(j,\cdot)}(\lambda) + c_j P_{\alpha;(j+1,\cdot)}(\lambda) = \lambda P_{\alpha;(j;\cdot)}(\lambda).$$

So, for any fixed $i = 0, \ldots, j$, the following equalities take place:

$$\sum_{k=0}^{j-1} a_{j-1;i,k} P_{\alpha;(j-1,k)}(\lambda) + \sum_{k=0}^{j} b_{j;i,k} P_{\alpha;(j,k)}(\lambda) + \sum_{k=0}^{j+1} c_{j;i,k} P_{\alpha;(j+1,k)}(\lambda) = \lambda P_{\alpha;(j;i)}(\lambda).$$

Since this equality takes place for all $\alpha = 0, 1, \ldots,$ we get

$$(47)\quad \sum_{k=0}^{j-1} a_{j-1;i,k} P_{\cdot;(j-1,k)}(\lambda) + \sum_{k=0}^{j} b_{j;i,k} P_{\cdot;(j,k)}(\lambda) + \sum_{k=0}^{j+1} c_{j;i,k} P_{\cdot;(j+1,k)}(\lambda) = \lambda P_{\cdot;(j;i)}(\lambda).$$

So, we can define elements of the matrix J in the same way as in (26), i.e., $\forall j, k \in \mathbb{N}_0$

$$(48)\qquad J_{j,k;l,m} = \int_{\mathbb{R}} \lambda \left(d\Sigma(\lambda) \overline{P_{\cdot;(k,m)}(\lambda)}, \overline{P_{\cdot;(j,l)}(\lambda)} \right)_{\ell_2}, \quad l = 0, \ldots, j, \quad m = 0, \ldots, k.$$

Since $P_{\cdot;(k,m)}(\lambda)$ is real-valued, we have: $\forall j = 0, 1, \ldots$

$$(49)\quad \begin{aligned} a_{j;l,m} &= J_{j+1,j;l,m} = (\lambda P_{\cdot;(j,m)}(\lambda), P_{\cdot;(j+1,l)}(\lambda))_{L^2}, \ \ l = 0, \ldots, j+1, \ \ m = 0, \ldots, j; \\ b_{j;l,m} &= J_{j,j;l,m} = (\lambda P_{\cdot;(j,m)}(\lambda), P_{\cdot;(j,l)}(\lambda))_{L^2}, \ \ l = 0, \ldots, j, \ \ m = 0, \ldots, j; \\ c_{j;l,m} &= J_{j,j+1;l,m} = (P_{\cdot;(j+1,m)}(\lambda), \lambda P_{\cdot;(j,l)}(\lambda))_{L^2}, \ \ l = 0, \ldots, j, \ \ m = 0, \ldots, j+1. \end{aligned}$$

Let us show that $b_j = (b_j)^T, a_j = (c_j)^T$. The matrix $d\Sigma(\cdot)$ is selfadjoint, i.e. $d\Sigma(\cdot) = (d\Sigma(\cdot))^* = (\overline{d\Sigma(\cdot)})^T$. Then $\forall j = 0, 1, \ldots$

$$\begin{aligned} a_{j;l,m} &= \int_{\mathbb{R}} \lambda (d\Sigma(\lambda) P_{\cdot;(j,m)}(\lambda), P_{\cdot;(j+1,l)}(\lambda))_{\ell_2} = \int_{\mathbb{R}} \lambda (P_{\cdot;(j,m)}(\lambda), \overline{d\Sigma(\lambda)} P_{\cdot;(j+1,l)}(\lambda))_{\ell_2} \\ &= \int_{\mathbb{R}} \lambda (d\Sigma(\lambda) P_{\cdot;(j+1,l)}(\lambda), P_{\cdot;(j,m)}(\lambda))_{\ell_2} = c_{j;m,l}, \ \ l = 0, \ldots, j+1, \ \ m = 0, \ldots, j. \end{aligned}$$

Therefore $a_j = (c_j)^T$. In the same way we can prove that $b_j = (b_j)^T$.

Now we formulate the following uniqueness theorem.

Theorem 6. *Elements of the matrix J in the form* (8) *with coefficients in the form* (30) *can be recovered by its spectral measure in a unique way.*

Proof. Let $d\Sigma(\cdot)$ be a spectral matrix of J and $P_{\alpha;(j,k)}(\lambda)$ be the respective polynomials. Then elements of the matrix J are defined by (48). Let us consider the vectors of polynomials (42), i.e., $Q_{j;k}(\lambda) \in L^2(\mathbb{R}, d\Sigma(\lambda))$. According to Theorem 2, $Q_{j;k}(\lambda) = P_{\cdot;(j,k)}(\lambda), j = 0, 1, \ldots, k = 0, \ldots, j$, and, therefore, $\lambda Q_{j;k}(\lambda) = \lambda P_{\cdot;(j,k)}(\lambda), j = 0, 1, \ldots, k = 0, \ldots, j$. So, equalities (48) have the form

$$J_{j,k;l,m} = \int_{\mathbb{R}} \lambda \left(d\Sigma(\lambda) Q_{k;m}(\lambda), Q_{j;l}(\lambda) \right)_{\ell_2}, \quad j, k \in \mathbb{N}_0, \quad l = 0, \ldots, j, \quad m = 0, \ldots, k$$

and, particularly,

$$
\begin{aligned}
&a_{j;l,m} = J_{j+1,j;l,m} = (\lambda Q_{j;m}(\lambda), Q_{j+1;l}(\lambda))_{L^2}, \quad l = 0, \dots, j+1, \quad m = 0, \dots, j; \\
&b_{j;l,m} = J_{j,j;l,m} = (\lambda Q_{j;m}(\lambda), Q_{j;l}(\lambda))_{L^2}, \quad l = 0, \dots, j, \quad m = 0, \dots, j; \\
&c_{j;l,m} = J_{j,j+1;l,m} = (Q_{j+1;m}(\lambda), \lambda Q_{j;l}(\lambda))_{L^2}, \quad l = 0, \dots, j, \quad m = 0, \dots, j+1.
\end{aligned} \tag{50}
$$

These formulas show that the coefficients of the matrix J are uniquely defined by $d\Sigma(\cdot)$. □

Remark 7. From Theorem 6, we obtain following result.

Let us consider some matrix J of form (8) with coefficients (30). Using the direct spectral problem for this matrix we obtain some spectral measure $\Sigma(\cdot)$. Since the measure $\Sigma(\cdot)$ is spectral, we can recover elements of some generalized Jacobi Hermitian matrix J' using the inverse spectral problem.

Then $J = J'$.

Now we formulate the main theorem for solution of the inverse spectral problem.

Theorem 7. *Let $d\Sigma(\lambda) = (d\sigma_{\alpha\beta}(\lambda))_{\alpha,\beta=0}^{\infty}$ be some non-negative operator-valued measure the values of which are bounded operators in ℓ_2. Then $d\Sigma(\lambda)$ is a spectral matrix of some matrix J of the form* (8) *with coefficients* (30) *if and only if the following conditions are satisfied:*

1) *relations* (35), (37), (38) *and* (39) *are valid, i.e. ii) and iii) hold true;*
2) *any system of vectors* (34) *is linearly independent in $L^2(\mathbb{R}, d\Sigma(\lambda))$ or, what is the same,* supp $\Sigma(\cdot)$ *has an infinite set of points.*

If 1)–2) are satisfied then the elements of the matrix J are correctly calculated by the formulas

$$
J_{j,k;l,m} = \int_{\mathbb{R}} \lambda \left(d\Sigma(\lambda) Q_{k;m}(\lambda), Q_{j;l}(\lambda) \right)_{\ell_2}, \quad j,k \in \mathbb{N}_0, \quad l = 0, \dots, j, \quad m = 0, \dots, k, \tag{51}
$$

i.e., all the coefficients are such that the matrix J has the form (8) *with elements $a_j, b_j, c_j, j = 0, 1, \dots,$ of type* (30) *defined by* (50).

Proof. **Necessity.** Let $d\Sigma(\lambda)$ be a spectral matrix of some operator J. Equalities (35), (37), (38) were proved earlier: the proof was given when these relations were formulated. The rest part of conditions 1) and 2) follows from Theorems 1 and 2.

Sufficiency. Let us define elements of the matrix J by (51).

First of all, we prove that all elements, which do not belong to the blocks a_j, b_j, c_j, are equal to zero. Indeed, let $|j-k| \geq 2$. For determinacy we consider the case $j \leq k-2$. According to (44), $Q_{j;l}(\lambda) = q_{j;l}\lambda^l\delta_{j-l} + S_{j;k}(\lambda)$. Consider $\lambda Q_{j;l}(\lambda)$. According to the construction of $Q_{j;k}(\lambda)$ we can state that $\lambda Q_{j;l}(\lambda)$ is a linear combination of the vectors

$$
\begin{array}{ccccc}
 & & \lambda^j \delta_0 & & \\
 & & & \lambda^{l+1}\delta_{j-l} & \\
 & \lambda^2\delta_0 & \vdots & \vdots & \\
\lambda\delta_0 & \lambda\delta_1 \quad \dots & \lambda\delta_{j-1} & \lambda\delta_j &
\end{array} \tag{52}
$$

Therefore,

$$
\begin{aligned}
\lambda Q_{j;l}(\lambda) = & \frac{q_{j;l}}{q_{j+1;l+1}} Q_{j+1;l+1}(\lambda) + *Q_{j+1;l}(\lambda) + \dots + *Q_{j+1;0}(\lambda) \\
& + *Q_{j;j}(\lambda) + \dots + *Q_{j;0}(\lambda) + \dots + *Q_{0;0}(\lambda),
\end{aligned} \tag{53}
$$

where $*$ denotes some real constants. Since $j \leq k-2$, from equality (51), representation (53) and orthogonality relations (43), we obtain that $J_{j,k;l,m} = 0$ for such j, k and all permitted l, m. It is easy to show that the matrix J is selfadjoint (in the same way as for the matrices defined by (49)), i.e., $J = J^* = J^T$ or, what is the same, $J_{j,k;l,m} = J_{k,j;m,l}$. Therefore, $J_{j,k;l,m} = 0$ for $j \geq k-2$ and all permitted l, m.

Let us show that for all j and all permitted $m \geq l+2$, $c_{j;l,m} = 0$. From definition (50), representation (53) and orthogonality relations (43) it is easy to see that $\lambda Q_{j;l}(\lambda) \perp Q_{j+1;m}(\lambda)$. So, $c_{j;l,m} = 0$. Since $a_j = (c_j)^*$, for all j and all permitted $l \geq m+2$ $a_{j;l,m} = 0$.

Now we show that for all $j = 0, 1, \ldots,$ $c_{j;l,l+1} = a_{j;l+1,l} > 0, l = 0, \ldots, j$. Since $a_j = (c_j)^*$, it sufficient to prove that $c_{j;l,l+1} > 0$. From equalities (50), representation (53) and orthogonality relations (43) we obtain that $Q_{j+1;l+1}(\lambda)$ is orthogonal in L^2 to all summands in the right-hand side of (53), except for the first one. Therefore, we get

$$c_{j;l,l+1} = \frac{q_{j;l}}{q_{j+1;l+1}} > 0. \tag{54}$$

And, all the other elements of the matrices a_j, b_j, c_j, i.e., not mentioned above, are some real constants in the general case. This is easy to see from definition (50), representation (53) and orthogonality relations (43).

Let us define $\forall j \in \mathbb{N}_0$

$$\Omega := \sum_{m=0}^{j-1} a_{j-1;l,m} Q_{j-1;m}(\lambda) + \sum_{m=0}^{j} b_{j;l,m} Q_{j;m}(\lambda) + \sum_{m=0}^{j+1} c_{j;l,m} Q_{j+1;m}(\lambda), \quad l = 0, \ldots, j,$$

where $Q_{-1;m}(\lambda) := 0$. Now we will show that $\forall j, p \in \mathbb{N}_0$

$$(\Omega, Q_{p;q}(\lambda))_{L^2} = (\lambda Q_{j;l}(\lambda), Q_{p;q}(\lambda))_{L^2}, \quad l = 0, \ldots, j, \quad q = 0, \ldots, p. \tag{55}$$

Indeed, let $|j-p| \geq 2$. Then $(\Omega, Q_{p;q}(\lambda))_{L^2} = 0$. Let, for determinacy, $p \geq j+2$. From representation (53) and orthogonality relations (43) we get that $(\lambda Q_{j;l}(\lambda), Q_{p;q}(\lambda))_{L^2} = 0$. It is easy to see (arguing in the same way as in above-mentioned proof for correctness of the coefficients) that the same equality takes place if $p \leq j+2$. Therefore, in this case, the equality (55) is valid.

Let $j = p$. Then from (43) we get that $(\Omega, Q_{p;q}(\lambda)) = b_{j;l,q}$. From equality (53) it follows that $(\lambda Q_{j;l}(\lambda), Q_{p;q}(\lambda))_{L^2} = b_{j;q,l}$. Since $b_j = (b_j)^*$, equality (55) holds true.

Let $j+1 = p$. Then from (43) and the definition of Ω, we obtain that $(\Omega, Q_{p;q}(\lambda)) = c_{j;l,q}$. Also, from equalities (53) and (50) it follow that $(\lambda Q_{j;l}(\lambda), Q_{p;q}(\lambda))_{L^2} = a_{j;q,l}$. So, (55) true.

And the last one, if $j-1 = p$, then $(\Omega, Q_{p;q}(\lambda))_{L^2} = a_{j-1;l,q} = c_{j-1;m,l} =$
$= (\lambda Q_{j;l}(\lambda), Q_{p;q}(\lambda))_{L^2}$. So, the proof of (55) is completed.

Since Ω and $\lambda Q_{j;l}(\lambda)$ are linear combinations of respective sets of vectors of type (34), they are linear combinations of the vectors $Q_{p;q}(\lambda)$. Since p and q are arbitrary in (55), $\Omega = \lambda Q_{j;l}(\lambda), \lambda \in \mathbb{R}$. Thus, $\forall j \in \mathbb{N}_0, l = 0, \ldots, j$

$$\sum_{m=0}^{j-1} a_{j-1;l,m} Q_{j-1;m}(\lambda) + \sum_{m=0}^{j} b_{j;l,m} Q_{j;m}(\lambda) + \sum_{m=0}^{j+1} c_{j;l,m} Q_{j+1;m}(\lambda) = \lambda Q_{j;l}(\lambda). \tag{56}$$

So, the sequence $Q_{j;l}(\lambda) \in L^2(\mathbb{R}, d\Sigma(\lambda))$, $j = 0, 1, \ldots, l = 0, \ldots, j$, is a solution of the equation (56) with the initial conditions $Q_{-1;l}(\lambda) = 0, Q_{j;0}(\lambda) = \delta_j$. On the other hand, the orthogonal polynomials $P_{\cdot;(j,l)}(\lambda)$ also satisfy the difference expression (56) (see (47)) and the same initial conditions. The solution of equation (56) is unique; therefore $Q_{j;l}(\lambda) = P_{\cdot;(j,l)}(\lambda)$, $j = 0, 1, \ldots, l = 0, \ldots, j$. From this fact and (43) it follows that $P_{\cdot;(j,l)}(\lambda)$ satisfy the orthogonality relations. Thus, the orthogonal polynomials constructed from difference expressions satisfy the orthogonality relations in the space $L^2(\mathbb{R}, d\Sigma(\cdot))$. Using Theorem 5 we can state that $d\Sigma(\cdot)$ is a spectral matrix (measure) constructed from the matrix J. $\square$

5. On the inverse spectral problem for a generalized Jacobi Hermitian matrix

In Section 4 we have investigated the inverse spectral problem for some type of generalized Jacobi Hermitian matrices. It is necessary to admit that the theory described in Section 4 can be easily transferred to such type of matrices where the procedure of calculating $P_{\cdot;(j,k)}(\lambda)$ is quite simple, in other words, if it is easy to understand the procedure of orthogonalization. For example, this is the case if c_n is a matrix with mixed rows in a matrix of form (30).

In article [5] we consider a direct spectral problem for generalized Jacobi Hermitian matrices. In this article we solved the inverse spectral problem just for the type of generalized Jacobi Hermitian matrices elements of which satisfy condition (30). So, a natural question arises: what is the situation in the general case?

First of all, it is necessary to say that the matrices J of type (8) with coefficients (30) also appear in article [3]. In that paper they were related to a complex moment problem. But there is a completely different situation with the direct and the inverse spectral problems. Since the introduced matrix induced a normal operator, in the direct spectral problem there appear only one boundary condition (in our case there is an infinite vector of them) and, therefore, the respective spectral measure is scalar-valued.

Secondly, it is easy to see that in the general case the vector of polynomials $P_{\cdot;(j,k)}(\lambda)$ for some j, k can be such that none of its coefficients is positive. Therefore, there is no way to obtain our polynomials by a standard orthogonalization of a system similar to (34).

And the last one, in general case the construction of polynomials $P_{\alpha;(j,k)}(\lambda)$ is quite complicated. It follows from (12) and examples 1, 2. So, if we even apply some pseudo-orthogonalization procedure, it is not quite clear what system is necessary to orthogonalize.

But in spite of all this difficulties, in my next works I will try to solve the inverse spectral problem in the general case or to prove that it is not possible to do.

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