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CONNECTED COMPONENTS OF PARTITION PRESERVING DIFFEOMORPHISMS

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Abstract. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a homogeneous polynomial and $\mathcal{S}(f)$ be the group of diffeomorphisms h of $\mathbb{R}^2$ preserving f, i.e. $f\circ h=f$. Denote by $\mathcal{S}_{\mathrm{id}}(f)^r$, $(0\le r\le\infty)$, the identity component of $\mathcal{S}(f)$ with respect to the weak Whitney C^r_W-topology. We prove that $\mathcal{S}_{\mathrm{id}}(f)^\infty=\cdots=\mathcal{S}_{\mathrm{id}}(f)^1$ for all f and that $\mathcal{S}_{\mathrm{id}}(f)^1\ne\mathcal{S}_{\mathrm{id}}(f)^0$ if and only if f is a product of at least two distinct irreducible over $\mathbb{R}$ quadratic forms.

1. Introduction

Let $f:\mathbb{R}^2\to\mathbb{R}$ be a homogeneous polynomial of degree $p\ge 1$. Thus up to sign we can write

$$f(x,y)=\pm\prod_{i=1}^{l}L_i^{\alpha_i}(x,y)\cdot\prod_{j=1}^{k}Q_j^{\beta_j}(x,y),\tag{1.1}$$

where every L_i is a linear function, Q_j is a positive definite quadratic form, $\alpha_i,\beta_j\ge 1$, and

$$\frac{L_i}{L_{i'}}\ne\mathrm{const}\quad\text{for}\quad i\ne i',\qquad \frac{Q_j}{Q_{j'}}\ne\mathrm{const}\quad\text{for}\quad j\ne j'.$$

Denote by $\mathcal{S}(f)=\{h\in\mathcal{D}(\mathbb{R}^2)\,:\,f\circ h=f\}$ the stabilizer of f with respect to the right action of the group $\mathcal{D}(\mathbb{R}^2)$ of C^∞-diffeomorphisms of $\mathbb{R}^2$ on the space $C^\infty(\mathbb{R}^2,\mathbb{R})$. It consists of diffeomorphisms of $\mathbb{R}^2$ preserving every level-set $f^{-1}(c)$ of f, $c\in\mathbb{R}$.

Let $\mathcal{S}_{\mathrm{id}}(f)^r$, $0\le r\le\infty$, be the identity component of $\mathcal{S}(f)$ with respect to the weak Whitney C^r_W-topology. Thus $\mathcal{S}_{\mathrm{id}}(f)^r$ consists of diffeomorphisms $h\in\mathcal{S}(f)$ isotopic in $\mathcal{S}(f)$ to $\mathrm{id}_{\mathbb{R}^2}$ via (an f-preserving isotopy) $H:\mathbb{R}^2\times I\to\mathbb{R}^2$ whose partial derivatives in $(x,y)\in\mathbb{R}^2$ up to order r continuously depend on (x,y,t), see Section 2 for a precise definition. Then it is easy to see that

$$\mathcal{S}_{\mathrm{id}}(f)^\infty\subset\cdots\mathcal{S}_{\mathrm{id}}(f)^r\subset\cdots\subset\mathcal{S}_{\mathrm{id}}(f)^1\subset\mathcal{S}_{\mathrm{id}}(f)^0.$$

It follows from results in [12, 13] that $\mathcal{S}_{\mathrm{id}}(f)^\infty=\mathcal{S}_{\mathrm{id}}(f)^1$. Moreover, it is actually proved in [9] that $\mathcal{S}_{\mathrm{id}}(f)^\infty=\mathcal{S}_{\mathrm{id}}(f)^0$ for $p\le 2$, see also [11]. The aim of this note is to prove the following theorem describing the relation between $\mathcal{S}_{\mathrm{id}}(f)^r$ for all $p\ge 1$.

Theorem 1.1. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be a homogeneous polynomial of degree $p\ge 1$. Then $\mathcal{S}_{\mathrm{id}}(f)^\infty=\cdots=\mathcal{S}_{\mathrm{id}}(f)^1$. Moreover, $\mathcal{S}_{\mathrm{id}}(f)^1\ne\mathcal{S}_{\mathrm{id}}(f)^0$ if and only if f is a product of at least two distinct definite quadratic forms, i.e. $f=Q_1^{\beta_1}\cdots Q_k^{\beta_k}$ for $k\ge 2$.*

This theorem is based on a rather general result about partition preserving diffeomorphisms, see Theorem 4.7. The applications of Theorem 1.1 will be given in another paper concerning smooth functions on surfaces with isolated singularities.

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Structure of the paper. In Section 2 we describe homotopies which induce continuous paths into functional spaces with the weak Whitney C^r_W-topologies. Section 3 introduces the so-called *singular partitions* of manifolds being the main object of the paper. Section 4 contains the main result, Theorem 4.7, about invariant contractions of singular partitions. In Section 5 an application of this theorem to local extremes of smooth functions is given. Section 6 contains a description of the group of linear symmetries of f. Finally in Section 7 we prove Theorem 1.1.

2. r-HOMOTOPIES

Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\overline{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty\}$.

Let M, N be two smooth manifolds of dimensions m and n respectively. Then for every $r \in \overline{\mathbb{N}}_0$ the space $C^r(M, N)$ admits the so-called *weak* Whitney topology denoted by C^r_W, see e.g. [8, 5].

Recall, e.g. [7, § 44.IV] that there exists a homeomorphism

$$C^0(I, C^0(M, N)) \approx C^0(M \times I, N)$$

with respect to the corresponding C^0_W-topologies (also called *compact open* ones) associating to every (continuous) path $w : I \to C^0(M, N)$ a homotopy $H : M \times I \to N$ defined by $H(x, t) = w(t)(x)$.

We will now describe homotopies inducing continuous paths $w : I \to C^r(M, N)$ with respect to C^r_W-topologies.

Definition 2.1. Let $H : M \times I \to N$ be a homotopy and $r \in \overline{\mathbb{N}}_0$. We say that H is an r**-homotopy** if

(1) $H_t : M \to N$ is C^r for every $t \in I$;
(2) partial derivatives of $H(x, t)$ with respect to x, up to order r, continuously depend on (x, t).

More precisely, let $z \in M \times I$. Then in some local coordinates at z we can regard H as a map

$$H = (H_1, \dots, H_n) : \mathbb{R}^m \times I \to \mathbb{R}^n$$

such that for every fixed t and i the function $H_i(x, t)$ is C^r. Condition (2) requires that for every $i = 1, \dots, n$ and every non-negative integer vector $k = (k_1, \dots, k_m)$ of norm $|k| = \sum_{j=1}^m k_i \leq r$ the function

$$\frac{\partial^{|k|} H_i}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}}(x_1, \dots, x_m, t)$$

continuously depend on (x, t).

Thus a 0-homotopy H is just the usual homotopy.

It easily follows from the definition of C^r_W-topologies that a path $w : I \to C^r(M, N)$ is continuous in the standard topology of I and C^r_W-topology of $C^r(M, N)$ if and only if the corresponding homotopy $H : M \times I \to N$ is an r-homotopy.

We can also define a C^r*-homotopy* as a C^r-map $M \times I \to N$. Evidently, every C^r-homotopy is an r-homotopy as well, but the converse is not true.

Example 2.2. Let $H : \mathbb{R} \times I \to \mathbb{R}$ be given by

$$H(x, t) = \begin{cases} t \ln(x^2 + t^2), & (x, t) \neq (0, 0), \\ 0, & (x, t) = (0, 0). \end{cases}$$

Then H is continuous, while $\frac{\partial H}{\partial x} = \frac{2tx}{x^2+y^2}$ is C^∞ for every fixed t as a function in x but discontinuous at $(0, 0)$ as a function in (x, t). In other words H is a 0-homotopy but not a 1-homotopy.

Moreover, define $G : \mathbb{R} \times I \to \mathbb{R}$ by $G(x,t) = \int_0^x H(y,t)\,dy$. Then G a 1-homotopy but not a 2-homotopy.

3. Singular partitions of manifolds

Let M be a smooth manifold equipped with a partition $\mathcal{P} = \{\omega_i\}_{i\in\Lambda}$, i.e., a family of subsets ω_i such that

$$M = \mathop{\cup}_{i\in\Lambda} \omega_i, \quad \omega_i \cap \omega_j = \varnothing \ (i \neq j).$$

In general Λ may be even uncountable and ω_i are not necessarily closed in M. Let also Λ' be a (possibly empty) subset of Λ and $\Sigma = \{\omega_i\}_{i\in\Lambda'}$ be a subfamily of $\mathcal{P}$ thought of as a set of "singular" elements. Then the pair $\Theta = (\mathcal{P}, \Sigma)$ will be called a *singular partition* of M.

Example 3.1. Let F be a vector field on M, $\mathcal{P}_F$ the set of orbits of F, and Σ_F the set of singular points of F. Then the pair $\Theta_F = (\mathcal{P}_F, \Sigma_F)$ will be called the *singular partition of* F.

Example 3.2. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map, $x \in \mathbb{R}^m$ a point, and $J(f,x)$ the Jacobi matrix of f at x. Then $x \in \mathbb{R}^m$ is called *critical* for f if *rank* $J(f,x) < \min\{m,n\}$. Otherwise x is *regular*. This definition naturally extends to maps between manifolds.

Let M, N be smooth manifolds and $f : M \to N$ a smooth map. Denote by Σ_f the set of critical points of f. Consider the following partition $\mathcal{P}_f$ of M: a subset $\omega \subset M$ belongs to $\mathcal{P}_f$ iff ω is either a critical point of f, or a connected component of the set of the form $f^{-1}(y) \setminus \Sigma_f$ for some $y \in N$. Then the pair $\Theta_f = (\mathcal{P}_f, \Sigma_f)$ will be called the *singular partition of* f. Evidently, every $\omega \in \mathcal{P}_f \setminus \Sigma_f$ is a submanifold of M.

Example 3.3. Assume that, in Example 3.2, $\dim M = \dim N + 1$ and both M and N are orientable. Then every element of $\mathcal{P}_f \setminus \Sigma_f$ is one-dimensional and orientations of M and N allow to *coherently orient all the elements of* $\mathcal{P}_f \setminus \Sigma_f$. Moreover, it is even possible to construct a vector field F on M such that the singular partitions Θ_f and Θ_F coincide.

In particular, let M be an orientable surface and $f : M \to \mathbb{R}$ be a smooth function. Then M admits a symplectic structure, and in this case we can assume that F is the corresponding Hamiltonian vector field of f.

Example 3.4. Let $\mathcal{F}$ be a foliation on M with singular leaves, $\mathcal{P}$ be the set of leaves of $\mathcal{F}$, and Σ be the set of its singular leaves (having non-maximal dimension). Then the pair $\Theta_{\mathcal{F}} = (\mathcal{P}_{\mathcal{F}}, \Sigma_{\mathcal{F}})$ will be called the *singular partition of* $\mathcal{F}$. This example generalizes all previous ones.

Let $\Theta = (\mathcal{P}, \Sigma)$ be a singular partition on M. For every open subset $V \subset M$ denote by $\mathcal{E}(\Theta, V)$ the subset of $C^\infty(V, M)$ consisting of maps $f : V \to M$ such that

(1) $f(\omega_i \cap V) \subset \omega_i$ for all $\omega_i \in \mathcal{P}$ and
(2) f is a local diffeomorphism at every point z belonging to some singular element $\omega \in \Sigma$.

Let also $\mathcal{D}(\Theta, V)$ be the subset of $\mathcal{E}(\Theta, V)$ consisting of *immersions*, i.e., *local diffeomorphisms.* For $V = M$ we abbreviate

$$\mathcal{E}(\Theta) = \mathcal{E}(\Theta, M), \quad \mathcal{D}(\Theta) = \mathcal{D}(\Theta, M).$$

For every $r \in \overline{\mathbb{N}}_0$ denote by $\mathcal{E}_{\mathrm{id}}(\Theta, V)^r$, resp. $\mathcal{D}_{\mathrm{id}}(\Theta, V)^r$, the path-component of the identity inclusion $i_V : V \subset M$ in $\mathcal{E}(\Theta, V)$, resp. in $\mathcal{D}(\Theta, V)$, with respect to the induced C^r_W-topology, see Section 2.

Thus $\mathcal{E}_{\mathrm{id}}(\Theta, V)^r$ (resp. $\mathcal{D}_{\mathrm{id}}(\Theta, V)^r$) consists of maps (resp. immersions) $V \subset M$ which are r-homotopic (r-isotopic) to $i_V : V \subset M$ in $\mathcal{E}(\Theta, V)$ (resp. in $\mathcal{D}(\Theta, V)$).

Evidently,

$$\mathcal{E}_{\mathrm{id}}(\Theta, V)^{\infty} \subset \cdots \subset \mathcal{E}_{\mathrm{id}}(\Theta, V)^{1} \subset \mathcal{E}_{\mathrm{id}}(\Theta, V)^{0}, \tag{3.1}$$

and similar relations hold for $\mathcal{D}_{\mathrm{id}}(\Theta, V)^r$.

The following notion turns out to be useful for studying singular partitions of vector fields.

3.5. **Shift-map of a vector field.** Let F be a vector field on M and

$$\Phi : M \times \mathbb{R} \supset \mathrm{dom}(\Phi) \to M$$

be the local flow of F defined on some open neighbourhood $\mathrm{dom}(\Phi)$ of $M \times 0$ in $M \times \mathbb{R}$. For every open subset $V \subset M$ let also

$$\mathrm{func}(\Phi, V) = \{\alpha \in C^{\infty}(V, \mathbb{R}) \ : \ \Gamma_\alpha \subset \mathrm{dom}(\Phi)\},$$

where $\Gamma_\alpha = \{(x, \alpha(x)) \ : \ x \in V\} \subset M \times \mathbb{R}$ is the graph of α. Then $\mathrm{func}(\Phi, V)$ is the largest subset of $C^{\infty}(V, \mathbb{R})$ on which the following *shift-map* is defined:

$$\varphi_V : \mathrm{func}(\Phi, V) \to C^{\infty}(V, M), \quad \varphi_V(\alpha)(x) = \Phi(x, \alpha(x)),$$

for $\alpha \in \mathrm{func}(\Phi, V)$, $x \in V$.

Lemma 3.6. *Let Θ_F be the singular partition of M by orbits of F. Then*

$$\mathrm{im}(\varphi_V) \subset \mathcal{E}_{\mathrm{id}}(\Theta_F, V)^{\infty}. \tag{3.2}$$

Moreover, if $\mathcal{D}_{\mathrm{id}}(\Theta_F, V)^r \subset \mathrm{im}(\varphi)$ for some $r \in \overline{\mathbb{N}}_0$, then

$$\mathcal{D}_{\mathrm{id}}(\Theta_F, V)^{\infty} = \cdots = \mathcal{D}_{\mathrm{id}}(\Theta_F, V)^{r+1} = \mathcal{D}_{\mathrm{id}}(\Theta_F, V)^{r}.$$

Proof. Let $\alpha \in \mathrm{func}(\Phi, V)$ and $f = \varphi(\alpha)$, i.e., $f(x) = \Phi(x, \alpha(x))$. Then $f(\omega \cap V) \subset \omega$ for every orbit of F. Moreover by [9, Lemma 20] f is a local diffeomorphism at a point $x \in V$ iff $d\alpha(F)(x) \neq -1$, where $d\alpha(F)(x)$ is the Lie derivative of α along F at x. Hence f is so at every singular point z of F, since $d\alpha(F)(z) = 0 \neq -1$, [9, Corollary 21]. Therefore $f \in \mathcal{E}(\Theta_F, V)$. Moreover an ∞-homotopy of f to $i_V : V \subset M$ in $\mathcal{E}(\Theta_F, V)$ can be given by $f_t(x) = \Phi(x, t\alpha(x))$. Thus $f \in \mathcal{E}_{\mathrm{id}}(\Theta_F, V)^{\infty}$.

Finally, suppose that $f \in \mathcal{D}_{\mathrm{id}}(\Theta_F, V)^r$. Then the restriction of f to any non-constant orbit ω of F is an orientation preserving local diffeomorphism. Therefore $d\alpha(F)(z) > -1$ on all of V. Hence $d(t\alpha)(F)(z) > -1$ for all $t \in I$ as well, i.e., $f_t \in \mathcal{D}(\Theta_F, V)$. This implies that $f \in \mathcal{D}_{\mathrm{id}}(\Theta_F, V)^{\infty}$. □

Example 3.7. Let A be a real non-zero $(m \times m)$-matrix, $F(x) = Ax$ be the corresponding linear vector field on $\mathbb{R}^m$, and V be a neighbourhood of the origin 0. Then the shift-map φ_V is given by

$$\varphi(\alpha)(x) = \Phi(x, \alpha(x)) = e^{A\alpha(x)}x.$$

It is shown in [9] that in this case $\mathrm{im}(\varphi_V) = \mathcal{E}_{\mathrm{id}}(\Theta_F, V)^0$. Hence for all $r \in \overline{\mathbb{N}}_0$ we have

$$\mathrm{im}(\varphi_V) = \mathcal{E}_{\mathrm{id}}(\Theta_F, V)^{\infty} = \cdots = \mathcal{E}_{\mathrm{id}}(\Theta_F, V)^{r},$$

$$\mathcal{D}_{\mathrm{id}}(\Theta_F, V)^{\infty} = \cdots = \mathcal{D}_{\mathrm{id}}(\Theta_F, V)^{r}.$$

4. Invariant contractions

Let $\Theta = (\mathcal{P}, \Sigma)$ be a singular partition on a manifold M. We will say that a subset $V \subset M$ is Θ-*invariant*, if it consists of full elements of Θ, i.e., if $\omega \in \mathcal{P}$ and $\omega \cap V \neq \varnothing$, then $\omega \subset V$.

Definition 4.1. Let $Z \subset M$ be a closed subset such that every point $z \in Z$ is a singular element of Θ, i.e., $\{z\} \in \Sigma$. Say that Θ has an **invariant r-contraction to Z** if there exists a closed Θ-invariant neighbourhood V of Z being a smooth submanifold of M and a homotopy $r : V \times I \to V$ such that:

(i) $r_1 = \mathrm{id}_V$;

(ii) r_0 is a proper retraction of V to Z, i.e., $r_0(V) = Z$, $r_0(z) = z$ for $z \in Z$, and $r_0^{-1}(K)$ is compact for every compact $K \subset Z$;
(iii) for every $t \in (0,1]$ the map r_t is a closed C^r-embedding of V into V such that for each $\omega \in \mathcal{P}$ (resp. $\omega \in \Sigma$) its image $r_t(\omega)$ is also an element of $\mathcal{P}$ (resp. Σ);
(iv) for each $z \in Z$ the set $V_z = r_0^{-1}(z)$ is Θ-invariant, and $r_t(V_z) \subset V_z$ for all $t \in I$.

Since V is Θ-invariant, it follows from (iii) that so is its image $r_t(V)$.

Example 4.2. Define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x_1, \ldots, x_m) = \sum_{i=1}^m x_i^2$. Evidently, the singular partition Θ_f consists of the origin 0 and concentric spheres centered at 0. For every $s > 0$ let $V_s = f^{-1}[0, s^2]$ be a closed m-disk of radius s. Then V_s is Θ_f-invariant and its invariant contraction of V_s to $Z = \{0\}$ can be given by $r(x,t) = tx$.

Example 4.3. The previous example can be parametrized as follows. Let $p : M \to Z$ be an m-dimensional vector bundle over a connected, smooth manifold Z. We will identify Z with the image $Z \subset M$ of the corresponding zero-section of p. Suppose that we are given a norm $\|\cdot\|$ on fibers such that the following function $f : M \to \mathbb{R}$ is smooth:

$$f(\xi, z) = \|\xi\|^2, \quad z \in Z, \quad \xi \in p^{-1}(z).$$

Define the following singular partition $\Theta = (\mathcal{P}, \Sigma)$ on M, where $\mathcal{P}$ consists of the subsets $\omega_{s,z} = f^{-1}(s) \cap p^{-1}(z)$ for $s \geq 0$ and $z \in Z$, and $\Sigma = \{\omega_{0,z} = \{z\} \ : \ z \in Z\}$ consists of points of Z. Thus every fiber $p^{-1}(z)$ is Θ-invariant, and the restriction of Θ to $p^{-1}(z)$ is the same as the one in Example 4.2.

Fix $s > 0$ and put $V = f^{-1}[0, s]$. Then V is Θ-invariant and a Θ-invariant contraction of V to Z can be given by $r(\xi, z, t) = (t\xi, z)$.

We will now generalize these examples. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function and suppose that there exists a neighbourhood V of 0 and smooth functions $\alpha_1, \ldots, \alpha_m : V \to \mathbb{R}$ such that

$$f = \alpha_1 \cdot f'_{x_1} + \cdots + \alpha_m \cdot f'_{x_m}. \tag{4.1}$$

Equivalently, let $\Delta(f, 0)$ be the *Jacobi* ideal of f in $C^\infty(V, \mathbb{R})$ generated by partial derivatives of f. Then (4.1) means that $f \in \Delta(f, 0)$.

For instance, let f be *quasi-homogeneous of degree d with weights $s_1, \ldots, s_m$*, i.e., $f(t^{s_1} x_1, \ldots, t^{s_m} x_m) = t^d f(x_1, \ldots, x_m)$ for $t > 0$, see e.g [1, §12]. Equivalently, we may require that the function

$$g(x_1, \ldots, x_m) = f(x_1^{s_1}, \ldots, x_m^{s_m})$$

be homogeneous of degree d. Then the following *Euler identity* holds true:

$$f = \frac{x_1}{s_1} f'_{x_1} + \ \cdots \ + \frac{x_1}{s_m} f'_{x_m}.$$

In particular, f satisfies (4.1). Moreover in the complex analytical case[1] the identity (4.1) characterizes quasi-homogeneous functions, see [21].

Lemma 4.4. *Let $f : M \to \mathbb{R}$ be a smooth function and $z \in M$ be an isolated local minimum of f. Suppose that f satisfies condition* (4.1) *at z, i.e., $f \in \Delta(f, z)$. Then the singular partition Θ_f admits an invariant contraction to z.*

Proof. Since the situation is local, we may assume that $M = \mathbb{R}^m$, $z = 0$ is a unique critical point of f being its global minimum, $f(0) = 0$, and there exists an $\varepsilon > 0$ such that $V = f^{-1}[0, \varepsilon]$ is a smooth compact m-dimensional manifold with boundary $L = f^{-1}(\varepsilon)$.

First we give a precise description of the partition $\mathcal{P}_f$ on V. Let F be any gradient like vector field on V for f, i.e., $df(F)(x) > 0$ for $x \neq 0$. Then following [15, Th. 3.1] we can construct a diffeomorphism

$$\eta : V \setminus \{0\} \to L \times (0, \varepsilon]$$

[1] I would like to thank V. A. Vasilyev for referring me to the paper [21] by K. Saito.

such that $f \circ \eta^{-1}(y,t) = t$ for all $(y,t) \in L \times (0,\varepsilon]$, see Figure 4.1.

Let $CL = L \times [0,\varepsilon]/\{L \times 0\}$ be the cone over L and $L_t = f^{-1}(t)$ for $t \in (0,\varepsilon]$. Since $\operatorname{diam}(L_t) \to 0$ when $t \to 0$, we obtain that η extends to a homeomorphism $\eta : V \to CL$ by $\eta(0) = \{L \times 0\}$.

For every $t \in (0,\varepsilon]$ put $L_t = f^{-1}(t)$. Then η diffeomorphically maps L_t onto $L \times \{t\}$.

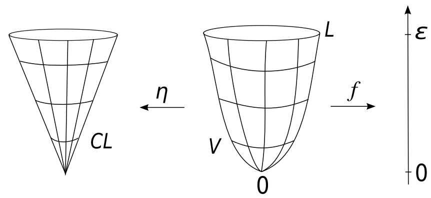


FIGURE 4.1

Lemma 4.5. *L is homotopy equivalent to S^{m-1}.*

We will prove this lemma below. Then it will follow from the generalized Poincaré conjecture that L is homeomorphic with the sphere S^{m-1}, and even diffeomorphic to S^{m-1} if $k = m-1 \neq 4$. For $k = 1,2$ this statement is rather elementary, for $k = 3$ this follows from a recent work of G. Perelman [18, 19], for $k = 4$ from M. Freedman [3], and for $k \geq 5$ from S. Smale [23], see also [14].

In particular, every L_t is connected, whence the partition $\mathcal{P}_f$ on V consists of a unique singular element $\{0\} \in \Sigma_f$ and sets L_t, $t \in (0,\varepsilon]$.

Let us recall the definition of η. Notice that every orbit of F starts at 0 and transversely intersects every L_t. For each $x \in V \setminus \{0\}$ denote by $q(x)$ a unique point of the intersection of the orbit of x with $L = L_\varepsilon = \partial V$. Then $\eta : V \setminus \{0\} \to L \times (0,\varepsilon]$ can be given by the following formula:

$$\eta(x) = (q(x), f(x)).$$

Also notice that if $\phi : [0,\varepsilon] \to [0,\varepsilon]$ is a (not necessarily surjective) C^∞ embedding such that $\phi(0) = 0$, then we can define the embedding

$$w_\phi : CL \to CL, \quad w_\phi(y,t) = (y, \phi(t))$$

and therefore the embedding $r_\phi = \eta \circ w_\phi \circ \eta^{-1} : V \to V$. Then r_ϕ is C^∞ on $V \setminus 0$ and diffeomorphically maps L_t onto $L_{q(t)}$ for all $t \in (0,\varepsilon]$. Moreover $r_\phi(0) = 0$, but in general r_ϕ is not even smooth at 0.

Suppose now that $f \in \Delta(f,0)$, i.e., we have a presentation (4.1). Consider the following vector field

$$F = \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_m \frac{\partial}{\partial x_m}.$$

Then (4.1) means that $f = df(F)$. Since $f(x) > 0$ for $x \neq 0$, it follows that F is a gradient like vector field for f. Therefore we can construct a homeomorphism $\eta : V \to CL$ using F as above. It follows from [10] that in our case this η has the following feature:

- if $\phi : [0,\varepsilon] \to [0,\varepsilon]$ is a C^∞ embedding such that $\phi(0) = 0$, then the corresponding embedding $r_\phi : V \to V$ is a *diffeomorphism onto its image*. Moreover, if ϕ_s, $(s \in I)$, is a C^∞ isotopy, then so is $r_{\phi_s} : V \to V$.

In particular, consider the following homotopy

$$\phi : [0,\varepsilon] \times I \to [0,\varepsilon], \quad \phi(t,s) = t(1-s)$$

which contracts $[0,\varepsilon]$ to a point and being an isotopy for $t > 0$. Then the induced homotopy $r : V \times I \to V$ is an invariant ∞-contraction of Θ_f to 0. □

Proof of Lemma 4.5. It suffices to establish that

$$\pi_k L \approx \pi_k S^{m-1} = \begin{cases} 0, & k = 0, \dots, m-2, \\ \mathbb{Z}, & k = m-1. \end{cases} \tag{4.2}$$

Then the generator $\mu : S^{m-1} \to L$ of $\pi_{m-1} L \approx \mathbb{Z}$ will yield isomorphisms of the homotopy groups $\pi_k S^{m-1} \approx \pi_k L$ for all $k \leq m-1 = \dim S^{m-1} = \dim L$. Now by the well-known Whitehead's theorem μ will be a homotopy equivalence between S^{m-1} and L.

For the calculation of homotopy groups of L consider the exact sequence of homotopy groups of the pair (V, L):

$$\cdots \to \pi_{k+1}(V, L) \to \pi_k L \to \pi_k V \to \cdots \tag{4.3}$$

Since V is homeomorphic with the cone CL, V is contractible, whence $\pi_k V = 0$ for all $k \geq 0$.

Moreover, $\pi_{k+1}(V, L) = 0$ for all $k = 0, \dots, m-2$. Indeed, let

$$\xi : (D^{k+1}, S^k) \to (V, L)$$

be a continuous map. We have to show that ξ is homotopic (as a map of pairs) to a map into L. Since $k+1 \leq m-1 < \dim V$, ξ is homotopic to a map into $V \setminus \{0\}$. But L is a deformation retract of $V \setminus \{0\}$, therefore ξ is homotopic to a map into L.

Now it follows from (4.3) that $\pi_k L = 0$ for $k = 0, \dots, m-2$. Hence from Hurewicz's theorem we obtain that $\pi_{m-1} L \approx H_{m-1}(L, \mathbb{Z})$. It remains to note that L is a connected closed orientable $(m-1)$-manifold, whence (by Poincaré duality) $H_{m-1}(L, \mathbb{Z}) \approx \mathbb{Z}$. This proves (4.2). $\square$

Remark 4.6. If f is a quasi-homogeneous function of degree d with weights $s_1, \dots, s_m$, then we can define an invariant contraction by

$$r(x_1, \dots, x_m, t) = (t^{s_1} x_1, \dots, t^{s_m} x_m).$$

If f is *homogeneous*, then we can even put $r(x, t) = tx$.

Theorem 4.7. *Let $\Theta = (\mathcal{P}, \Sigma)$ be a singular partition on a manifold M, and $Z \subset M$ be a closed subset such that every $z \in Z$ is a singular element of $\mathcal{P}$, i.e., $\{z\} \in \Sigma$. Suppose that Θ has an invariant ∞-contraction to Z defined on a Θ-invariant neighbourhood V of Z. Let also $h \in \mathcal{E}(\Theta, V)$ be a map fixed outside some neighbourhood U of z such that $\overline{U} \subset \mathrm{Int} V$. Then $h \in \mathcal{E}_{\mathrm{id}}(\Theta, V)^0$, though h not necessarily belongs to $\mathcal{E}_{\mathrm{id}}(\Theta, V)^r$ for some $r \geq 1$.*

Remark 4.8. Let Θ_f be the singular partition of $f(x) = \|x\|^2$ as in Example 4.2, V be the unit n-disk centered at 0 and $r : V \times I \to V$ be the invariant contraction of Θ_f to a point $Z = \{0\}$ defined by $r(x, t) = tx$. Then a 0-homotopy between h and id_V can be defined by

$$H_t(x) \begin{cases} t h(\frac{x}{t}), & t > 0, \\ 0, & t = 0. \end{cases}$$

c.f. [5, Ch. 4, Theorems 5.3 & 6.7]. Theorem 4.7 generalizes this example.

Proof of Theorem 4.7. Let $r : V \times I \to V$ be an invariant ∞-contraction of Θ_F to Z. Define the following map $H : V \times I \to M$ by

$$H(x, t) = \begin{cases} r_t \circ h \circ r_t^{-1}(x), & \text{if } t > 0 \text{ and } x \in r_t(V), \\ x, & \text{otherwise.} \end{cases}$$

We claim that H is a 0-homotopy (i.e. just a homotopy) between h and the identity inclusion $i_V : V \subset M$ in $\mathcal{E}(\Theta, V)$. To make this more obvious we rewrite the formulas for H in another way.

The homotopy r can be regarded as the composition

$$r = p_1 \circ \widetilde{r} : \ V \times I \xrightarrow{\widetilde{r}} V \times I \xrightarrow{p_1} V,$$

where $\widetilde{r}$ is the following level-preserving map

$$\widetilde{r} : V \times I \to V \times I, \quad \widetilde{r}(x,t) = (r(x,t),t),$$

and $p_1 : V \times I \to V$ is the projection to the first coordinate. It follows from the definition

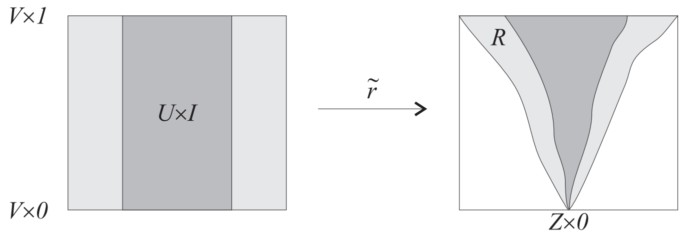


FIGURE 4.2

that r yields a level-preserving embedding $V \times (0,1]$ to $V \times I$, see Figure 4.2. Denote

$$R' = \widetilde{r}(V \times (0,1]), \quad R = \widetilde{r}(V \times I).$$

Then $R \setminus R' = Z \times 0$. Define also the following map

$$\widetilde{h} : V \times I \to V \times I, \quad \widetilde{h}(x,t) = (h(x),t).$$

In these terms, the homotopy H is defined by

$$H = p_1 \circ \widetilde{H} : \ V \times I \xrightarrow{\widetilde{H}} V \times I \xrightarrow{p_1} V,$$

where $\widetilde{H} : V \times I \to V \times I$ is a level-preserving map given by

$$\widetilde{H}(x,t) = \begin{cases} \widetilde{r} \circ \widetilde{h} \circ \widetilde{r}^{\,-1}(x,t), & (x,t) \in R', \\ \qquad (x,t), & (x,t) \in (V \times I) \setminus R'. \end{cases}$$

Now we can prove that H has the desired properties.

Since $r_1 = \mathrm{id}_V$, we have $H_1 = h$. Moreover $H_0 = \mathrm{id}_V$.

1. Continuity of $\widetilde{H}$ on $V \times (0,1]$. Notice that $\widetilde{r} \circ \widetilde{h} \circ \widetilde{r}^{-1}$ is well-defined and continuous on R'. Moreover, since h is fixed on $V \setminus U$, it follows that $\widetilde{h}$ is fixed on $(V \setminus U) \times I$, whence $\widetilde{r} \circ \widetilde{h} \circ \widetilde{r}^{-1}$ is fixed on the subset $\widetilde{r}\big((V \setminus U) \times (0,1]\big) \subset R'$. This implies that $\widetilde{H}$ is continuous on $V \times (0,1]$.

2. Continuity of $\widetilde{H}$ when $t \to 0$. Let $z \in V$. Then $\widetilde{H}(z,0) = (z,0)$.

Suppose that $z \in V \setminus Z$. Since Z is closed in V, $\widetilde{H}$ is also fixed and therefore continuous on some neighbourhood of $(z,0)$ in $(V \times I) \setminus R$.

Let $z \in Z$ and let W be a neighbourhood of $(z,0)$ in $V \times I$. We have to find another neighbourhood W' of $(z,0)$ such that $\widetilde{H}(W') \subset W$.

Recall that for every $y \in Z$ we denoted $V_y = r_0^{-1}(y)$. Then V_y is compact and Θ-invariant.

Claim. *There exist $\varepsilon > 0$ and an open neighbourhood N of z in V such that $\overline{N} \times [0,\varepsilon] \subset W$ and*

$$\widetilde{r}(V_y \times [0,\varepsilon]) \subset W, \quad (y \in \overline{N} \cap Z). \tag{4.4}$$

Proof. Let N be an open neighbourhood of z such that $\overline{N}$ is compact and $\overline{N} \times 0 \subset W$. Denote

$$Q \ = \ r_0^{-1}(\overline{N} \cap Z) \ = \ \bigcup_{y \in \overline{N} \cap Z} r_0^{-1}(y) \ = \ \bigcup_{y \in \overline{N} \cap Z} V_y.$$

Then Q is a compact subset of V, and $\widetilde{r}^{-1}(W)$ is an open neighbourhood of $Q \times 0$ in $V \times I$. Hence there exists $\varepsilon > 0$ such that $Q \times [0, \varepsilon] \subset \widetilde{r}^{-1}(W)$. This implies (4.4). Decreasing ε if necessary we can also assume that $\overline{N} \times [0, \varepsilon] \subset W$ as well. □

Denote $W' = N \times [0, \varepsilon)$. We claim that $\widetilde{H}(W') \subset W$.

Let $(x, t) \in W'$. If either $(x, t) \in W' \setminus R'$ or $t = 0$, then $\widetilde{H}(x, t) = (x, t) \in W' \subset W$.

Suppose that $(x, t) \in W' \cap R'$. Then $t > 0$. Let also $y = r_0(x) \in Z$. Then $\widetilde{r}^{\,-1}(x, t) \in V_y \times t$ for all $t \in I$. Hence

$$\widetilde{r} \circ \widetilde{h} \circ \widetilde{r}^{\,-1}(x, t) \;\in\; \widetilde{r} \circ \widetilde{h}(V_y \times t) \;\subset\; \widetilde{r}(V_y \times t) \;\overset{(4.4)}{\subset}\; W.$$

In the second inclusion we have used a Θ-invariance of V_y and the assumption that $h \in \mathcal{E}(\Theta, V)$.

3. Proof that $H_t \in \mathcal{E}(\Theta, V)$ **for** $t \in I$**.** We have to show that (i) for every $t \in I$ the mapping H_t is C^∞, (ii) $H_t(\omega) \subset \omega$ for every element $\omega \in \mathcal{P}$ included in V, and (iii) H_t is a local diffeomorphism at every point z belonging to some $\omega \in \Sigma$.

(i) Since r_t, $t > 0$, is C^∞ and h is fixed on $V \setminus U$, it follows that H_t is C^∞ as well.

(ii) Let $\omega \subset V$ be an element of $\mathcal{P}$ (resp. Σ).

If $\omega \subset V \setminus r_t(V)$, then H_t is fixed on ω, whence $H_t(\omega) = \omega \in \mathcal{P}$ (resp. Σ).

Suppose that $\omega \subset r_t(V)$. Since $r_t(V)$ is Θ-invariant, $\omega = r_t(\omega')$ for some another element $\omega' \in \mathcal{P}$ (resp. Σ). Then $h(\omega') \subset \omega'$, whence

$$H_t(\omega) \;=\; r_t \circ h \circ r_t^{-1}(\omega) \;=\; r_t \circ h(\omega') \;\subset\; r_t(\omega') \;=\; \omega.$$

(iii) Suppose that $\omega \in \Sigma$ and let $x \in \omega$.

If $x \in V \setminus r_t(U)$, then H_t is fixed in a neighbourhood of x, and therefore it is a local diffeomorphism at x.

Suppose that $x = r_t(x') \in r_t(U)$ for some $x' \in U$ and let $\omega' \in \Sigma$ be the element containing x'. Then h is a local diffeomorphism at x', whence $H_t = r_t \circ h \circ r_t^{-1}$ is a local diffeomorphism at x. □

5. Stabilizers of smooth functions

Let $B^m \subset \mathbb{R}^m$ be the unit disk centered at the origin 0, $S^{m-1} = \partial B^m$ be its boundary sphere, $f : B^m \to \mathbb{R}$ be a C^∞ function, and Θ_f be the singular partition of f.

Let $\mathcal{S}(f) = \{h \in \mathcal{D}(B^m) \,:\, f \circ h = f\}$ be the stabilizer of f with respect to the right action of the group $\mathcal{D}(B^m)$ of diffeomorphisms of B^m on the space $C^\infty(B^m, \mathbb{R})$. Denote by $\mathcal{S}^+(f)$ the subgroup of $\mathcal{S}(f)$ consisting of orientation preserving diffeomorphisms. For $r \in \overline{\mathbb{N}}_0$ let also $\mathcal{S}_{\mathrm{id}}(f)^r$ be the identity component of $\mathcal{S}(f)$ with respect to the C^r-topology. Then

$$\mathcal{S}_{\mathrm{id}}(f)^\infty \;\subset \cdots \subset\; \mathcal{S}_{\mathrm{id}}(f)^{r+1} \;\subset\; \mathcal{S}_{\mathrm{id}}(f)^r \;\subset \cdots \subset\; \mathcal{S}_{\mathrm{id}}(f)^0 \;\subset\; \mathcal{S}^+(f).$$

Theorem 5.1. *Let $m \geq 2$, $f : B^m \to [0, 1]$ be a C^∞ function such that 0 is a unique critical point of f being its global minimum, $f(0) = 0$, and $f(S^{m-1}) = 1$. Denote by $\mathcal{S}$ the subgroup of $\mathcal{S}(f)$ consisting of diffeomorphisms h such that $h|_{S^{m-1}} : S^{m-1} \to S^{m-1}$ is C^∞-isotopic to $\mathrm{id}_{S^{m-1}}$. Suppose also that the singular partition Θ_f of f has an invariant ∞-contraction to 0. Then $\mathcal{S} \subset \mathcal{S}_{\mathrm{id}}(f)^0$.*

If $m = 2, 3, 4$, then $\mathcal{S} = \mathcal{S}_{\mathrm{id}}(f)^0 = \mathcal{S}^+(f)$.

For the proof we need the following two simple standard statements concerning smoothing homotopies at the beginning and at the end, see e.g. [20, pp. 74 & 118] and [16, p. 205]. Let M be a closed smooth manifold.

Claim 5.2. *Let $a, b, c \in \mathbb{R}$ be numbers such that $0 < a < b < c$, and $N = M \times (0, c]$. Then we have a foliation on N by submanifolds $M \times t$, $t \in (0, c]$. Let also $h : N \to N$ be a C^∞ leaf preserving diffeomorphism, i.e., $h(x, t) = (\phi(x, t), t)$ for some C^∞ map $\phi : M \times (0, c] \to M$ such that for every $t \in (0, c]$ the map $\phi_t : M \to M$ is a diffeomorphism.*

Then there exists a leaf preserving isotopy relatively to $M \times (0, a]$ of h to a diffeomorphism $\hat{h}(x, t) = (\hat{\phi}(x, t), t)$ such that $\hat{\phi}_t = \phi_c$ for all $t \in [b, c]$.

Proof. Let $\mu : (0, c] \to (0, c]$ be a C^∞ function such that $\mu(t) = t$ for $t \in (0, a]$ and $\mu(t) = c$ for $t \in [b, c]$, see Figure 5.1a). Define the following lead preserving isotopy $H : N \times I \to N$ by

$$H_s(x, t) = (\phi(x, (1 - s)t + s\mu(t)), t).$$

Then it easy to see that $H_0 = \mathrm{id}_N$, $H_t = h$ on $M \times (0, a]$ and $\hat{h} = H_1$ satisfies conditions of our claim. □

Claim 5.3. *Let $d < e \in (0, 1)$ and $G : M \times I \to M$ be a C^∞ homotopy (isotopy). Then there exists another C^∞ homotopy (isotopy) $G' : M \times I \to M$ such that $\hat{G}_t = G_0$ for $t \in [0, d]$ and $\hat{G}_t = G_1$ for $t \in [e, 1]$.*

Proof. Take any C^∞ function $\nu : I \to I$ such that $\nu[0, d] = 0$ and $\nu[e, 1] = 1$, and put $\hat{G}_t = G_{\nu(t)}$, see Figure 5.1b). □

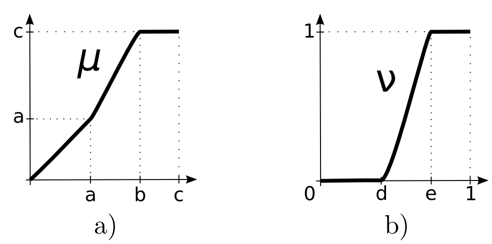


FIGURE 5.1

Proof of Theorem 5.1. Since 0 is a unique critical point of f and f is constant on S^{m-1}, it follows from the arguments of the proof of Lemma 4.4 that there exists a diffeomorphism $\eta : D^m \setminus 0 \to S^{m-1} \times (0, 1]$ such that $f \circ \eta^{-1}(y, t) = t$. In particular, for every $t \in [0, 1]$ the set $f^{-1}(t)$ is diffeomorphic with S^{m-1}. Since S^{m-1} is connected, we obtain that $\mathcal{D}(\Theta_f) = \mathcal{S}(f)$.

Therefore $\mathcal{D}_{\mathrm{id}}(\Theta_f)^r = \mathcal{S}_{\mathrm{id}}(f)^r$ for all $r \in \overline{\mathbb{N}}_0$.

By assumption there exists an invariant ∞-contraction of Θ_f to 0 defined on some Θ_f-invariant neighbourhood V of 0. Therefore we can assume that $V = f^{-1}[0, 2c]$ for some $c \in (0, \frac{1}{2})$.

Lemma 5.4. *There exists a C^∞-isotopy of h in $\mathcal{S}$ to a diffeomorphism $\hat{h}$ fixed on $f^{-1}[c, 1]$. Then is follows from Theorem 4.7 that $\hat{h}$ and therefore h belong to $\mathcal{D}_{\mathrm{id}}(\Theta_f)^0 = \mathcal{S}_{\mathrm{id}}(f)^0$.*

Proof. Since h preserves f, it follows that the following diffeomorphism

$$g = \eta \circ h \circ \eta^{-1} : S^{m-1} \times (0, 1] \to S^{m-1} \times (0, 1]$$

is leaf preserving, i.e., $h(S^{m-1} \times t) = S^{m-1} \times t$ for all $t \in (0, 1]$. Then by Claim 5.2 we can assume that $g|_{S^{m-1} \times t} = h|_{S^{m-1}}$ for all $t \in [0.5, 1]$.

Take any $a \in (0, c)$. It suffices to find a leaf preserving isotopy relatively to $S^{m-1} \times (0, a]$ of g to a diffeomorphism $\hat{g}$ fixed on $S^{m-1} \times [c, 1]$. This isotopy will yield an isotopy relatively to $f^{-1}[0, a]$ of h in $\mathcal{S}$ to a diffeomorphism $\hat{h}$ which is fixed on $f^{-1}[c, 1]$.

By assumptions of our theorem there exists a C^∞ isotopy $G : S^{m-1} \times [1, 2] \to S^{m-1}$ such that $G_1 = h|_{S^{m-1}}$ and $G_2 = \mathrm{id}_{S^{m-1}}$. By Claim 5.3 we can assume that $G_t = h|_{S^{m-1}}$

for all $t \in [1, 1.5]$. Hence g and G yield the following C^∞ leaf preserving diffeomorphism:

$$T : S^{m-1} \times (0, 2] \to S^{m-1} \times (0, 2], \quad T(y, s) = \begin{cases} g(y, s), & s \in (0, 1], \\ G(y, s), & s \in [1, 2]. \end{cases}$$

Notice that $T(y, 2) = G(y, 2) = y$. Then, by Claim 5.3, T is isotopic via a leaf preserving isotopy relatively $S^{m-1}\times(0, a]$ to a diffeomorphism $\hat{T}$ which is fixed on $S^{m-1}\times (c, 2]$. Denote $\hat{g} = \hat{T}|_{S^{m-1}\times(0,1]}$. The restriction of this isotopy to $S^{m-1} \times (0, 1]$ gives a leaf preserving isotopy relatively $S^{m-1} \times (0, a]$ of g to a diffeomorphism $\hat{g}$ with desired properties. The construction of homotopy is schematically presented in Figure 5.2. □

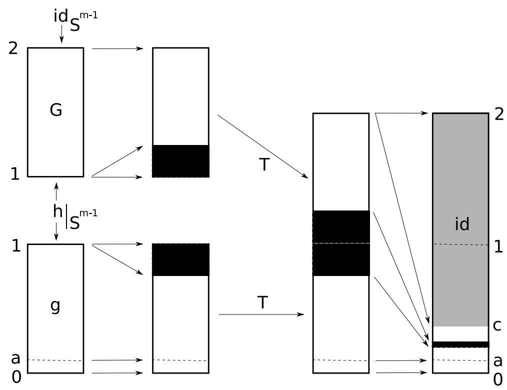


Figure 5.2

Suppose now that $m = 2, 3, 4$. Then every orientation preserving diffeomorphism of S^{m-1} is C^∞-isotopic to $\mathrm{id}_{S^{m-1}}$, whence $\mathcal{S} = \mathcal{S}^+(f)$, and therefore $\mathcal{S} = \mathcal{S}_{\mathrm{id}}(f)^0 = \mathcal{S}^+(f)$. For $m = 2$ this is rather trivial, for $m = 3$ is proved by S. Smale [22], and for $m = 4$ by A. Hatcher [4].

If $m \geq 5$, then $\mathcal{D}^+(S^{m-1})$ is not connected in general, see e.g. [17], and therefore Theorem 4.7 is not applicable. □

6. Linear symmetries of homogeneous polynomials

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p \geq 2$ given by (1.1),

$$f(x, y) = \pm \prod_{i=1}^{l} L_i^{\alpha_i}(x, y) \cdot \prod_{j=1}^{k} Q_j^{\beta_j}(x, y).$$

Denote

$$\mathcal{LS}(f) = \mathcal{S}^+(f) \cap \mathrm{GL}^+(2, \mathbb{R}).$$

Thus $\mathcal{LS}(f)$ consists of preserving orientation linear automorphisms $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \circ h = f$. Also notice that $\mathcal{LS}(f)$ is a closed subgroup of $\mathrm{GL}^+(2, \mathbb{R})$, and therefore it is a Lie group. Denote by $\mathcal{LS}(f)_0$ the connected component of the unit matrix $\mathrm{id}_{\mathbb{R}^2}$ in $\mathcal{LS}(f)$.

In this section we recall the structure of $\mathcal{LS}(f)$. Notice that we may make linear changes of coordinates to reduce f to a convenient form. Then $\mathcal{LS}(f)$ will change to a conjugate subgroup in $\mathrm{GL}^+(2, \mathbb{R})$.

The following statement is evident.

Lemma 6.1. *If* $\deg f$ *is even, then* $f(-z) \equiv f(z)$*, i.e.,* $-\mathrm{id}_{\mathbb{R}^2} \in \mathcal{LS}(f)$*. Therefore in this case* $\mathcal{LS}(f)$ *is a non-trivial group.*

We will distinguish the following five cases of f.

(A) $l=1$, $k=0$, $f=L_1^{\alpha_1}$. By a linear change of coordinates we can assume that $L_1(x,y)=y$ and thus $f(x,y)=y^{\alpha_1}$. Then

$$\mathcal{LS}_0(y^{\alpha_1}) = \left\{ \left(\begin{smallmatrix} a & 0 \\ b & 1 \end{smallmatrix}\right) \ : \ a>0 \right\}.$$

If α_1 is odd then $\mathcal{LS}_0 = \mathcal{LS}(f)$, otherwise, $\mathcal{LS}(f)$ consists of two connected components $\mathcal{LS}_0$ and $-\mathcal{LS}_0$.

(B) $l=2$, $k=0$, $f=L_1^{\alpha_1} L_2^{\alpha_2}$. By a linear change of coordinates we can assume that $L_1(x,y)=x$, $L_2(x,y)=y$ and thus $f(x,y)=x^{\alpha_1} y^{\alpha_2}$. Then

$$\mathcal{LS}_0(x^{\alpha_1} y^{\alpha_2}) = \left\{ \begin{pmatrix} e^{\alpha_2 t} & 0 \\ 0 & e^{-\alpha_1 t} \end{pmatrix} \ : \ t\in\mathbb{R} \right\}.$$

Moreover, $\mathcal{LS}(f)/\mathcal{LS}_0$ is isomorphic with some subgroup of $\mathbb{Z}_4$ generated by the rotation of $\mathbb{R}^2$ by $\pi/2$.

(C) $l=0$, $k=1$, $f=Q_1^{\beta_1}$. By a linear change of coordinates we can assume that $Q_1(x,y)=x^2+y^2$, whence $f(x,y)=(x^2+y^2)^{\beta_1}$. Then

$$\mathcal{LS}(x^2+y^2) = SO(2,\mathbb{R}) = \left\{ \left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right) \ : \ t\in[0,2\pi) \right\}.$$

The above statements are elementary and we left them to the reader. Notice also that, in the cases (A)–(C), $l+2k\leq 2$. The remaining two cases are the following:

(D) $l=0$, $k\geq 2$, $f=Q_1^{\beta_1}\cdots Q_k^{\beta_k}$. In this case $\deg f$ is even, whence $\mathcal{LS}(f)$ is non-trivial.

(E) $l\geq 1$, $l+2k\geq 3$.

Lemma 6.2. *In the cases* (D) *and* (E), *$\mathcal{LS}(f)$ is a finite cyclic subgroup of* $\mathrm{GL}^+(2,\mathbb{R})$. *Moreover, in the case* (E), *$\mathcal{LS}(f)$ is a subgroup of $\mathbb{Z}_{2l}$.*

Proof. In fact cyclicity of $\mathcal{LS}(f)$ for the case $l+2k-1\geq 2$ can be extracted from the paper of W. C. Huffman, [6], where the symmetries of complex binary forms are classified. Regard $f:\mathbb{C}^2\to\mathbb{C}$ as a complex polynomial with real coefficients. Then by [6] the subgroup $\mathcal{LS}_{\mathbb{C}}(f)$ of $GL(2,\mathbb{C})$ consisting of *complex* symmetries of f turned out to be of one of the following types: cyclic, dihedral, tetrahedral, octahedral, and icosahedral. Notice $\mathcal{LS}(f)$ is the subgroup of $\mathcal{LS}_{\mathbb{C}}(f)$ consisting of *preserving orientation real* symmetries of f, i.e., automorphisms which also leave invariant the 2-plane $\mathbb{R}^2\subset\mathbb{C}^2$ of real coordinates and preserve its orientation. Then it follows from the structure of symmetries of regular polyhedrons that $\mathcal{LS}(f)$ must be cyclic.

Nevertheless, since we need a very particular case of [6] and for the sake of completeness, we will present a short elementary proof. It suffices to show that $\mathcal{LS}(f)$ is finite, see 6.6. This will imply that $\mathcal{LS}(f)$ is isomorphic to a finite subgroup of $SO(2)$, and therefore is cyclic. Also notice that the fact that $\mathcal{LS}(f)$ is discrete also follows from [12]. First we establish the following three statements.

Claim 6.3. *Let $h\in\mathrm{GL}^+(2,\mathbb{R})$ and Q be a positive definite quadratic form such that $Q\circ h = t\,Q$ for some $t>0$. Then $t=\det(h)$.*

Proof. By a linear change of coordinates we can assume that $Q(z)=|z|^2$. Then $h(z)=\sqrt{t}e^{i\psi}z$ for some $\psi\in\mathbb{R}$, hence $\det(h)=t$. □

Claim 6.4. *Let Q_1,Q_2 be a positive definite quadratic form such that $\frac{Q_1}{Q_2}\not\equiv$ const. Let also $h\in\mathrm{GL}^+(2,\mathbb{R})$ be such that $Q_i\circ h = tQ_i$ for $i=1,2$, where $t=\det(h)$. Then $h(z)=\pm\sqrt{t}\,z$.*

Proof. We can assume that $Q_1(x,y)=x^2+y^2$ and $Q_2(x,y)=ax^2+by^2$, where $a,b>0$ and either $a\neq 1$ or $b\neq 1$. Denote $g(z)=h(z)/\sqrt{t}$. Then $Q_i\circ g=Q_i$, i.e., g preserves every circle $x^2+y^2=$ const and every ellipse $ax^2+by^2=$ const. Therefore $g=\pm\mathrm{id}_{\mathbb{R}^2}$, and $h(z)=\pm\sqrt{t}z$. □

Claim 6.5. *If* $t \cdot \mathrm{id}_{\mathbb{R}^2} \in \mathcal{LS}(f)$ *for some* $t \in \mathbb{R}$, *then* $t = \pm 1$.

Proof. Let $z \in \mathbb{R}^2$ be such that $f(z) \neq 0$. Since f is homogeneous, we have $f(z) = f(tz) = t^{\deg f} f(z)$, whence $t = \pm 1$. □

Let $h \in \mathcal{LS}(f)$. Since L_i and Q_j are irreducible over $\mathbb{R}$, so are $L_i \circ h$ and $Q_j \circ h$. Therefore the identity $f \circ h = f$ implies that "h permutes L_i and Q_j up to non-zero multiples". This means that for every i there exist i' and $s_i \in \mathbb{R} \setminus \{0\}$, and for every j there exist j' and $t_j > 0$ such that

$$L_i(h(z)) = s_i\, L_{i'}(z), \quad Q_j(h(z)) = t_j\, Q_{j'}(z).$$

Denote by $\mathrm{Sym}(r)$ the group of permutations of r symbols. Then we have a well-defined homomorphism

$$\mu : \mathcal{LS}(f) \to \mathrm{Sym}(l) \times \mathrm{Sym}(k)$$

associating to every $h \in \mathcal{LS}$ its permutations of L_i and Q_j.

Claim 6.6. *If* $l + 2k \geq 3$, *then* $\ker \mu \subset \{\pm \mathrm{id}_{\mathbb{R}^2}\}$, *whence* $\mathcal{LS}(f)$ *is a finite group.*

Proof. Let $h \in \ker \mu$. Thus $L_i \circ h = s_i L_i$ and $Q_j \circ h = t_j Q_j$ for all i, j. We will show that $h = t \cdot \mathrm{id}_{\mathbb{R}^2}$ for some $t \neq 0$. Then it will follow from Claim 6.5 that $h = \pm\mathrm{id}$.

Notice that h preserves every line $\{L_i = 0\}$ and thus has l distinct eigen directions.

a) Therefore if $l \geq 3$, then $h = t \cdot \mathrm{id}_{\mathbb{R}^2}$ for some $t \in \mathbb{R}$.

b) Moreover, if $k \geq 2$, then by Claim 6.4, we also have $h = \pm\mathrm{id}_{\mathbb{R}^2}$.

c) Suppose that $1 \leq l \leq 2$ and $k = 1$. We can assume that $Q(z) = |z|^2$ and that $h(z) = t\, e^{i\psi} z$ for some $t > 0$ and $\psi \in \mathbb{R}$. Since h has $l \geq 1$ eigen directions, we obtain that $h(z) = \pm\sqrt{t} z$. □

Thus $\mathcal{LS}(f) \approx \mathbb{Z}_n$ for some $n \in \mathbb{N}$. Let h be a generator of $\mathcal{LS}(f)$. Then we can assume that $h(z) = e^{2\pi i/n} z$. It remains to prove the latter statement.

Claim 6.7. *Suppose that* $l \geq 1$. *Then* n *divides* $2l$, *whence* $\mathcal{LS}(f)$ *is isomorphic to a subgroup of* $\mathbb{Z}_{2l}$.

Proof. Since $f \circ h = f$, it follows that $h(f^{-1}(0)) = f^{-1}(0)$. By assumption $l \geq 1$, whence $f^{-1}(0) = \mathop{\cup}\limits_{i=1}^{l} \{L_i = 0\}$ is the union of l lines passing through the origin. This set can be viewed as the union of $2l$ rays starting at the origin, and these rays are cyclically shifted by h. Moreover, if h^t preserves at least one of these rays, then $h^t = \mathrm{id}_{\mathbb{R}^2}$. Therefore n divides $2l$. □

Lemma 6.2 is completed. □

7. Proof of Theorem 1.1.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p \geq 1$ given by (1.1)

$$f(x, y) = \pm \prod_{i=1}^{l} L_i^{\alpha_i}(x, y) \cdot \prod_{j=1}^{k} Q_j^{\beta_j}(x, y).$$

We will refer to the cases (A)–(E) of f considered in the previous section. We have to show that $\mathcal{S}_{\mathrm{id}}(f)^{\infty} = \cdots = \mathcal{S}_{\mathrm{id}}(f)^1$ and that $\mathcal{S}_{\mathrm{id}}(f)^1 \neq \mathcal{S}_{\mathrm{id}}(f)^0$ iff f is of the case (D).

Our first aim is to identify $\mathcal{S}_{\mathrm{id}}(f)^r$ with the group $\mathcal{D}_{\mathrm{id}}(\Theta_G)^r$ for some vector field G on $\mathbb{R}^2$, see Lemma 7.1. Then we will use the shift map of G. Denote

$$D = \pm \prod_{i=1}^{l} L_i^{\alpha_i - 1} \cdot \prod_{j=1}^{k} Q_j^{\beta_j - 1}.$$

Then

$$f = L_1 \cdots L_l \cdot Q_1 \cdots Q_q \cdot D$$

and it is easy to see that D is the greatest common divisor of f'_x and f'_y in the ring $\mathbb{R}[x,y]$.

Let $F = -f'_y \frac{\partial}{\partial x} + f'_x \frac{\partial}{\partial x}$ be the Hamiltonian vector field of f on $\mathbb{R}^2$ and

$$G = F/D = -(f'_y/D)\,\frac{\partial}{\partial x} + (f'_x/D)\,\frac{\partial}{\partial x}.$$

We will call G the *reduced* Hamiltonian vector field of f. Notice that

$$\deg G = l + 2k - 1$$

and the coordinate functions of G are relatively prime in $\mathbb{R}[x,y]$.

As noted in Example 3.3 the singular partitions Θ_f and Θ_F coincide. Let us describe the singular partition $\Theta_G = (\mathcal{P}_G, \Sigma_G)$. Recall that elements of $\mathcal{P}_G$ are the orbits of G and Σ_G consists of zeros of G. Consider the following cases, see Figure 7.1.

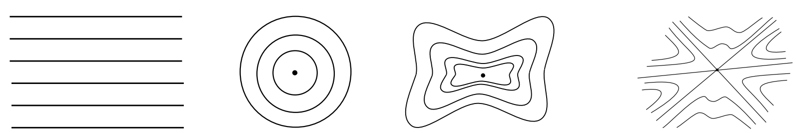
(A) $f = y^{\alpha_1}$. Then $D = y^{\alpha_1 - 1}$ and $F(x,y) = \alpha_1 y^{\alpha_1-1}\frac{\partial}{\partial y}$. Hence $G(x,y) = \alpha_1 \frac{\partial}{\partial y}$ is a constant vector field and the partition Θ_G consists of horizontal lines $\{y = \mathrm{const}\}$ being non-singular elements of Θ_G.

(C) and (D) $f = Q_1^{\beta_1} \cdots Q_k^{\beta_k}$. In this case $\Theta_F = \Theta_G$. The origin is a unique singular element of Θ_G. All other elements of Θ_G are level-sets $f^{-1}(c)$ of f for $c > 0$.

(B) and (E) either $l = 2$ and $k = 0$ or $l \geq 1$ and $k \geq 1$. In both cases the set of singular points of Θ_F consist of the origin and the set

$$D^{-1}(0) = \underset{i\,:\,\alpha_i \geq 2}{\cup} \{L_i = 0\}$$

of zeros of D being the union of those lines $\{L_i = 0\}$ for which L_i is a multiple factor of f. Since after division of F by D the coordinate functions of $G = F/D$ are relatively prime, it follows that 0 is a unique singular element of Θ_G. Hence non-singular elements of Θ_G are the connected components $f^{-1}(c)$ for $c \neq 0$ and the half-lines in $f^{-1}(0) \setminus 0$, see Figure 7.2.



Case (A) Case (C) Cases (D) Cases (B) and (E)

FIGURE 7.1

a) $F(x,y) = -2xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$ b) $G(x,y) = -2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

FIGURE 7.2. Case (B). Hamiltonian and reduced Hamiltonian vector fields for $f(x,y) = x\,y^2$.

Since f is constant along orbits of F and G, it follows that

$$\mathcal{D}(\Theta_F) \subset \mathcal{D}(\Theta_G) \subset \mathcal{S}(f). \tag{7.1}$$

Also notice that $\mathcal{D}(\Theta_F)$ consists of those $h \in \mathcal{D}(\Theta_G)$ which fixes every critical point of f.

Lemma 7.1. $\mathcal{D}_{\mathrm{id}}(\Theta_G)^r = \mathcal{S}_{\mathrm{id}}(f)^r$ *for all* $r \in \overline{\mathbb{N}}_0$.

Proof. It follows from (7.1) that $\mathcal{D}_{\mathrm{id}}(\Theta_G)^r \subset \mathcal{S}_{\mathrm{id}}(f)^r$.

Conversely, let $h \in \mathcal{S}_{\mathrm{id}}(f)^r$, so there exists an r-isotopy $h_t : \mathbb{R}^2 \to \mathbb{R}^2$ between $h_0 = \mathrm{id}_{\mathbb{R}^2}$ and $h_1 = h$ in $\mathcal{S}(f)$, i.e.,

$$f \circ h_t = f, \quad t \in I. \tag{7.2}$$

We claim that every $h_t \in \mathcal{D}(\Theta_G)$, i.e., $h_t(\omega) = \omega$ for every element ω of Θ_G. This will mean that $\{h_t\}$ is an r-isotopy in $\mathcal{D}(\Theta_G)$, whence $h \in \mathcal{D}_{\mathrm{id}}(\Theta_G)^r$.

It follows from (7.2) that $h_t(f^{-1}(c)) = f^{-1}(c)$ for every $c \in \mathbb{R}$ and $h_t(\Sigma_f) = \Sigma_f$. Since $h_0 = \mathrm{id}_{\mathbb{R}^2}$ preserves every connected component ω of $f^{-1}(c) \setminus \Sigma_f$, so does h_t, $t \in I$. If either $c \neq 0$, or $c = 0$ but f is of either the cases (A), (C), or (D), then by definition every such ω is an element of Θ_G.

Let $c = 0$. We claim that in the cases (B) and (E) $h_t(0) = 0$ for all $t \in I$. Indeed, in these cases the origin is "the most degenerate point among all other points of $f^{-1}(0)$". This means the following.

For every $z \in f^{-1}(0)$ denote by p_z the least number such that p_z-jet of f at z does not vanish, i.e., $j^{p_z-1}(f, z) = 0$ while $j^{p_z}(f, z) \neq 0$. In other words, the Taylor series of f at z starts with terms of order p_z. It is easy to see that for the origin $p_0 = \deg f$, while for all other points $z \in f^{-1}(0)$ we have that $p_z < \deg f$. Also notice that this number p_z is preserved by any diffeomorphism $h \in \mathcal{S}(f)$, i.e., $p_{h(z)} = p_z$. It follows that $h_t(0) = 0$.

It remains to note that by continuity every h_t preserves connected components of $D^{-1}(0) \setminus \{0\}$. Hence $h_t \in \mathcal{D}(\Theta_G)$ for all $t \in I$. □

Now we can complete Theorem 1.1. Let Φ be the local flow on $\mathbb{R}^2$ generated by G, and φ be the shift map of G, see Section 3.5. The following statement was established in [9].

Lemma 7.2. *In the cases* (A)-(C) *(i.e., when* $\deg G \leq 1$*) for every* $h \in \mathcal{E}_{\mathrm{id}}(\Theta_G)^0$ *there exists a smooth function* $\sigma : \mathbb{R}^2 \to \mathbb{R}$ *such that* $h(z) = \Phi(x, \sigma(x))$*, i.e.,* $\mathrm{im}(\varphi) = \mathcal{E}_{\mathrm{id}}(\Theta_G)^0$.

It follows from Lemmas 3.6 and 7.1 that, in the cases (A)–(C),

$$\mathcal{S}_{\mathrm{id}}(f)^\infty = \mathcal{D}_{\mathrm{id}}(\Theta_G)^\infty = \cdots = \mathcal{D}_{\mathrm{id}}(\Theta_G)^0 = \mathcal{S}_{\mathrm{id}}(f)^0. \tag{7.3}$$

The following lemma is a consequence of results of [12, 13].

Lemma 7.3. *In the cases* (D) *and* (E) *(i.e., when* $\deg G \geq 2$*)* $\mathrm{im}(\varphi)$ *consists of all* $h \in \mathcal{E}(\Theta_G)$ *whose tangent map* $T_0 h : T_0\mathbb{R}^2 \to T_0\mathbb{R}^2$ *at* 0 *is the identity.*

Corollary 7.4. $\mathcal{E}_{\mathrm{id}}(\Theta_G)^1 \subset \mathrm{im}(\varphi)$*, whence similarly to* (7.3) *we get* $\mathcal{S}_{\mathrm{id}}(f)^\infty = \cdots = \mathcal{S}_{\mathrm{id}}(f)^1$.

Proof of Corollary. Let $h \in \mathcal{E}_{\mathrm{id}}(\Theta_G)^1$. Then there exists a 1-homotopy h_t between $h_0 = \mathrm{id}_{\mathbb{R}^2}$ and $h_1 = h$ in $\mathcal{E}(\Theta_G)$. In particular, $T_0 h_t$ is continuous in t.

Since f is homogeneous, it follows from [9, Lemma 36], that $T_0 h_t$ regarded as a linear automorphism of $\mathbb{R}^2$ also must preserve f, i.e., $T_0 h_t \in \mathcal{LS}(f)$. Therefore the family of maps $T_0 h_t$ can be regarded as a homotopy in $\mathcal{LS}(f)$. But by Lemma 6.2 in the cases (D) and (E) the group $\mathcal{LS}(f)$ is discrete, whence all $T_0 h_t$ coincide with the identity map $T_0 h_0 = \mathrm{id}_{\mathbb{R}^2}$. In particular, $T_0 h = \mathrm{id}_{\mathbb{R}^2}$, whence by Lemma 7.3 $h \in \mathrm{im}(\varphi)$. □

It remains to show that $\mathcal{S}_{\mathrm{id}}(f)^1$ and $\mathcal{S}_{\mathrm{id}}(f)^0$ coincide in the case (E) and are distinct in the case (D).

(D) Let f be a product of at least two distinct definite quadratic forms. Then by Theorem 5.1 $\mathcal{S}_{\mathrm{id}}(f)^0 = \mathcal{S}^+(f)$. Moreover, since $\deg f$ is even, we have that $-\mathrm{id}_{\mathbb{R}^2} \in \mathcal{S}^+(f) = \mathcal{S}_{\mathrm{id}}(f)^0$, see Lemma 6.1. On the other hand by Lemma 7.3 for every $h \in \mathcal{S}_{\mathrm{id}}(f)^1$ its tangent map $T_0 h = \mathrm{id}_{\mathbb{R}^2} \neq -\mathrm{id}_{\mathbb{R}^2}$. Hence $\mathcal{S}_{\mathrm{id}}(f)^1 \neq \mathcal{S}_{\mathrm{id}}(f)^0$.

(E) In this case $f^{-1}(0)$ is a union of $l \geq 1$ straight lines $\{L_i = 0\}$ passing through the origin. Let $h \in \mathcal{D}_{\mathrm{id}}(\Theta_G)^0$. Since there exists a homotopy between h and $\mathrm{id}_{\mathbb{R}^2}$ in

$\mathcal{D}_{\mathrm{id}}(\Theta_G)^0$, it follows that h preserves every half-line of $f^{-1}(0) \setminus \{0\}$. Therefore so does $T_0 h \in \mathcal{LS}(f)$. Then it follows from Claim 6.7 that $T_0 h = \mathrm{id}_{\mathbb{R}^2}$, whence by Lemma 7.3 $h \in \mathrm{im}(\varphi)$. Thus $\mathcal{D}_{\mathrm{id}}(\Theta_G)^0 \subset \mathrm{im}(\varphi)$.

Now we get from Lemma 3.6 that

$$\mathcal{S}_{\mathrm{id}}(f)^1 = \mathcal{D}_{\mathrm{id}}(\Theta_G)^1 = \mathcal{D}_{\mathrm{id}}(\Theta_G)^0 = \mathcal{S}_{\mathrm{id}}(f)^0.$$

Theorem 1.1 is completed.

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