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REPRESENTATION OF COMMUTANTS FOR COMPOSITION OPERATORS INDUCED BY A HYPERBOLIC LINEAR FRACTIONAL AUTOMORPHISMS OF THE UNIT DISK

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Abstract. We describe the commutant of the composition operator induced by a hyperbolic linear fractional transformation of the unit disk onto itself in the class of linear continuous operators which act on the space of analytic functions. Two general classes of linear continuous operators which commute with such composition operators are constructed.

The composition of functions is a fundamental operation in mathematics, and operators of the composition have an important role in complex analysis and the theory of operators. Composition operators naturally arise in a description of commutants for operators of multiplication by functions in different function spaces.

Let G be an arbitrary domain in the complex plane. By $\mathcal{H}(G)$ we denote the space of all functions analytic in G and endowed with the topology of compact convergence [1]; by $\mathcal{L}(\mathcal{H}(G))$ we denote the set of all linear continuous operators that act on $\mathcal{H}(G)$. For a fixed function $\varphi(z) \in H(G)$ with $\varphi(G) \subset G$ the formula $(K_\varphi f)(z) = f(\varphi(z))$ defines a composition operator $K_\varphi \in \mathcal{L}(\mathcal{H}(G))$ induced by φ.

Over the last thirty years the regular research of the properties of composition operators on the spaces of Hardy, Bergman and Dirichlet was carried out. These Banach spaces consist of functions analytical in the disk $D = \{z \in \mathbb{C} : |z| < 1\}$ with certain restrictions on their module. A main task of these researches is the investigation of properties of the composition operator K_φ in terms of conditions on the generating function $\varphi(z)$. For the operators K_φ, the estimates on their norms are obtained, and in a some cases exact formulas for evaluating the norms are found, conditions of compactness are studied, the structure of invariant subspaces of such operators is investigated, etc. Results of these researches are stated by E. A. Nordgren [2], J. H. Shapiro [3], R. K. Singh and J. S. Manhas [4], C. C. Cowen and B. D. MacCluer [5]. In the review of P. Rosenhtal [6] on monographs [3], [4] it is remarked that the problem of determining the commutant for the composition operator in these spaces is difficult enough. B. Cload in [7], [8] investigated the commutant of the composition operator K_φ on the Hardy space H^2 in the case when the function $\varphi(z)$ is an automorphism of D. B. Cload has described the operators of multiplication on analytic functions that commute with composition operators induced by an elliptic disk automorphism. In conclusions to [8], the author remarks that the problem of a description of the commutant of K_φ on the space H^2 for hyperbolic and parabolic disk automorphisms is not solved. The commutants of composition operators on H^2 were studied by T. Worner [9]. She gave examples of functions $\varphi(z)$ and operators from the class $\mathcal{L}(H^2)$ that commute with K_φ on H^2, but are not generated by operators of multiplication on analytical functions and operators of the composition, that is, are not represented as the sums of products of such operators.

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In connection with this, there arises a problem on an investigation of images of the commutants for composition operators that act on the space $\mathcal{H}(G)$, where G is an arbitrary simply connected domain in the complex plane and G is distinct from $\mathbb{C}$. This problem is important and interesting for the composition operators induced by conformal automorphisms of the domain G. Since, by the Riemann Mapping Theorem, for each such domains there exists a conformal mapping of the corresponding domain to the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, it is enough to solve this problem for composition operators which act on the space $\mathcal{H}(D)$. The general representation of a conformal map of the unit disk D onto itself is defined by

$$\varphi(z) = e^{i\alpha} \frac{z - z_0}{1 - \overline{z_0} z}, \tag{1}$$

where $|z_0| < 1$ and $\alpha \in \mathbb{R}$.

A method for solving the problem on a description of the commutant for K_φ and the final result essentially depend on the class of the function $\varphi(z)$, which could be elliptic, hyperbolic or parabolic. In this paper we solve this problem for composition operators induced by hyperbolic automorphisms of the unit disk D. The main result of this paper is announced in [10].

Recall that the linear fractional transformation $\varphi(z)$ (1) is hyperbolic if and only if the condition $|z_0| > |\sin \frac{\alpha}{2}|$ is satisfied. A hyperbolic linear fractional transformation has two different fixed points that belong to the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

In [1], a one-to-one correspondence between the operators $T \in \mathcal{L}(\mathcal{H}(G))$ and its characteristic functions $t(\lambda, z) = T\left[\frac{1}{\lambda - \tilde{z}}\right]$ is established. For solutions of the main problem of this paper it is convenient to use the special characteristic functions for operators from the class $\mathcal{L}(\mathcal{H}(G))$.

1. Suppose that $T \in \mathcal{L}(\mathcal{H}(D))$ and let $t(\lambda, z) = T\left[\frac{1}{\lambda - \tilde{z}}\right]$ be a characteristic function, in the sense of Köthe, of the operator T. Then $t(\lambda, z)$ is a locally analytical function on the set $\complement D \times D$. This means that there exists a sequence of positive numbers (r_n) that monotonically increase to 1 and there exists a monotonically increasing sequence of numbers $(N(n))$ such that the function $t(\lambda, z)$ is analytical when $|\lambda| > r_{N(n)}$, $|z| < r_n$; $t(\infty, z) = 0$ for $|z| < r_n$. For an arbitrary function $f \in \mathcal{H}(D)$ we have

$$(Tf)(z) = \frac{1}{2\pi i} \int_{\gamma_n} t(\lambda, z) f(\lambda)\, d\lambda, \tag{2}$$

where $\gamma_n = \{\lambda : |\lambda| = r_{N(n)+1}\}$, $|z| < r_n$ (see [1]).

By B_n we denote a Banach space of all functions $f(z)$ that are continuous on the set $|z| \le r_n$ and analytic in the disk $|z| < r_n$ with the norm $||f|| = \max\limits_{|z| \le r_n} |f(z)|$. Then, as it was shown in [1], the operator T uniquely extends to an operator that is linear continuous and acts from $B_{N(n)}$ into B_n. From the proof of Theorem 19 [1] it follows that the formula (2) is correct for an arbitrary function $f(z)$ analytic in the disk $|z| < r$, where $r > r_{N(n)+1}$. Let μ be an arbitrary nonzero complex number for which $|\mu| < \frac{1}{r_{N(n)+1}}$. Then, the function $f_\mu(z) = \frac{z - \mu}{1 - \mu z}$ is analytic for $|z| < \frac{1}{|\mu|}$ and

$$(Tf_\mu)(z) = \frac{1}{2\pi i} \int_{\gamma_n} t(\lambda, z) \frac{\lambda - \mu}{1 - \mu\lambda} d\lambda \tag{3}$$

when $|z| < r_n$. By $t_1(\mu, z)$ we denote $(Tf_\mu)(z)$. Calculating the integral (3) we obtain

$$t_1(\mu, z) = -\operatorname*{res}_{\lambda = \frac{1}{\mu}} t(\lambda, z) \frac{\lambda - \mu}{1 - \mu\lambda} - \operatorname*{res}_{\lambda = \infty} t(\lambda, z) \frac{\lambda - \mu}{1 - \mu\lambda} = \frac{1 - \mu^2}{\mu^2} t\left(\frac{1}{\mu}, z\right) - \frac{1}{\mu} \varphi_0(z),$$

where $\varphi_0(z) = T1$. If $\mu = 0$, then $t_1(0, z) = \varphi_1(z)$, where $\varphi_1(z) = Tz$. Therefore, the function $t_1(\mu, z)$ is analytic when $|\mu| < \frac{1}{r_{N(n)+1}}$, $|z| < r_n$, and

$$t_1(\mu, z) = \frac{1-\mu^2}{\mu^2} t\left(\frac{1}{\mu}, z\right) - \frac{1}{\mu}\varphi_0(z). \tag{4}$$

Hence, the function $t_1(\mu, z)$ is locally analytic on the set $\overline{D} \times D$, that is, $t_1(\mu, z)$ is analytic on a set $\mathcal{K}= \bigcup_{n=1}^{\infty} K_{\rho_{N(n)}} \times K_{r_n}$, where $K_{r_n} = \{z \in \mathbb{C} : |z| < r_n\}$, $K_{\rho_n} = \{\mu \in \mathbb{C} : |\mu| < \rho_n\}$, (r_n) and (ρ_n) are sequences of positive numbers each of them is monotonically increases and monotonically decreases to 1 respectively, and $(N(n))$ is a monotonically increasing sequence of natural numbers (in ours case $\rho_{N(n)} = \frac{1}{r_{N(n)+1}}$, $n = 1, 2, \ldots$). From (4) it follows that

$$t_1(1, z) + t_1(-1, z) = 0, \quad z \in D. \tag{5}$$

Conversely, let $t_1(\mu, z)$ be a locally analytic function on $\overline{D} \times D$ and satisfy (5). Substituting $\mu = \frac{1}{\lambda}$ in (4), and using $\varphi_0(z) = t_1(-1, z)$, we see that the formula

$$t(\lambda, z) = \frac{1}{\lambda^2 - 1} t_1\left(\frac{1}{\lambda}, z\right) + \frac{\lambda}{\lambda^2 - 1} t_1(-1, z)$$

defines a locally analytic function on the set $\complement D \times D$. According to (2) for an arbitrary function $f \in \mathcal{H}(D)$ we have

$$(Tf)(z) = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{1}{\lambda^2 - 1} t_1\left(\frac{1}{\lambda}, z\right) f(\lambda)\, d\lambda,$$

where $\Gamma_n = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{\rho_{N(n)+1}}\}$, $|z| < r_n$. The number $N(n)$ is found for the number n by the definition of a function $t_1(\mu, z)$ locally analytic on $\overline{D} \times D$. Since $T\left[\frac{\tilde{z}-\mu}{1-\mu\tilde{z}}\right] = t_1(\mu, z)$, we have that $t_1(\mu, z)$ is the characteristic function of the operator T. Thus, the following statement is valid.

Theorem 1. *There exists a one-to-one correspondence between operators $T \in \mathcal{L}(\mathcal{H}(D))$ and its characteristic functions $t_1(\lambda, z) = T\left[\frac{\tilde{z}-\lambda}{1-\lambda\tilde{z}}\right]$ that are locally analytic on the set $\overline{D} \times D$ and satisfy the condition* (5).

Remark 1. If $g_1(\mu, z)$ is an arbitrary locally analytic function on the set $\overline{D} \times D$, then the formula

$$t_1(\mu, z) = g_1(\mu, z) - \frac{1}{2} g_1(1, z) - \frac{1}{2} g_1(-1, z)$$

defines a function locally analytic on $\overline{D} \times D$ that satisfies (5). From Theorem 1 it follows that $t_1(\mu, z)$ is the characteristic function for the same operator $T \in \mathcal{L}(\mathcal{H}(D))$.

2. The function $\varphi(z) = \frac{z-z_0}{1-z_0 z}$, where $z_0 \in \mathbb{R}$ and $0 < |z_0| < 1$ is an important partial case of a hyperbolic automorphism of the unit disk D. This map plays an important role in the definition of a distance in the model of Lobachevsky geometry, which was proposed by A. Poincare [11].

Let us describe the commutant of the operator K_φ in the class $\mathcal{L}(\mathcal{H}(D))$. Suppose that $T \in \mathcal{L}(\mathcal{H}(D))$ commutes with K_φ. Then the equality

$$TK_\varphi = K_\varphi T \tag{6}$$

holds. By $t_1(\lambda, z) = T\left[\frac{\tilde{z}-\lambda}{1-\tilde{z}\lambda}\right]$ we denote the characteristic function of the operator T. Let $\tilde{t}_1(\lambda, z)$ and $\tilde{\tilde{t}}_1(\lambda, z)$ be characteristic functions of the operators TK_φ and $K_\varphi T$ respectively. Fix an arbitrary natural number n and $N(n)$ is found by the definition of the function $t_1(\lambda, z)$ locally analytic on the set $\overline{D} \times D$. Then,

$$(Tg)(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\tau^2 - 1} t_1\left(\frac{1}{\tau}, z\right) g(\tau)\, d\tau, \tag{7}$$

where $\Gamma_n = \{\tau \in \mathbb{C} : |\tau| = \frac{1}{\rho_{N(n)+1}}\}$, $|z| < r_n$. Formula (7) is correct for an arbitrary function $g(z)$ analytic in the disk $|z| < r$, where $r > \frac{1}{\rho_{N(n)+1}}$ (see the proof of Theorem 1).

Suppose $C = \max\{|\varphi(z)| : |z| \le \frac{1}{\rho_{N(n)+1}}\}$ and $C_1 = \min\{|\varphi(\lambda)| : |\lambda| = \rho_{N(n)+1}\}$; then $C < 1 < C_1$. We choose $N_1(n)$ so that $\rho_{N_1(n)} < \min\{C_1, \frac{1}{C}, \frac{1}{|z_0|}\}$. Fix an arbitrary $\lambda \in \mathbb{C}$, $|\lambda| < \rho_{N_1(n)}$; then $\frac{1}{|\lambda|} > C$. Suppose $r(\lambda) = 1$ for $\frac{1}{|\lambda|} \ge 1$, and $r(\lambda) = \min\{\varphi^{-1}(t) : |t| = \frac{1}{|\lambda|}, t \in \mathbb{C}\}$ for $\frac{1}{|\lambda|} < 1$; then from the definition of the number C it follows that $r(\lambda) > \frac{1}{\rho_{N(n)+1}}$ and the function $g_\lambda(z) = \frac{\varphi(z)-\lambda}{1-\lambda\varphi(z)}$ is analytic in the disk $|z| < r(\lambda)$. Using (7), we get

$$\tilde{t}_1(\lambda, z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\tau^2 - 1} t_1\left(\frac{1}{\tau}, z\right) \frac{\varphi(\tau) - \lambda}{1 - \lambda\varphi(\tau)}\, d\tau,$$

when $|z| < r_n$ and $|\lambda| < \rho_{N_1(n)}$. The inequality $|\varphi(\tau)| \cdot |\lambda| \le C\rho_{N_1(n)} < 1$ holds for $|\tau| = \frac{1}{\rho_{N(n)+1}}$ and $|\lambda| < \rho_{N_1(n)}$. Therefore,

$$\frac{\varphi(\tau) - \lambda}{1 - \lambda\varphi(\tau)} = \frac{\tau - \varphi^{-1}(\lambda)}{1 - \tau\varphi^{-1}(\lambda)}$$

for $|\tau| = \frac{1}{\rho_{N(n)+1}}$ and $|\lambda| < \rho_{N_1(n)}$.

Indeed, we obtain

$$\tilde{t}_1(\lambda, z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\tau^2 - 1} t_1\left(\frac{1}{\tau}, z\right) \frac{\tau - \varphi^{-1}(\lambda)}{1 - \tau\varphi^{-1}(\lambda)}\, d\tau, \tag{8}$$

when $|z| < r_n$, $|\lambda| < \rho_{N_1(n)}$. If $|\lambda| < \rho_{N_1(n)}$, then $|\lambda| < C_1|\varphi^{-1}(\lambda)| < \rho_{N(n)+1}$. From (8) it follows that $\tilde{t}_1(\lambda, z) = t_1(\varphi^{-1}(\lambda), z)$ for $|z| < r_n$, $|\lambda| < \rho_{N_1(n)}$.

Since $\tilde{\tilde{t}}_1(\lambda, z)$ is the characteristic function of the operator $K_\varphi T$, we see that for the chosen number n there exists a number $N_2(n)$ for which $\tilde{\tilde{t}}_1(\lambda, z) = t_1(\lambda, \varphi(z))$, where $|z| < r_n$, $|\lambda| < \rho_{N_2(n)}$. The equality (6) implies that $\tilde{t}_1(\lambda, z) = \tilde{\tilde{t}}_1(\lambda, z)$ on $\mathcal{K}$. Therefore, we have

$$t_1(\varphi^{-1}(\lambda), z) = t_1(\lambda, \varphi(z)), \tag{9}$$

when $|z| < r_n$, $|\lambda| < \rho_{N_3(n)}$, where $N_3(n) = \max\{N_1(n), N_2(n)\}$. Let us make the substitution $\lambda = \varphi(\mu)$ in (9). Similarly, there exists $N_4(n)$ such that $|\varphi(\mu)| < \rho_{N_3(n)}$ for $|\mu| < \rho_{N_4(n)}$. Then, from (9), it follows that for $|z| < r_n$,$|\mu| < \rho_{N_4(n)}$, and the equality $t_1(\mu, z) = t_1(\varphi(\mu), \varphi(z))$ holds.

Thus, the characteristic function of the operator TK_φ coincides with the characteristic function of the operator $K_\varphi T$ if and only if the last equality is correct on the set $\mathcal{K}$. The following statement holds.

Theorem 2. *Let $\varphi(z) = \frac{z-z_0}{1-z_0 z}$, $z_0 \in \mathbb{R}$, $0 < |z_0| < 1$, be a hyperbolic automorphism of the unit disk D. An operator $T \in \mathcal{L}(\mathcal{H}(D))$ commutes with the operator K_φ if and only if its characteristic function $t_1(\lambda, z)$ satisfies the equality*

$$t_1(\varphi(\lambda), \varphi(z)) = t_1(\lambda, z) \tag{10}$$

on the set $\mathcal{K}$.

Remark 2. In the proof of Theorem 2 we essentially use that the function $k(z, \lambda) = \frac{z-\lambda}{1-\lambda z}$ satisfies the equality $k(\varphi(z), \varphi(\lambda)) = k(\lambda, z)$ when $|\lambda z| < 1$.

We find the solutions of the equation (10) in the class of functions $t_1(\lambda, z)$ that are locally analytic on $\overline{D} \times D$.

Let $t_1(\lambda, z)$ be a locally analytic function on $\overline{D} \times D$ and satisfy (10) on the set $\mathcal{K}$. Make the substitution

$$\mu = \frac{z-\lambda}{1-\lambda z} \tag{11}$$

in (10). For each number n by G_n we denote the open set $G_n = \{\frac{z-\lambda}{1-\lambda z} : |\lambda| < \rho_{N(n)}, |z| < r_n\}$. Let $G = \bigcup_{n=1}^{\infty} G_n$. Clearly, G is open in $\mathbb{C}$. Since for $\lambda \in K_{\rho_{N(1)}}$ the points $(\lambda, 0) \in K_{\rho_{N(1)}} \times K_{r_1}$, then $K_{\rho_{N(1)}} \subset G_1$. Thus,

$$K_{\rho_{N(1)}} \subset G. \tag{12}$$

If $|\lambda| = 1$, $z \in K_{r_n}$, then $(\lambda, z) \in \mathcal{K}$, and $\{(\lambda, z) : |\lambda| = 1, |z| < 1\} \subset \mathcal{K}$. Therefore, $\{\frac{z-\lambda}{1-\lambda z} : |\lambda| = 1, |z| < 1\} \subset G$. We have

$$\left\{\frac{z-i}{1-iz} : |z| < 1\right\} = \{\mu : \operatorname{Im}\mu < 0\}, \left\{\frac{z+i}{1+iz} : |z| < 1\right\} = \{\mu : \operatorname{Im}\mu > 0\}$$

for $\lambda = \pm i$. Thus, we see that

$$\{\mu \in \mathbb{C} : \operatorname{Im}\mu \neq 0\} \subset G. \tag{13}$$

For an arbitrary positive ε by G_ε we denote the set $G_\varepsilon = \mathbb{C} \setminus \{\mu \in \mathbb{R} : |\mu| \geq 1+\varepsilon\}$. From (12) and (13) it follows that there exists $\varepsilon > 0$ such that $G_\varepsilon \subset G$.

Substituting λ with (11) into (10) we obtain

$$t_1\left(\varphi\left(\frac{z-\mu}{1-\mu z}\right), \varphi(z)\right) = t_1\left(\frac{z-\mu}{1-\mu z}, z\right) \tag{14}$$

for $\mu \in G_n, |z| < r_n$. We have

$$\varphi\left(\frac{z-\mu}{1-\mu z}\right) = \frac{\varphi(z)-\mu}{1-\mu\varphi(z)}$$

when $\mu \in G_n$, $z \in K_{r_n}$. Therefore, (14) can be presented in the form

$$t_1\left(\frac{\varphi(z)-\mu}{1-\mu\varphi(z)}, \varphi(z)\right) = t_1\left(\frac{z-\mu}{1-\mu z}, z\right), \tag{15}$$

$z \in K_{r_n}$, $\mu \in G_n$. By $t_2(\mu, z)$ we denote $t_1\left(\frac{z-\mu}{1-\mu z}, z\right)$. The function $t_2(\mu, z)$ satisfies the equality

$$t_2(\mu, \varphi(z)) = t_2(\mu, z) \tag{16}$$

for $\mu \in G_n$, $z \in K_{r_n}$. The function $t_2(\mu, z)$ is analytic on the set $G \times D$ since this function is analytic on the set $G_n \times K_{r_n}$ for each number n. Returning to the function $t_1(\lambda, z)$ we obtain

$$t_1(\lambda, z) = t_2\left(\frac{z-\lambda}{1-\lambda z}, z\right) \tag{17}$$

if $|\lambda| < \rho_{N(n)}$, $|z| < r_n$.

Thus, there exists $\varepsilon > 0$ such that $t_2(\mu, z)$ is an analytic function on the set $G_\varepsilon \times D$ and the equality (16) holds on this set, which proves the *necessity*.

Theorem 3. *Let $\varphi(z) = \frac{z-z_0}{1-z_0 z}$, $z_0 \in \mathbb{R}$, $|z_0| < 1$, be a hyperbolic automorphism of the unit disk D. A function $t_1(\lambda, z)$, locally analytic on the set $\overline{D} \times D$, is a solution of the equation* (10) *if and only if it has the form* (17)*, where $t_2(\mu, z)$ is an analytic function on the same set $G_\varepsilon \times D$ and satisfies* (16).

Proof. **Sufficiency.** Assume that there exists $\varepsilon > 0$ such that $t_2(\mu, z)$ is an analytic function on $G_\varepsilon \times D$ and satisfies on this set (16). Let us show that for any number r, $0 < r < 1$, there exists $\rho > 1$ for which $G(r, \rho) \subset G_\varepsilon$, where $G(r, \rho) = \left\{\frac{z-\lambda}{1-\lambda z} : |z| < r, |\lambda| < \rho\right\}$. Indeed, fix an arbitrary $r \in (0, 1)$ and choose ρ such that $1 < \rho < \min\left\{\frac{1}{r}, 1+\varepsilon, \frac{1+\varepsilon+r}{1+r+\varepsilon r}\right\}$.

We show that for this ρ the inclusion $G(r,\rho) \subset G_\varepsilon$ holds. Suppose the contrary. Assume that the inclusion $G(r,\rho) \subset G_\varepsilon$ does not hold. Then there are numbers z_0, λ_0 for which $|z_0| < r$, $|\lambda_0| < \rho$, and $\frac{z_0-\lambda_0}{1-\lambda_0 z_0} = x$, where $x \in \mathbb{R}$, $|x| > 1+\varepsilon$. Without lost of a generality it can be assumed that $x > 0$, then $x > 1+\varepsilon$. We have $z_0 = \frac{\lambda_0+x}{1+\lambda_0 x}$.

Let $\psi(\lambda) = \frac{\lambda+x}{1+\lambda x}$ be the auxiliary linear fractional transformation. By K denote the image of the circle $\{\lambda \in \mathbb{C} : |\lambda| = \rho\}$ under the map $\psi(\lambda)$. The points $\psi(\rho) = \frac{\rho+x}{1+\rho x}$ and $\psi(-\rho) = -\frac{x-\rho}{\rho x-1}$ are diametrically opposite for K. The disk $\{\lambda \in \mathbb{C} : |\lambda| < \rho\}$ is mapped on the exterior of K . By the choice of the number ρ we have $-\frac{x-\rho}{\rho x-1} < -r < r < \frac{\rho+x}{1+\rho x}$. Then, the circle K contains the disk $\{z \in \mathbb{C} : |z| < r\}$ and consequently the equality $z_0 = \frac{\lambda_0+x}{1+\lambda_0 x}$ is impossible. This contradicts the assumption, so the inclusion $G(r,\rho) \subset G_\varepsilon$ is proved. Then, the formula (17) defines a function $t_1(\lambda, z)$, locally analytic on the set $\overline{D} \times D$, which satisfies (10). Theorem 3 is proved. □

From (17) it follows that $t_1(1,z) = t_2(-1,z)$, $t_1(-1,z) = t_2(1,z)$. Therefore, for the function $t_1(\lambda,z)$, which is defined by (17), relation (5) is equivalent to

$$t_2(-1,z) + t_2(1,z) = 0 \tag{18}$$

for $|z| < 1$. Thus, Theorems 1–3 imply that an operator $T \in \mathcal{L}(\mathcal{H}(D))$ commutes with the operator K_φ if and only if its characteristic function $t_1(\lambda,z)$ has the form (17), where $t_2(\mu,z)$ is a function analytic on some set $G_\varepsilon \times D$ and satisfies (16), (18). Using the equality

$$t(\lambda,z) = \frac{1}{\lambda^2-1} t_2\left(\frac{\lambda z-1}{\lambda-z}, z\right) + \frac{\lambda}{\lambda^2-1} t_2(1,z), \tag{19}$$

we restore $T \in \mathcal{L}(\mathcal{H}(D))$ by means of its characteristic function $t(\lambda,z)$, that is, the following statement holds.

Theorem 4. *Let $\varphi(z) = \frac{z-z_0}{1-z_0 z}$, $z_0 \in \mathbb{R}$, $0 < |z_0| < 1$, be a hyperbolic automorphism of the unit disk D. An operator $T \in \mathcal{L}(\mathcal{H}(D))$ commutes with the operator K_φ if and only if*

$$(Tf)(z) = \frac{1}{2\pi i} \int_{\gamma_z} \frac{1}{\lambda^2-1} t_2\left(\frac{\lambda z-1}{\lambda-z}, z\right) f(\lambda)\, d\lambda. \tag{20}$$

Here $t_2(\mu,z)$ is a function analytic on some set $G_\varepsilon \times D$ and satisfies (16), (18). *The contour of integration γ_z is chosen for $z \in D$ by definition of the function $t(\lambda,z)$, locally analytic on $\complement D \times D$, that is defined by* (19).

Remark 3. Let $t_2(\mu,z)$ be a function analytic on the some set $G_\varepsilon \times D$ and satisfy (16). Theorem 3 and Remark 1 imply that the function $\tilde{t}_1(\lambda,z) = t_2\left(\frac{z-\lambda}{1-\lambda z}, z\right) - \frac{1}{2}t_2(1,z) - \frac{1}{2}t_2(-1,z)$ is analytic on $G_\varepsilon \times D$ also, and $\tilde{t}_1(1,z) + \tilde{t}_1(-1,z) = 0$ when $|z| < 1$. Therefore, $\tilde{t}_1(\lambda,z)$ is a characteristic function of an operator $T \in \mathcal{L}(\mathcal{H}(D))$ that commutes with the operator K_φ. From Theorem 4 it follows that formula (20) defines this operator for an arbitrary function $f \in \mathcal{H}(D)$.

Suppose $t_2(\mu,z) = \mu^n$, n is an arbitrary fixed natural number. Then the formula

$$(T_n f)(z) = \frac{1}{2\pi i} \int_{\gamma_z} \frac{(\lambda z-1)^n}{(\lambda^2-1)(\lambda-z)^n} f(\lambda)\, d\lambda, \tag{21}$$

where $\gamma_z = \{\lambda : |\lambda| = r\}$, and r is such that $|z| < r < 1$, defines an operator $T_n \in \mathcal{L}(\mathcal{H}(D))$. Remark 3 and Theorem 4 imply that T_n commutes with the operator K_φ.

Let $t_2(\mu,z)$ be the entire function on μ, analytic in z in the disk D and satisfy (16) for $\mu \in \mathbb{C}$, $z \in D$. Then, from Remark 3 and Theorem 4 it follows that the formula (3) defines an operator $T \in \mathcal{L}(\mathcal{H}(D))$ that commutes with the operator K_φ. In this case γ_z is the circle $|\lambda| = r$, $|z| < r < 1$. We expand the function $t_2(\mu, z)$ into a series

$$t_2(\mu, z) = \sum_{n=0}^{\infty} \psi_n(z)\mu^n, \tag{22}$$

$\mu \in \mathbb{C}$, $z \in D$. Then $(\psi_n(z))$ is a sequence of analytic functions in the disk D. For the function $t_2(\mu, z)$ the condition (16) is equivalent to

$$\psi_n(\varphi(z)) = \psi_n(z), \quad z \in D, \quad n = 0, 1, 2, \ldots \tag{23}$$

Substituting (22) in (20) and integrating the corresponding series, we have

$$(Tf)(z) = \sum_{n=1}^{\infty} \psi_n(z)(T_n f)(z) \tag{24}$$

for an arbitrary functions $f \in \mathcal{H}(D)$, $|z| < 1$. We prove that $(\psi_n(z))$ satisfies the condition

$$\forall r < 1 \lim_{n\to\infty} \sqrt[n]{\max_{|z|\le r}|\psi_n(z)|} = 0. \tag{25}$$

Fix an arbitrary $r < 1$ and $0 < \rho < \infty$. From Cauchy's inequalities for coefficients of the expansion of the function $t_2(\mu, z)$ entire in μ in series (22) it follows that

$$|\psi_n(z)| \le \frac{\max_{|\mu|=\rho}|t_2(\mu, z)|}{\rho^n}, \quad n = 0, 1, \ldots,$$

when $|z| \le r$. Therefore,

$$\max_{|z|\le r}|\psi_n(z)| \le \frac{C}{\rho^n}, \quad n = 0, 1, \ldots,$$

where $C = \max\{|t_2(\mu, z)| : |\mu| = \rho, |z| \le r\}$. The last inequalities implies $\overline{\lim}_{n\to\infty} \sqrt[n]{\max_{|z|\le r}|\psi_n(z)|} \le \frac{1}{\rho}$. Suppose $\rho \to \infty$ in this inequalities; then (25) holds.

Conversely, let $(\psi_n(z))$ be a sequence functions that are analytic in the disk D and satisfy (23), (25). We show that the formula (24) defines some operator $T \in \mathcal{L}(\mathcal{H}(D))$ that commutes with the operator K_φ.

Fix numbers r_1, r such that $0 < r_1 < r < 1$. From (21) it follows that

$$|(T_n f)(z)| \le \frac{r(r_1 r + 1)^n}{(1 - r^2)(r - r_1)^n} \max_{|\lambda|=r}|f(\lambda)|$$

for an arbitrary function $f \in \mathcal{H}(D)$, $|z| \le r_1$. Therefore,

$$\max_{|z|\le r_1}|(T_n f)(z)| \le \frac{r}{1 - r^2}\left(\frac{r_1 r + 1}{r - r_1}\right)^n \max_{|\lambda|=r}|f(\lambda)|.$$

Choose a number h, $0 < h < \frac{1}{2}\left(\frac{r-r_1}{r_1 r+1}\right)$. From (25) it follows that there exists a constant $C > 0$ such that $\max_{|z|\le r_1}|\psi_n(z)| \le Ch^n$, $n = 0, 1, \ldots$ From last inequalities we have

$$\max_{|z|\le r_1}(|\psi_n(z)||(T_n f)(z)|) \le \frac{Cr}{1 - r^2}\left(\frac{1}{2}\right)^n \max_{|\lambda|=r}|f(\lambda)|, n = 0, 1, \ldots \tag{26}$$

Using (26), we see that the series in the right-hand side of (24) converges uniformly on the disk $|z| \le r_1$. This series converges uniformly inside D since $r_1 < 1$ is arbitrary. Weierstrass' theorem on uniformly convergent series of analytic functions implies that the formula (24) defines some operator T that acts on the space $\mathcal{H}(D)$. The linearity of T follows from linearity of the operators T_n. Continuity of T follows from estimates (26). Hence, indeed, the formula (24) defines an operator $T \in \mathcal{L}(\mathcal{H}(D))$. By the immediate check we see that T commutes with K_φ. Thus, the following statement holds.

Corollary 1. *Let $(\psi_n(z))$ be a sequence of functions analytic on the disk D that satisfy conditions* (23), (25), *and the operators T_n be defined by formula* (21). *Then the operator T that is defined by the formula* (24) *belongs to $\mathcal{L}(\mathcal{H}(D))$ and commutes with the operator K_φ.*

Using the residues and calculating the integral (21), we can write the operator T_n in an explicit form, that is, in the form of a differential operator of finite order with variable coefficients.

For an arbitrary function $f \in \mathcal{H}(D)$, we have

$$\begin{aligned}(T_n f)(z) &= \frac{1}{2\pi i}\int_{|\lambda|=r}\frac{(\lambda z-1)^n}{(\lambda-z)^n}\frac{f(\lambda)}{\lambda^2-1}d\lambda = \frac{1}{(n-1)!}\left((\lambda z-1)^n\frac{f(\lambda)}{\lambda^2-1}\right)^{(n-1)}\Big|_{\lambda=z}\\ &= \sum_{k=0}^{n-1} C_{n-1}^k\frac{n}{(k+1)!}z^{n-1-k}(z^2-1)^{k+1}\left(\frac{f(z)}{z^2-1}\right)^{(k)}\end{aligned}$$

for $|z|<r<1$. But

$$\begin{aligned}\left(\frac{f(z)}{z^2-1}\right)^{(k)} &= \frac{1}{2}\left[\left(f(z)\frac{1}{z-1}\right)^{(k)} - \left(f(z)\frac{1}{z+1}\right)^{(k)}\right]\\ &= \frac{1}{2}\sum_{s=0}^{k} C_k^s(-1)^{k-s}(k-s)!\left[\frac{1}{(z-1)^{k-s+1}}-\frac{1}{(z+1)^{k-s+1}}\right]f^{(s)}(z).\end{aligned}$$

Therefore,

$$\begin{aligned}(T_n f)(z) = \frac{1}{2}\sum_{k=0}^{n-1}\sum_{s=0}^{k} C_{n-1}^k C_k^s\frac{n(k-s)!}{(k+1)!}(-1)^{k-s}z^{n-1-k}&\\ \times\left[(z+1)^{k+1}(z-1)^s-(z-1)^{k+1}(z+1)^s\right]f^{(s)}(z).&\end{aligned}$$

Interchanging the order of summing, we obtain

$$\begin{aligned}(T_n f)(z) = \frac{1}{2}\sum_{s=0}^{n-1}\sum_{k=s}^{n-1} C_{n-1}^k C_k^s\frac{n(k-s)!}{(k+1)!}(-1)^{k-s}z^{n-1-k}&\\ \times\left[(z+1)^{k+1}(z-1)^s-(z-1)^{k+1}(z+1)^s\right]f^{(s)}(z).&\end{aligned}$$

This formula implies

$$\begin{gathered}(T_1 f)(z) = f(z),\\ (T_2 f)(z) = (z^2-1)f'(z),\\ (T_3 f)(z) = f(z) + z(z^2-1)f'(z) + \frac{(z^2-1)^2}{2}f''(z).\end{gathered}$$

Each of the functions $\psi_n(z)$ that satisfies the condition $\psi_n(\varphi(z)) = \psi_n(z)$, $z \in D$, is automorphic with respect to the cyclic group generated by the fractional linear transformation $\varphi(z)$. Let us remark that in [12] classes of such automorphic functions are constructed with Blaschke products.

We give one more class of operators from $\mathcal{L}(\mathcal{H}(D))$ that commute with K_φ. Fix $\mu_0 \in \mathbb{R}$, $|\mu_0| > 1$. Let $t_2(\mu, z)$ be an analytic function satisfying the equality $t_2(\mu, \varphi(z)) = t_2(\mu, z)$, $\mu \in \overline{\mathbb{C}} \setminus \{\mu_0\}$, $z \in D$. Theorem 4 and Remark 3 imply that the formula (20) defines an operator $T \in \mathcal{L}(\mathcal{H}(D))$ that commutes with the operator K_φ. We represent the function $t_2(\mu, z)$ in the form

$$t_2(\mu, z) = \sum_{n=1}^{\infty}\frac{\psi_n(z)}{(\mu-\mu_0)^n}. \tag{27}$$

Then, from the analyticity condition on $t_2(\mu, z)$ and relations $t_2(\mu, \varphi(z)) = t_2(\mu, z)$ it follows that the functions $\psi_n(z)$ are analytic in the disk D and satisfy the equality $\psi_n(\varphi(z)) = \psi_n(z)$, $z \in D$, $n = 1, 2, \dots$ From (20) we obtain

$$(Tf)(z) = \sum_{n=1}^{\infty} \psi_n(z)(\tilde{T}_n f)(z) \tag{28}$$

for an arbitrary function $f \in \mathcal{H}(D)$, $|z| < 1$, where

$$(\tilde{T}_n f)(z) = \frac{1}{(z-\mu_0)^n} \frac{1}{2\pi i} \int_{\gamma_z} \frac{(\lambda - z)^n}{\left(\lambda - \frac{1-\mu_0 z}{z-\mu_0}\right)^n} \frac{f(\lambda)}{\lambda^2 - 1} d\lambda, \tag{29}$$

$\gamma_z = \{\lambda : |\lambda| = r\}$ and r is chosen such that the point $\frac{1-\mu_0 z}{z-\mu_0} = \frac{z - \frac{1}{\mu_0}}{1 - \frac{1}{\mu_0} z}$ belongs to $|\lambda| < r$. Theorem 4 implies that each of the operators $\tilde{T}_n$, $n = 1, 2, \dots$, commutes with K_φ.

Using Cauchy's inequalities for the coefficients of the expansion of an analytic function into Laurent series (27), we obtain

$$\forall r < 1 \lim_{n\to\infty} \sqrt[n]{\max_{|z|\le r} |\psi_n(z)|} = 0. \tag{30}$$

Conversely, let a sequence of functions $(\psi_n(z))$ from $\mathcal{H}(D)$ satisfies the condition (30) and $\psi_n(\varphi(z)) = \psi_n(z)$ for $|z| < 1$, $n = 1, 2, \dots$ Show that the formula (28) defines the some operator $T \in \mathcal{L}(\mathcal{H}(D))$ that commutes with K_φ. Fix an arbitrary r_1, $0 < r_1 < 1$. Under the map $w = \frac{z - \frac{1}{\mu_0}}{1 - \frac{1}{\mu_0} z}$ the disk $|z| \le r_1$ will be mapped in some disk that contains in $|z| < 1$. Therefore, the number $\rho = \max\{|\frac{z - \frac{1}{\mu_0}}{1 - \frac{1}{\mu_0} z}| : |z| \le r_1\}$ satisfies the inequality $\rho < 1$. Choose r, $\rho < r < 1$. If we take $\gamma_z = \{\lambda : |\lambda| = r\}$ in (29), then this formula is correct for an arbitrary function $f \in \mathcal{H}(D)$ and for any z, $|z| \le r_1$. Estimating integrals of (29) we obtain

$$|(\tilde{T}_n f)(z)| \le \frac{1}{(|\mu_0| - r_1)^n} \frac{(r + r_1)^n}{(r - \rho)^n} \frac{r}{(1 - r^2)} \max_{|\lambda| = r} |f(\lambda)| \tag{31}$$

for an arbitrary function $f \in \mathcal{H}(D)$, $|z| \le r_1$, $n = 1, 2, \dots$ Using (30), (31) and the immediate check, we see that the formula (28) defines some operator T from the class $\mathcal{L}(\mathcal{H}(D))$ that T commutes with K_φ.

Corollary 2. *Let $(\psi_n(z))$ be a sequence of functions analytic in the disk D, $\psi_n(\varphi(z)) = \psi_n(z)$, $z \in D$, $n = 1, 2, \dots$, and $\psi_n(z)$ satisfy* (30) *. Then the operator T defined by* (28) *belongs to $\mathcal{L}(\mathcal{H}(D))$ and commutes with the operator K_φ.*

Calculating the integral of (29) for an arbitrary $f \in \mathcal{H}(D)$, we have

$$\begin{aligned}(\tilde{T}_n f)(z) &= \frac{1}{2} \sum_{s=0}^{n-1} \sum_{k=s}^{n-1} C_{n-1}^k C_k^s \frac{n}{(k+1)!} (k-s)! (-1)^{k-s} \frac{1}{(z-\mu_0)^{n+s}} \\ &\times \left[\frac{(1-z)^s (1+z)^{k+1}}{(1+\mu_0)^{k-s+1}} - \frac{(1+z)^s (1-z)^{k+1}}{(1-\mu_0)^{k-s+1}} \right] f^{(s)} \left(\frac{z\mu_0 - 1}{\mu_0 - z} \right), \quad n = 1, 2, \dots,\end{aligned}$$

$|z| < 1$. For $n = 1, 2$ we get

$$(T_1 f)(z) = \frac{1}{1 - \mu_0^2} f\left(\frac{z\mu_0 - 1}{\mu_0 - z} \right),$$

$$(T_2 f)(z) = \frac{2}{(1-\mu_0^2)^2} f\left(\frac{z\mu_0 - 1}{\mu_0 - z} \right) + \frac{1}{1 - \mu_0^2} \frac{1 - z^2}{(z - \mu_0)^2} f'\left(\frac{z\mu_0 - 1}{\mu_0 - z} \right).$$

Note that the operators T_n and $\tilde{T}_n$, $n = 1, 2, \ldots$, are universal in the following sense: they commute with all operators of composition, K_φ, where φ is an arbitrary function of the form $\varphi(z) = \frac{z - z_0}{1 - z_0 z}$, $z_0 \in \mathbb{R}$, $0 < |z_0| < 1$.

3. The linear fractional transformation $\varphi(z) = \frac{z - z_0}{1 - z_0 z}$, $z_0 \in \mathbb{R}$, $|z_0| < 1$, is a particular case of a hyperbolic map of the unit disk D onto itself. We shall describe a commutant of the composition operator K_{φ_1} induced by an arbitrary hyperbolic transformation $\varphi_1(z) = e^{i\alpha_1} \frac{z - z_1}{1 - \overline{z}_1 z}$, $|\sin \frac{\alpha_1}{2}| < |z_1| < 1$ of the unit disk D.

Lemma 1. *Let $\varphi_1(z) = e^{i\alpha_1} \frac{z - z_1}{1 - \overline{z}_1 z}$, $0 < |z_1| < 1$, $\alpha_1 \in \mathbb{R}$, be a hyperbolic automorphism of the unit disk D . Then on the space $\mathcal{H}(D)$ the operator K_{φ_1} is equivalent to the operator K_{φ_2} , where $\varphi_2(z) = e^{i\alpha_1} \frac{z - |z_1|}{1 - |\overline{z}_1| z}$. For the isomorphism K_{ψ_1} with $\psi_1(z) = e^{-i\beta} z$, $\beta = \arg z_1$, the following identity holds:*

$$K_{\varphi_1} K_{\psi_1} = K_{\psi_1} K_{\varphi_2}. \tag{32}$$

Proof. Since $\psi_1 \circ \varphi_1 = \varphi_2 \circ \psi_1$, we have (32).

Consider the operator K_{φ_2}, where $\varphi_2(z) = e^{i\alpha_1} \frac{z - |z_1|}{1 - |z_1| z}$, $|\sin \frac{\alpha_1}{2}| < |z_1| < 1$. We show that on the space $\mathcal{H}(D)$ the operator K_{φ_2} is equivalent to some operator K_{φ_3}, where $\varphi_3(z) = \frac{z - z_3}{1 - \overline{z}_3 z}$, $0 < |z_3| < 1$. For this we prove that there exists a function $\psi_2(z) = \frac{z - z_2}{1 - \overline{z}_2 z}$, $|z_2| < 1$, that satisfies the equality

$$K_{\varphi_2} K_{\psi_2} = K_{\psi_2} K_{\varphi_3}. \tag{33}$$

If $\sin \frac{\alpha_1}{2} = 0$ then we take $\psi_2(z) = z$. Suppose $\sin \frac{\alpha_1}{2} \neq 0$; then $e^{i\alpha_1} \neq 1$. The relation (33) is equivalent to $\psi_2 \circ \varphi_2 \circ \psi_2^{-1} = \varphi_3$. But $(\psi_2 \circ \varphi_2)(z) = e^{i\alpha_1} \frac{1 + |z_1| z_2 e^{-i\alpha_1}}{1 + |z_1| \overline{z}_2 e^{i\alpha_1}} \frac{z - z^*}{1 - \overline{z}^* z}$, where $z^* = \frac{e^{i\alpha_1} |z_1| + z_2}{e^{i\alpha_1} + z_2 |z_1|}$. Since $\psi_2^{-1}(z) = \frac{z + z_2}{1 + \overline{z}_2 z}$, then $((\psi_2 \circ \varphi_2) \circ \psi_2^{-1})(z) = k \frac{z - z_3}{1 - \overline{z}_3 z}$, where

$$k = \frac{z_2 |z_1| - |z_2|^2 + e^{i\alpha_1}(1 - \overline{z}_2 |z_1|)}{1 - z_2 |z_1| + e^{i\alpha_1}(\overline{z}_2 |z_1| - |z_2|^2)}, \tag{34}$$

$$z_3 = \frac{e^{i\alpha_1}(|z_1| - z_2) + z_2(1 - z_2 |z_1|)}{e^{i\alpha_1}(1 - \overline{z}_2 |z_1|) + z_2(|z_1| - \overline{z}_2)}. \tag{35}$$

We prove that there exists a complex number z_2, $|z_2| < 1$, for which the number k is equal to one, that is, the equality

$$2 z_2 |z_1| - 1 - |z_2|^2 + e^{i\alpha_1}(1 - 2\overline{z}_2 |z_1| + |z_2|^2) = 0 \tag{36}$$

holds. The solution of this equation can be found in the form $z_2 = r e^{i\varphi}$, $r \in \mathbb{R}$, $\varphi \in \mathbb{R}$. Then, the equation (36) with respect to r and φ becomes

$$(e^{i\alpha_1} - 1) r^2 + 2r|z_1| \left(e^{i\varphi} - e^{i(\alpha_1 - \varphi)} \right) + e^{i\alpha_1} - 1 = 0.$$

After elementary transformations we have

$$r^2 \sin \frac{\alpha_1}{2} + 2r|z_1| \sin \left(\varphi - \frac{\alpha_1}{2} \right) + \sin \frac{\alpha_1}{2} = 0.$$

Suppose $\varphi = \frac{\alpha_1}{2} - \frac{\pi}{2}$; then this equation is reduced to

$$r^2 \sin \frac{\alpha_1}{2} - 2r|z_1| + \sin \frac{\alpha_1}{2} = 0.$$

One of solutions of the last equation is the number $r = \frac{|z_1| - \sqrt{|z_1|^2 - \sin^2 \frac{\alpha_1}{2}}}{\sin \frac{\alpha_1}{2}}$. By the assumption, $0 < |\sin \frac{\alpha_1}{2}| < |z_1|$, and we see that $r \in \mathbb{R}$ and $0 < |r| < 1$.

Thus,

$$z_2 = e^{i\left(\frac{\alpha_1}{2} - \frac{\pi}{2}\right)} \frac{|z_1| - \sqrt{|z_1|^2 - \sin^2 \frac{\alpha_1}{2}}}{\sin \frac{\alpha_1}{2}} \tag{37}$$

is a solution of equation (36), and $0 < |z_2| < 1$. □

The following statement is correct.

Lemma 2. *Let* $\varphi_2(z) = e^{i\alpha_1}\frac{z-|z_1|}{1-|z_1|z}$, $0 < |\sin\frac{\alpha_1}{2}| < |z_1| < 1$, *be a hyperbolic automorphism of the unit disk D. Then, on the space* $\mathcal{H}(D)$ *the operator* K_{φ_2} *is equivalent to the operator* K_{φ_3} *where* $\varphi_3(z) = \frac{z-z_3}{1-\overline{z}_3 z}$, *and the number* z_3 *is defined by* (35), (37). *The equality* (33) *with* $\psi_2(z) = \frac{z-z_{2i}}{1-\overline{z}_2 z}$ *also holds.*

Remark 4. From (33) it follows that the number z_3 is not equal to zero, since otherwise, we get $\varphi_2(z) = z$.

Using Lemma 1 for the map $\varphi_3(z) = \frac{z-z_3}{1-\overline{z}_3 z}$, we see that on the space $\mathcal{H}(D)$ the operator K_{φ_3} is equivalent to the operator K_{φ_4}, $\varphi_4(z) = \frac{z-|z_3|}{1-|z_3|z}$. Thus,

$$K_{\varphi_3}K_{\psi_3} = K_{\psi_3}K_{\varphi_4}, \tag{38}$$

with $\psi_3(z) = e^{-i\beta_1}z$, $\beta_1 = \arg z_3$. From (32),(33) and (38) it follows that $\psi\circ\varphi_1 = \varphi_4\circ\psi$, where $\psi = \psi_3\circ\psi_2\circ\psi_1$. Thus,

$$K_{\varphi_1}K_{\psi} = K_{\psi}K_{\varphi_4}. \tag{39}$$

The formula (39) and Lemma 1 [13] imply the following result.

Theorem 5. *Let* $\varphi_1(z) = e^{i\alpha_1}\frac{z-z_1}{1-\overline{z_1}z}$, $|\sin\frac{\alpha_1}{2}| < |z_1| < 1$, *be an arbitrary hyperbolic automorphism of the unit disk D. An operator* $\tilde{T} \in \mathcal{L}(\mathcal{H}(D))$ *commutes with the operator* K_{φ_1} *if and only if* $\tilde{T} = K_\psi T K_\psi^{-1}$, *where* $T \in \mathcal{L}(\mathcal{H}(D))$ *and commutes with* K_{φ_4}.

The commutant of the operator K_{φ_4} is described in **2.**

The commutant of the composition operator induced by an arbitrary elliptic automorphism of the unit disk is described in [13]. If $\varphi(z)$ is a parabolic automorphism of the unit disk, then the commutant for the operator K_φ can be described in a similar way.

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