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ON SOLVABILITY OF A PARTIAL INTEGRAL EQUATION IN THE SPACE $L_2(\Omega \times \Omega)$

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Abstract. In this paper we investigate solvability of a partial integral equation in the space $L_2(\Omega \times \Omega)$, where $\Omega = [a, b]^\nu$. We define a determinant for the partial integral equation as a continuous function on Ω and for a continuous kernels of the partial integral equation we give explicit description of the solution.

In the models of solid state physics [1] and also in the lattice field theory [2], there appear so called discrete Schrodinger operators, which are lattice analogies of usual Schrodinger operators in continuous space. The study of spectra of lattice Hamiltonians (that is discrete Schrodinger operators) is an important matter of mathematical physics. Nevertheless, on studying spectral properties of discrete Schrodinger operators there appear partial integral equations in a Hilbert space of multi-variable functions [1, 3]. Therefore, on the investigation of spectra of Hamiltonians considered on a lattice, the study of a solvability problem for partial integral equations in L_2 is essential (and even interesting from the point of view of functional analysis).

A question on the existence of a solution of partial integral equations for functions of two variables was considered in [4–8] and others. In this paper we consider an integral equation on the space of functions of two variables $L_2(\Omega \times \Omega)$, where $\Omega = [a, b]^\nu \subset \mathbb{R}^\nu$, with one partial integral operator. We define a determinant for the partial intergal equation (PIE) as a continuous function on Ω, which helps to obtain the classical Fredholm theorems for a PIE, and for a continuous kernels of the PIE we give explicit description of the solution.

Let $\mathcal{H} = L_2(\Omega \times \Omega)$ ($\mathcal{H}_0 = L_2(\Omega)$) be a Hilbert space of measurable and quadratic integrable functions on $\Omega \times \Omega$ (on Ω), where $\Omega = [a, b]^\nu$. We denote by μ the Lebesgue measure on Ω and define the measure $\widehat{\mu}$ on $\Omega \times \Omega$ by $\widehat{\mu} = \mu \otimes \mu$. In the space $\mathcal{H}$, we consider a partial integral operator (PIO) T_1 defined by

$$T_1 f = \int_\Omega k(x, s, y) f(s, y)\, ds, \quad f \in \mathcal{H},$$

where $k(x, s, y) \in L_2(\Omega^3)$. The function $k(x, s, y)$ is called *kernel* of the PIO T_1.

If there exists a number M such that

$$(I) \qquad b(t) \le M \quad \text{for almost all} \quad t \in \Omega,$$

then the operator T_1 is a linear bounded operator on $\mathcal{H}$ and it is uniquely defined by its kernel $k(x, s, y)$, where

$$b(t) = \int_\Omega \int_\Omega |k(x, s, t)|^2 dx ds.$$

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A kernel $\overline{k(s,x,y)}$ corresponds to the adjoint operator T_1^*, i.e.

$$T_1^* f = \int_\Omega \overline{k(s,x,y)} f(s,y)\, ds, \quad f \in \mathcal{H}.$$

Consider a family of operators $\{K_\alpha\}_{\alpha\in\Omega}$ in $\mathcal{H}_0$ associated with T_1 by the following formula

$$K_\alpha \varphi = \int_\Omega k(x,s,\alpha)\varphi(s)\, ds, \quad \varphi \in \mathcal{H}_0,$$

where $k(x,s,y)$ is the kernel of T_1.

Further, if a set of integrability in the integral is absent, then we mean integrability by the set Ω. First, we consider certain properties of PIO T_1 with the kernel $k(x,s,y) \in L_2(\Omega^3)$ satisfying the condition (I) and then we study solvability of the PIE with the kernel $k(x,s,y) \in C(\Omega^3)$.

Lemma 1. *Let $f \in \mathcal{H}$ and $\varphi_y(x) = f(x,y)$, where $y \in \Omega$ is fixed. Then for an arbitrary $\varepsilon > 0$, there exists a subset $\Omega_\varepsilon \subset \Omega$ such that $\mu(\Omega_\varepsilon) \geq \mu(\Omega) - \varepsilon$ and $\varphi_\alpha \in \mathcal{H}_0$, $\alpha \in \Omega_\varepsilon$. Moreover, $\|\varphi_\alpha\| \leq C$, $\alpha \in \Omega_\varepsilon$ for some $C > 0$.*

Proof. Let $f \in \mathcal{H}$ and $d = \|f\|^2 \neq 0$. Define two sequences of measurable subsets in Ω by the following equalities:

$$A_n = \left\{ y : \int |f(x,y)|^2 dx < n,\ y \in \Omega \right\}, \quad n \in \mathbb{N},$$

$$B_n = \left\{ y : \int |f(x,y)|^2 dx \geq n,\ y \in \Omega \right\}, \quad n \in \mathbb{N}.$$

The sequences of subsets $\{A_n\}$ and $\{B_n\}$ hold the following properties:
1^o. $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ and $B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$;
2^o. $\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$ and $\lim_{n\to\infty} B_n = \bigcap_{n\in\mathbb{N}} B_n$;
3^o. $\Omega = A_n \cup B_n$ and $A_n \cap B_n = \varnothing$, $n \in \mathbb{N}$.

Further, we define two bounded sequences of non-negative numbers a_n and b_n by

$$a_n = \int_{A_n} dy \int_\Omega |f(x,y)|^2 dx \quad \text{and} \quad b_n = \int_{B_n} dy \int_\Omega |f(x,y)|^2 dx.$$

The sequences of numbers a_n and b_n have the properties:
4^o. a_n is increasing and b_n is decreasing;
5^o. $a_n + b_n = d$, $n \in \mathbb{N}$.

From the boundedness and monotonicity of the sequences a_n and b_n we infer that both of them have finite limit. By the property 5^o and by the construction of the set B_n we obtain that $d - a_n \geq 0$, $n \in \mathbb{N}$ and $d \geq a_n + n\mu(B_n)$, $n \in \mathbb{N}$. Then $\mu(B_n) \leq (d - a_n)/n$, $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} \mu(B_n) = 0$. By the property 3^o we have $\mu(A_n) = \mu(\Omega) - \mu(B_n)$, $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \mu(A_n) = \mu(\Omega)$, i.e. for an arbitrary small $\varepsilon > 0$ there exists a number $n_0 \in \mathbb{N}$ such that $\mu(\Omega) - \varepsilon \leq \mu(A_{n_0}) \leq \mu(\Omega)$ and $0 \leq \mu(B_{n_0}) < \varepsilon$. Moreover, this means that

$$\int |\varphi_\alpha(x)|^2 dx = \int |f(x,\alpha)|^2 dx < n_0, \quad \alpha \in A_{n_0}.$$

Then, for the set $\Omega_\varepsilon = A_{n_0}$ we have $\varphi_\alpha \in \mathcal{H}_0$, $\alpha \in \Omega_\varepsilon$ and $\|\varphi_\alpha\| \leq C$, $\alpha \in \Omega_\varepsilon$ for all $C \geq n_0$. □

Corollary 1. *Let $f \in \mathcal{H}$, $\|f\| = 1$ and $\varphi_y(x) = f(x,y)$, where $y \in \Omega$ is fixed. Then there exists a measurable subset $\Omega_0 \subset \Omega$ such that, $\mu(\Omega_0) > 0$ and the family $\{\varphi_\alpha\}_{\alpha\in\Omega}$ of functions on Ω has the following property: $\varphi_\alpha \in \mathcal{H}_0$, $\alpha \in \Omega_0$ and $0 < \|\varphi_\alpha\| \leq C$, $\alpha \in \Omega_0$ for some $C > 0$.*

Corollary 2. *Let $f \in \mathcal{H}$. Then there exists a decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$ and*

(a) *for each $n \in \mathbb{N}$ there exists a measurable subset $\Omega_n \subset \Omega$ with the property $\mu(\Omega_n) > \mu(\Omega) - \varepsilon_n$ such that $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_n \subset \ldots$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$;*

(b) *for each $n \in \mathbb{N}$, $\varphi_\alpha^{(n)} \in \mathcal{H}_0$, $\alpha \in \Omega_n$ and there exists a positive number C_n such that $\|\varphi_\alpha^{(n)}\| \le C_n$, $\forall \alpha \in \Omega_n$, where $\varphi_\alpha^{(n)}(x) = f(x, \alpha)$, $\alpha \in \Omega_n$;*

(c) *for any $n \in \mathbb{N}$, the function*

$$f_n(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in \Omega \times \Omega_n, \\ 0, & \text{otherwise} \end{cases}$$

belongs to $\mathcal{H}$ and $\lim_{n\to\infty} f_n(x,y) = f(x,y)$.

Proposition 1. *The following two conditions are equivalent:*

(i) *A number $\lambda \in \mathbb{C}$ is an eigenvalue for the operator T_1;*

(ii) *A number $\lambda \in \mathbb{C}$ is an eigenvalue for operators $\{K_\alpha\}_{\alpha\in\Omega_0}$, where Ω_0 is some subset of Ω such that $\mu(\Omega_0) > 0$.*

Proof. We start with the implication $(i) \Rightarrow (ii)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of operator T_1, i.e. $T_1 f_0 = \lambda f_0$ for some $f_0 \in \mathcal{H}$, $\|f_0\| = 1$. We define $\varphi_\alpha = \varphi_\alpha(x) = f_0(x,\alpha)$, $\alpha \in \Omega$. Therefore, we have a family $\{\varphi_\alpha\}_{\alpha\in\Omega}$ of functions on Ω. Then, by Corollary 1, there exists a subset $\Omega_0 \subset \Omega$ such that $\mu(\Omega_0) > 0$ and $\varphi_\alpha \in \mathcal{H}_0$, $\alpha \in \Omega_0$, $\|\varphi_\alpha\| \ne 0$, $\forall \alpha \in \Omega_0$. For an arbitrary $\alpha \in \Omega_0$ we have

$$K_\alpha \varphi_\alpha = \int k(x,s,\alpha)\varphi_\alpha(s)\,ds = \int k(x,s,\alpha) f_0(s,\alpha)\,ds = \lambda f_0(x,\alpha) = \lambda\varphi_\alpha(x),$$

i.e. the number λ is an eigenvalue for K_α, $\alpha \in \Omega_0$.

Now, we prove the implication $(ii) \Rightarrow (i)$. Suppose that there exists a subset Ω_0 in Ω with $\mu(\Omega_0) > 0$ and a number $\lambda \in \mathbb{C}$ is an eigenvalue for operators K_α, $\alpha \in \Omega_0$. Since K_α is a compact operator for all $\alpha \in \Omega$, then there exists a function $f_0 \in \mathcal{H}$, $f_0 \ne 0$ [9] such that $T_1 f_0 = \lambda f_0$. □

Proposition 2. *If $\lambda \in \mathbb{C}$ is an eigenvalue of the operator T_1, then the number $\overline{\lambda}$ is an eigenvalue of the operator T_1^*.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of the operator T_1. Then there exists a subset $\Omega_0 \subset \Omega$, $\mu(\Omega_0) > 0$ such that λ is an eigenvalue of the every compact operator K_α, $\alpha \in \Omega_0$. Therefore the number $\overline{\lambda}$ is an eigenvalue of every operator K_α^*, $\alpha \in \Omega_0$

$$K_\alpha^* \varphi = \int \overline{k(s,x,\alpha)}\varphi(s)\,ds, \quad \varphi \in \mathcal{H}_0.$$

By Proposition 1, the number $\overline{\lambda}$ is an eigenvalue of the adjoint operator T_1^*. □

Proposition 3. *Every eigenvalue of the operator T_1 has infinite multiplicity.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of T_1. Hence, there exists an element $f_0 \in \mathcal{H}$, $\|f_0\| = 1$ such that $T_1 f_0 = \lambda f_0$. We consider a subspace $L_0 \subset \mathcal{H} : L_0 = \{\widetilde{f} \in \mathcal{H} : \widetilde{f}(x,y) = b(y) f_0(x,y)$, where $b = b(y)$ is an arbitrary bounded measurable function on $\Omega\}$. For every $\widetilde{f} \in L_0$ we have $T_1\widetilde{f} = \lambda\widetilde{f}$, i.e. $L_0 \subset M_\lambda$, where M_λ is the eigen-subspace corresponding to λ. But, the subspace L_0 is infinite dimensional, therefore, M_λ is also infinite dimensional subspace of $\mathcal{H}$. □

Now we consider the equation

$$f - \varkappa T_1 f = g_0 \tag{1}$$

in the space $\mathcal{H}$, where f is an unknown function from $\mathcal{H}$, $g_0 \in \mathcal{H}$ is given (known) function, $\varkappa \in \mathbb{C}$ is a parameter of the equation, T_1 is PIO with a kernel $k(x,s,y)$ continuous on Ω^3.

It is clear that, if $k(x,s,y) \in C(\Omega^3)$ then for all $\alpha \in \Omega$ the integral operators K_α on $\mathcal{H}_0$ are compact. For each $\alpha \in \Omega$ we denote by $\Delta_\alpha^{(1)}(\varkappa)$ and $M_\alpha^{(1)}(x,s;\varkappa)$, respectively, the Fredholm determinant and the Fredholm minor of the operator $E-\varkappa K_\alpha$, $\varkappa \in \mathbb{C}$ [10], where E is the identity operator in $\mathcal{H}_0$. According to the continuity of the kernel $k(x,s,y)$ and uniform convergence of the series for $\Delta_\alpha^{(1)}(\varkappa)$ and $M_\alpha^{(1)}(x,s;\varkappa)$ for every $\varkappa \in \mathbb{C}$ we obtain [10] that the function $D_1(y) = D_1(y;\varkappa)$ on Ω and the function $M_1(x,s,y) = M_1(x,s,y;\varkappa)$ on Ω^3, which are given respectively by the equalities

$$D_1(y;\varkappa) = \Delta_y^{(1)}(\varkappa), \;\; y \in \Omega \quad \text{and} \quad M_1(x,s,y;\varkappa) = M_y^{(1)}(x,s;\varkappa), \;\; y \in \Omega,$$

are continuous functions on Ω and Ω^3 for every $\varkappa \in \mathbb{C}$.

The continuous function $D_1(y) = D_1(y;\varkappa)$ ($M_1(x,s,y) = M_1(x,s,y;\varkappa)$) is called *a determinant* (*a minor*) of the operator $E-\varkappa T_1$, $\varkappa \in \mathbb{C}$.

Definition 1. If for a number $\varkappa_0 \in \mathbb{C}$ $D_1(y;\varkappa_0) \neq 0$ for all $y \in \Omega$, then $\varkappa_0$ is called *a regular number* of the PIE (1). A set of all regular numbers of the PIE (1) is denoted by $\mathcal{R}_{T_1}$.

Definition 2. If for a number $\varkappa_0 \in \mathbb{C}$ there exists a point $y_0 \in \Omega$ such that

$$D_1(y_0;\varkappa_0) = 0,$$

then $\varkappa_0$ is called *a singular number* of the PIE (1). A set of all singular numbers of the the PIE (1) is denoted by $\mathcal{S}_{T_1}$.

Definition 3. If for a number $\varkappa_0 \in \mathbb{C}$ there exists a measurable subset $\Omega_0 \subset \Omega$ with $\mu(\Omega_0) > 0$ such that $D_1(y;\varkappa_0) = 0$, $\forall y \in \Omega_0$, then $\varkappa_0$ is called *a characteristic number* of the the PIE (1). A set of all characteristic numbers of the PIE (1) is denoted by $\mathcal{X}_{T_1}$.

Definition 4. A number $\varkappa_0 \in \mathbb{C}$ is called *an essential number* of the PIE (1) if $\varkappa_0 \in \mathcal{S}_{T_1} \setminus \mathcal{X}_{T_1}$. A set of all essential numbers of the PIE (1) is denoted by $\mathcal{E}_{T_1}$.

Thus, for a parameter $\varkappa$ of the PIE (1), there exist subsets $\mathcal{R}_{T_1}$, $\mathcal{S}_{T_1}$, $\mathcal{X}_{T_1}$, and $\mathcal{E}_{T_1}$ in $\mathbb{C}$, which have the following relations:

(i) $\mathcal{R}_{T_1} \cup \mathcal{S}_{T_1} = \mathbb{C}$ and $\mathcal{R}_{T_1} \cap \mathcal{S}_{T_1} = \varnothing$;

(ii) $\mathcal{X}_{T_1} \cup \mathcal{E}_{T_1} = \mathcal{S}_{T_1}$ and $\mathcal{X}_{T_1} \cap \mathcal{E}_{T_1} = \varnothing$.

From Definitions 1, 2, 3 and 4 one gets that for an arbitrary non-zero PIO T_1 sets $\mathcal{R}_{T_1}$ and $\mathcal{E}_{T_1}$ are non-empty, but $\mathcal{X}_{T_1}$ may be empty. For example, consider a PIE in the space $L_2([0,1]^2)$

$$f(x,y) - \varkappa \int_0^1 e^{x-s} e^y f(s,y)\, ds = g_0(x,y),$$

where f is an unknown function in $L_2([0,1]^2)$, $g_0 \in L_2([0,1]^2)$ is an arbitrary given function. For this PIE, the determinant has a simple form $D_1(y;\varkappa) = 1 - \varkappa e^y$, $y \in [0,1]$. Therefore $\mathcal{S}_{T_1} = \left[e^{-1}, 1\right]$ and $\mathcal{X}_{T_1} = \varnothing$.

From Proposition 1 and Definition 3 it follows

Theorem 1. *A number $\lambda \in \mathbb{C}$, $\lambda \neq 0$, is an eigenvalue of the operator T_1 if and only if $\lambda^{-1} \in \mathcal{X}_{T_1}$.*

Theorem 2. a) *if $\varkappa_0 \in \mathcal{E}_{T_1}$, then $\overline{\varkappa_0} \in \mathcal{E}_{T_1^*}$;*

b) *if $\varkappa_0 \in \mathcal{X}_{T_1}$, then $\overline{\varkappa_0} \in \mathcal{X}_{T_1^*}$.*

Proof. Let $\varkappa_0 \in \mathcal{E}_{T_1}$. Then there exists a point $y_0 \in \Omega$ with $D_1(y_0;\varkappa_0) = 0$ and we have $\mu\{y \in \Omega : D_1(y;\varkappa_0) = 0\} = 0$. But using a property of the determinant $D_1(y;\varkappa_0)$ we obtain that $\overline{D_1(y;\varkappa_0)} = \widetilde{D}_1(y;\overline{\varkappa}_0)$, where $\widetilde{D}_1(y;\overline{\varkappa}_0)$ is a determinant of the operator $E - \overline{\varkappa}_0 T_1^*$. Therefore, we have $\widetilde{D}_1(y_0;\overline{\varkappa}_0) = 0$ and $\mu\left\{y \in \Omega : \widetilde{D}_1(y;\overline{\varkappa}_0) = 0\right\} = 0$, i.e. the number $\overline{\varkappa}_0$ is an essential number of the adjoint equation $f - \overline{\varkappa}_0 T_1^* f = g_0$, and the proof of property a) is complete. The proof of the property b) can be proceeded analogously. □

Theorem 3. *If $\varkappa_0 \in \mathcal{R}_{T_1}$ then for every $g_0 \in \mathcal{H}$ the PIE (1) has a unique solution on $\mathcal{H}$ and it is of the form $f = g_0 + \varkappa_0 B g_0$, where an operator $B = B(\varkappa_0)$ acts in $\mathcal{H}$ by the formula*

$$Bg = \int \frac{M_1(x,s,y;\varkappa_0)}{D_1(y;\varkappa_0)} g(s,y)\,ds, \quad g \in \mathcal{H}, \tag{2}$$

but the corresponding homogeneous equation $f - \varkappa_0 T_1 f = 0$ has only trivial solution (zero solution). Here $D_1(y;\varkappa_0)$ and $M_1(x,s,y;\varkappa_0)$ are the determinant and the minor of the operator $E - \varkappa_0 T_1$, respectively.

Proof. Let $\varkappa_0 \in \mathcal{R}_{T_1}$ and $\varkappa_0 \neq 0$. First, we prove that PIE (1) is solvable in $\mathcal{H}$. By Corollary 2, for the function g_0 there exists a decreasing sequence of non-negative numbers ε_n and a sequence of increasing measurable subsets $\Omega_n \subset \Omega$, which satisfy the properties (a), (b) and (c) with $\lim_{n\to\infty} \varepsilon_n = 0$. For every Ω_n we define a subspace $L_2^{(n)} = L_2^{(n)}(\Omega \times \Omega)$ as follows: a function $\widetilde{f} \in \mathcal{H}$ belongs to the subspace $L_2^{(n)}$, if it satisfies the following conditions:

(i) $\varphi_\alpha^{(n)}(x) = \widetilde{f}(x,\alpha) \in \mathcal{H}_0$, $\forall \alpha \in \Omega_n$;

(ii) there exists a positive number C_n such that $\|\varphi_\alpha^{(n)}\| \leq C_n$, $\forall \alpha \in \Omega_n$;

(iii) $\widetilde{f}(x,y) = 0$ if $(x,y) \in \Omega \times (\Omega \setminus \Omega_n)$.

For every $f \in \mathcal{H}$, there exists a sequence $f_n \in L_2^{(n)}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} f_n = f$. Therefore, first we find a solution of the equation (1) in the space $L_2^{(n)}$ and we can find a solution of the equation (1) in the space $\mathcal{H}$ as the limit $f(x,y) = \lim_{n\to\infty} \widetilde{f}_n(x,y)$, where $\widetilde{f}_n$ are solutions of the equation (1) in the space $L_2^{(n)}$. Thus, the equation (1) in $L_2^{(n)}$ reduces to the following one:

$$\widetilde{f}_n(x,y) - \varkappa_0 T_1 \widetilde{f}_n(x,y) = g_n(x,y), \tag{3}$$

where g_n is an element of $L_2^{(n)}$ corresponding to the function $g_0(x,y)$.

Hence, by the property (b) of Corollary 2, for each fixed $y \in \Omega$, the equation (3) reduces to the following second type Fredholm integral equation in $\mathcal{H}_0$:

$$\varphi_\alpha^{(n)}(x) - \varkappa_0 K_\alpha \varphi_\alpha^{(n)}(x) = h_\alpha^{(n)}(x), \quad \alpha \in \Omega, \tag{3$'$}$$

where $\varphi_\alpha^{(n)}(x) = \widetilde{f}_n(x,\alpha)$ is an unknown function in $\mathcal{H}_0$, $h_\alpha^{(n)}(x) = g_n(x,\alpha)$ is a given function in $\mathcal{H}_0$.

By the first fundamental Fredholm theorem, the equation (3$'$) for every $\alpha \in \Omega_n$ has the only solution

$$\varphi_\alpha^{(n)} = \varphi_\alpha^{(n)}(x) = h_\alpha^{(n)}(x) + \varkappa_0 B_\alpha h_\alpha^{(n)}(x),$$

where the operator $B_\alpha = B_\alpha(\varkappa_0)$ acts in $\mathcal{H}_0$ by the formula

$$B_\alpha \varphi = \int \frac{M_\alpha^{(1)}(x,s;\varkappa_0)}{\Delta_\alpha^{(1)}(\varkappa_0)} \varphi(s)\,ds \quad (\alpha \in \Omega_n)$$

and B_α is compact. Here $\Delta_\alpha^{(1)}(\varkappa_0)$ and $M_\alpha^{(1)}(x,s;\varkappa_0)$ are the Fredholm determinant and the Fredholm minor of the operator $E - \varkappa_0 K_\alpha$, respectively.

It is clear, that if $\alpha \in \Omega \setminus \Omega_n$ then the equation (3') has the solution $\varphi_\alpha^{(n)}(x) = 0$. Hence, the function $\widetilde{f}_n(x,y) = \varphi_y^{(n)}(x)$ belongs to the subspace $L_2^{(n)}$ and it is a solution of the equation (3), where $\varphi_\alpha^{(n)}(x)$, $\varphi \in \Omega$ the solutions of the equation (3'). We define the function $f_0 \in \mathcal{H}$ by the equality $f_0(x,y) = (E + \varkappa_0 B) g_0(x,y)$, where the operator

$B = B(\varkappa_0)$ acts in $\mathcal{H}$ by the formula (2) and it is a bounded operator. But, if $y \in \Omega_n$ then we have

$$f_0(x,y) = g_0(x,y) + \varkappa_0 B g_0(x,y) = g_n(x,y) + \varkappa_0 B g_n(x,y) = \\ = h_y^{(n)}(x) + \varkappa_0 B_y h_y^{(n)}(x) = \varphi_y^{(n)}(x) = \widetilde{f}_n(x,y)$$

and for every $y \in \Omega \setminus \Omega_n$ we have $f_0(x,y) = \varphi_y^{(n)}(x) = 0$. Thus, by the property (c) of Corollary 2 we obtain $f_0(x,y) = \lim_{n\to\infty} \widetilde{f}_n(x,y)$. Therefore the function $f(x,y) = f_0(x,y) = (E + \varkappa_0 B) g_0(x,y)$ is a solution of the equation (1).

Thus, we have proved that the equation (1) is solvable. Now we prove uniqueness of the solution of the equation (1). Suppose, $f_1 \in \mathcal{H}$ and $f_2 \in \mathcal{H}$ are solutions of the equation (1), where $f_1 \neq f_2$. Then, for the function $\widehat{f} = f_1 - f_2 \neq 0$ we have $\widehat{f} - \varkappa_0 T_1 \widehat{f} = 0$, i.e. the homogeneous equation $f - \varkappa_0 T_1 f = 0$ has a solution $\widehat{f} \neq 0$. Hence, the number $\varkappa_0^{-1}$ is an eigenvalue of T_1, then by Theorem 1 we obtain that $\varkappa_0 \in \mathcal{X}_{T_1}$. But this is impossible since $\varkappa_0 \in \mathcal{R}_{T_1}$.

Using Proposition 1 we can show that for $\varkappa_0 \in \mathcal{R}_{T_1}$ the homogeneous equation $f - \varkappa_0 T_1 f = 0$ has only trivial solution. The proof is complete. □

Theorem 4. *Let $\varkappa_0 \in \mathcal{E}_{T_1}$. If the free term g_0 of the PIE (1) satisfies the condition*

$$(II) \qquad \int \frac{\int |g_0(s,y)|^2 ds}{|D_1(y;\varkappa_0)|^2}\, dy < \infty,$$

then PIE (1) has a unique solution on $\mathcal{H}$ and it has a form $f = g_0 + \varkappa_0 B g_0 \in \mathcal{H}$, but corresponding homogeneous equation $f - \varkappa_0 T_1 f = 0$ has only trivial solution, where the operator B is given by (2).

Proof. Let $\varkappa_0 \in \mathcal{E}_{T_1}$. Put $\Omega' = \{y \in \Omega : D_1(y;\varkappa_0) = 0\}$. It is evident that $\Omega' \neq \varnothing$ and $\mu(\Omega') = 0$. However, for every $y \in \Omega \setminus \Omega'$ the function $f_0(x,y) = g_0(x,y) + \varkappa_0 B g_0(x,y)$ satisfies the equation (1). Now it is enough to show that $f_0 \in \mathcal{H}$. Suppose that g_0 satisfies the condition (II). We have

$$\int\int |Bg_0(x,y)|^2 dxdy = \int\int \left|\int \frac{M_1(x,s,y;\varkappa_0)}{D_1(y;\varkappa_0)} g_0(s,y)\, ds\right|^2 dxdy \\ \leq \int\int \left(\frac{\int |M_1(x,s,y;\varkappa_0)| \cdot |g_0(s,y)|\, ds}{|D_1(y;\varkappa_0)|}\right)^2 dxdy \\ \leq N_0^2 \int\int \frac{\left(\int |g_0(s,y)|\, ds\right)^2}{|D_1(y;\varkappa_0)|^2}\, dxdy \leq N_0^2 \mu(\Omega) \int \frac{\left(\int |g_0(s,y)|\, ds\right)^2}{|D_1(y;\varkappa_0)|^2}\, dy,$$

where $N_0 = \max_{x,s,y\in\Omega} |M_1(x,s,y;\varkappa_0)|$.

But for the function $g_0(x,y)$ from the Cauchy-Schwartz inequality for almost all $y \in \Omega$ we have

$$\int |g_0(s,y)|\, ds \leq \sqrt{\mu(\Omega)} \cdot \sqrt{\int |g_0(s,y)|^2 ds}.$$

Hence, we obtain

$$\int\int |Bg_0(x,y)|^2 dxdy \leq (N_0 \cdot \mu(\Omega))^2 \cdot \int \frac{\int |g_0(s,y)|^2 ds}{|D_1(y;\varkappa_0)|^2}\, dy < \infty,$$

i.e. $Bg_0 \in \mathcal{H}$, therefore $f_0 = g_0 + \varkappa_0 B g_0 \in \mathcal{H}$ and f_0 is a solution of the equation (1).

Uniqueness of the solution follows from Theorem 1. Using Proposition 1 one can also show that the homogeneous equation $f - \varkappa_0 T_1 f = 0$ has only the trivial solution. □

Remark 1. The condition (II) in Theorem 4 is natural.

For example, for the equation

$$f(x,y) - \varkappa \int_0^1 e^{x-s} y f(s,y)\,ds = e^x y^{1/2} \tag{4}$$

in the space $L_2([0,1]^2)$, we have $D_1(y;\varkappa) = 1 - \varkappa y$, $y \in [0,1]$ and $M_1(x,s,y;\varkappa) = e^{x-s}y$. Hence, $\mathcal{S}_{T_1} = \mathcal{E}_{T_1} = [1,\infty)$. For each $\varkappa \notin [1,\infty)$, the equation (4) has the solution

$$f_0(x,y) = \frac{e^x y^{1/2}}{1 - \varkappa y} \in L_2([0,1]^2). \tag{5}$$

If $\varkappa_0 \in [1,\infty)$, then the function (10) is a continuous function on the set $\Omega' = [0,1] \times ([0,1] \setminus \{1/\varkappa_0\})$ with $\widehat{\mu}(\Omega') = \widehat{\mu}([0,1] \times [0,1])$ and for every $y \in [0,1] \setminus \{1/\varkappa_0\}$ the function (10) satisfies the equation (4), but $f_0 \notin L_2([0,1]^2)$.

Remark 2. Let $k(x,s,y) \in C(\Omega^3)$. Then, in the case of $\varkappa_0 \in \mathcal{E}_{T_1}$, the set of all functions $g \in \mathcal{H}$ (see Theorem 4), which satisfies the inequality

$$\int \frac{\int |g(s,y)|^2 ds}{|D_1(y;\varkappa_0)|^2}\,dy < \infty,$$

is infinite dimensional subspace in $\mathcal{H}$.

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