Methods of Functional Analysis and Topology
Vol. 14 (2008), no. 4, pp. 314–322

FUNCTOR OF SEMIADDITIVE FUNCTIONALS

D. E. DAVLETOV AND G. F. DJABBAROV

Abstract. In the present paper we describe semiadditive functionals and establish that the construction generated by semiadditive functionals forms a covariant functor. We show that the functor of semiadditive functionals is a normal functor acting in category of compact sets.

1. Introduction

The general theory of covariant functors on the category Comp of compact Hausdorff topological spaces and their continuous mappings originates in the fundamental work by E. Shchepin [1], where he distinguished some elementary properties of covariant functors in the category of compacts and defined the concept of normal functor.

The following functors are well investigated among normal functors: the functor P of probability measures; the functor exp of a hyperspace (see. [2]–[8]).

In [9], T. Radul introduced the functor O of weakly additive order-preserving normed functionals in the category of compacts. He proved that the functor O is not normal. The functor O has the following important property: the functors P, exp and λ can be realized as subfunctors of the functor O. In [10], A. Zaitov extended the functor O up to the functor $O \circ \beta$ and investigated categorical properties of $O \circ \beta$. In [11], R. Beshimov extended the functor O up to the functor O_β in the category of Tychonoff spaces and their continuous mappings. Categorical properties of positively-homogeneous functionals were studied in [12], [13].

In this paper we will investigate the functor of semiadditive functionals.

A general form of semiadditive functionals will be given in Section 3 (Theorem 3.3).

In Section 4 we investigate categorical properties of the functor of semiadditive functionals OS. We will prove that OS is a normal functor (Theorem 4.10).

2. Preliminaries

Let X be a compactum. By $C(X)$ we denote the set of all continuous functions $f : X \to \mathbb{R}$ with the usual (pointwise) operations and sup-norm, i.e., with the norm $\|f\| = \sup\{|f(x)| : x \in X\}$. For each $c \in \mathbb{R}$, by c_X we denote the constant function defined by the formula $c_X(x) = c$, $x \in X$. Let $\varphi, \psi \in C(X)$. The inequality $\varphi \leq \psi$ means $\varphi(x) \leq \psi(x)$ for all $x \in X$.

Definition 2.1. [8]. A functional $\nu : C(X) \to \mathbb{R}$ is called:

1) *weakly additive* if the equality $\nu(\varphi + c_X) = \nu(\varphi) + c \cdot \nu(1_X)$ holds for each $c \in \mathbb{R}$ and $\varphi \in C(X)$;
2) *order-preserving* if for all $\varphi, \psi \in C(X)$ with $\varphi \leq \psi$ we have $\nu(\varphi) \leq \nu(\psi)$;
3) *normed* if $\nu(1_X) = 1$;
4) *positively-homogeneous* if $\nu(t\varphi) = t\nu(\varphi)$ for all $\varphi \in C(X)$, $t \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0, +\infty)$;

2000 *Mathematics Subject Classification.* Primary: 54B30, 54C35, 18C15.
Key words and phrases. Normal functor, semiadditive functionals, probability measures.

5) *semiadditive* if $\nu(f+g) \le \nu(f) + \nu(g)$ for all $f, g \in C(X)$.

For a compactum X by $O(X)$ we denote the set of all weakly additive, order-preserving, normed functionals. By $OH(X)$ we denote the set of all positively-homogeneous functionals from $O(X)$, and by $OS(X)$ we denote the set of semiadditive functionals from $OH(X)$. For brevity, we'll call elements of the set $OS(X)$ *semiadditive functionals*. This set is endowed with the topology of pointwise convergence. The base of neighborhoods for the functional $\nu \in OS(X)$ is formed by the sets

$$\langle \nu; \varphi_1, ...\varphi_k; \varepsilon \rangle = \left\{ \nu' \in OS(X) : |\nu'(\varphi_i) - \nu(\varphi_i)| < \varepsilon, i = \overline{1,k} \right\},$$

where $\varphi_i \in C(X)$, $i = \overline{1,k}$, $k \in \mathbb{N}$, $\varepsilon > 0$.

By $\mathbf{2} = \{0, 1\}$ we denote the two-point set with discrete topology.

Preliminaries.

Let δ_i $(i = 0, 1)$ be the Dirac functionals on $C(\mathbf{2})$. We define the functionals $\delta_0 \vee \delta_1$ and $\delta_0 \wedge \delta_1$ by the following rules

$$(\delta_0 \vee \delta_1)(f) = \max\{f(0), f(1)\}, \quad f \in C(\mathbf{2})$$

and

$$(\delta_0 \wedge \delta_1)(f) = \min\{f(0), f(1)\}, \quad f \in C(\mathbf{2}).$$

It is known [12] that $OH(\mathbf{2})$ is affine homeomorphic to the square with the vertices in points δ_0, δ_1, $\delta_0 \vee \delta_1$, $\delta_0 \wedge \delta_1$. Moreover, any element of $OH(\mathbf{2})$ has one of the following forms

$$\nu = t_0\delta_0 + t_1\delta_1 + t_2(\delta_0 \vee \delta_1) \tag{1}$$

or

$$\nu' = t_0\delta_0 + t_1\delta_1 + t_2(\delta_0 \wedge \delta_1), \tag{2}$$

where $t_i \ge 0$, $i = \overline{0,2}$, $\sum_{i=0}^{2} t_i = 1$.

It is clear that a functional ν in the form (1) belongs to $OS(\mathbf{2})$. Let's show that at $t_2 \ne 0$ the functional ν' in the form (2) doesn't belong to $OS(\mathbf{2})$.

Indeed, let $\nu' = t_0\delta_0 + t_1\delta_1 + t_2(\delta_0 \wedge \delta_1)$, $t_2 \ne 0$. Take $f, g \in C(\mathbf{2})$ such that $f(0) = 1$, $f(1) = 0$, $g(0) = 0$, $g(1) = 1$. Then $\nu'(f+g) = t_0 + t_1 + t_2$ and $\nu'(f) = t_0$, $\nu(g) = t_1$, which implies $\nu'(f+g) > \nu'(f) + \nu'(g)$.

Thus, each element of $OS(\mathbf{2})$ has the form (1). Hence, $OS(\mathbf{2})$ is affine homeomorphic to the triangle with the vertices δ_0, δ_1, $\delta_0 \vee \delta_1$.

3. Description of semiadditive functionals

In this paragraph we obtain a general form of semiadditive functionals on $C(X)$.

Proposition 3.1. *For any compactum X, the space $OS(X)$ is a convex compactum.*

Proof. Since $OS(X) \subset OH(X)$ and $OH(X)$ is a convex compact set [12, Theorem 1], it is sufficient to show that $OS(X)$ is a convex closed set of $OH(X)$. Let $\nu_1, \nu_2 \in OS(X)$ and $t \in [0, 1]$. We have

$$\begin{aligned}(t\nu_1 + (1-t)\nu_2)(\varphi + \psi) &= t\nu_1(\varphi + \psi) + (1-t)\nu_2(\varphi + \psi) \\ &\le t\nu_1(\varphi) + t\nu_1(\psi) + (1-t)\nu_2(\varphi) + (1-t)\nu_2(\psi) \\ &= (t\nu_1 + (1-t)\nu_2)(\varphi) + (t\nu_1 + (1-t)\nu_2)(\psi),\end{aligned}$$

i.e., the functional $t\nu_1 + (1-t)\nu_2$ is semiadditive. This implies that $t\nu_1 + (1-t)\nu_2 \in OS(X)$.

Let now $(\nu_\alpha)_{\alpha\in I} \subset OS(X)$ be an arbitrary convergent net and $\nu_0 \in OH(X)$ be the limit of this net. We show that $\nu_0 \in OS(X)$. Since $\nu_\alpha(\varphi+\psi) \le \nu_\alpha(\varphi)+\nu_\alpha(\psi)$ for each $\varphi, \psi \in C(X)$, $\alpha \in I$, we have

$$\lim \nu_\alpha(\varphi+\psi) \le \lim(\nu_\alpha(\varphi)+\nu_\alpha(\psi)) = \lim \nu_\alpha(\varphi) + \lim \nu_\alpha(\psi) = \nu_0(\varphi)+\nu_0(\psi).$$

On the other hand,

$$\lim \nu_\alpha(\varphi+\psi) = \nu_0(\varphi+\psi).$$

The last equation shows that $\nu_0(\varphi+\psi) \le \nu_0(\varphi)+\nu_0(\psi)$. So, $\nu_0 \in OS(X)$. The proposition is proved. □

Let $P(X)$ be a space of all positive normed linear functionals on $C(X)$, A be a non-empty subset of $P(X)$, and $f \in C(X)$. Then $|\mu(f)| \le \|f\|$ for each $\mu \in A$, and therefore the set $\{\mu(f) : \mu \in A\}$ is bounded above. Hence, for each $f \in C(X)$ there exists

$$\nu_A(f) = \sup\{\mu(f) : \mu \in A\}, \quad f \in C(X). \tag{3}$$

Proposition 3.2. *Let A be a non-empty set of $P(X)$. Then*

a) *the functional $\nu_A : C(X) \to \mathbb{R}$ belongs to $OS(X)$;*
b) $\nu_A = \nu_{\mathrm{co}(A)}$ *where* $\mathrm{co}(A)$ *is a convex envelope of* A;
c) $\nu_A = \nu_{\mathrm{cl}(A)}$ *where* $\mathrm{cl}(A)$ *is the closure of* A;
d) $\nu_A = \nu_{\mathrm{cl}(\mathrm{co}(A))}$.

Proof. a)

1) Let $f \in C(X)$ and $c \in \mathbb{R}$. We have $\nu_A(f+c_X) = \sup\{\mu(f+c_X) : \mu \in A\} = \sup\{\mu(f)+c : \mu \in A\} = \sup\{\mu(f) : \mu \in A\} + c = \nu_A(f)+c$.
2) Take $f, g \in C(X)$ such that $f \le g$. Then $\nu_A(f) = \sup\{\mu(f) : \mu \in A\} \le \sup\{\mu(g) : \mu \in A\} = \nu_A(g)$.
3) $\nu_A(1_X) = \sup\{\mu(1_X) : \mu \in A\} = \sup\{1 : \mu \in A\} = 1$.
4) Let $f \in C(X)$ and $t \in \mathbb{R}_+$. Then $\nu_A(tf) = \sup\{\mu(tf) : \mu \in A\} = \sup\{t\mu(f) : \mu \in A\} = t\sup\{\mu(f) : \mu \in A\} = t\nu_A(f)$.
5) Let $f, g \in C(X)$. Then $\nu_A(f+g) = \sup\{\mu(f+g) : \mu \in A\} = \sup\{\mu(f)+\mu(g) : \mu \in A\} \le \sup\{\mu(f) : \mu \in A\} + \sup\{\mu(g) : \mu \in A\} = \nu_A(f)+\nu_A(g)$.

b) Since $A \subset \mathrm{co}(A)$, $\nu_A(f) \le \nu_{\mathrm{co}(A)}(f)$ for each $f \in C(X)$. We show that $\nu_A(f) \ge \nu_{\mathrm{co}(A)}(f)$ for each $f \in C(X)$.

Let $f \in C(X)$ and $\varepsilon > 0$. Then there is $\mu_\varepsilon \in \mathrm{co}(A)$ such that $\mu_\varepsilon(f) \ge \nu_{\mathrm{co}(A)}(f) - \varepsilon$. Since $\mu_\varepsilon \in \mathrm{co}(A)$, μ_ε has the form $\sum\limits_{i=1}^{n} t_k\mu_k$ where $\mu_k \in A$, $t_k \ge 0$, $k = \overline{1,n}$, $\sum\limits_{k=1}^{n} t_k = 1$. Since $\mu_k(f) \le \nu_A(f)$,

$$\nu_{\mathrm{co}(A)}(f) - \varepsilon \le \mu_\varepsilon(f) = \sum_{k=1}^{n} t_k\mu_k(f) \le \sum_{k=1}^{n} t_k\nu_A(f) = \nu_A(f),$$

i.e., $\nu_{\mathrm{co}(A)}(f) - \varepsilon \le \nu_A(f)$. By virtue of arbitrariness of $\varepsilon > 0$, we have $\nu_{\mathrm{co}(A)}(f) \le \nu_A(f)$.

c) Take $f \in C(X)$ and $\varepsilon > 0$. Then there exists $\mu_\varepsilon \in \mathrm{cl}(A)$ such that $\mu_\varepsilon(f) \ge \nu_{\mathrm{cl}(A)}(f) - \varepsilon$. Since $\mu_\varepsilon \in \mathrm{cl}(A)$, there exists $\mu_0 \in A$ such that

$$|\mu_\varepsilon(f) - \mu_0(f)| < \varepsilon.$$

Hence,

$$\nu_{\mathrm{cl}(A)}(f) - \varepsilon \le \mu_\varepsilon(f) < \mu_0(f) + \varepsilon \le \nu_A(f) + \varepsilon,$$

i.e., $\nu_{\mathrm{cl}(A)}(f) - 2\varepsilon < \nu_A(f)$. By virtue of arbitrariness of $\varepsilon > 0$, we have $\nu_{\mathrm{cl}(A)}(f) \le \nu_A(f)$.

d) immediately follows from b) and c).

Proposition is proved. □

The following result shows that formula (3) gives a general form of functionals from $OS(X)$.

Theorem 3.3. *For each $\nu \in OS(X)$ there exists a non-empty convex compactum A in $P(X)$ such that $\nu = \nu_A$ where ν_A is a functional in form (3), in addition, for each $f \in C(X)$ there exists $\mu \in A$ such that $\nu(f) = \mu(f)$.*

Proof. Let ν be an arbitrary element of $OS(X)$. Put $A = \{\mu \in P(X) : \mu \leq \nu\}$ where $\mu \leq \nu$ means $\mu(f) \leq \nu(f)$ for all $f \in C(X)$. Obviously, the set A is a convex closed subset of $P(X)$. Hence, A is a convex compactum in $P(X)$.

Let us show that A is the set sought for.

It is clear that $\nu_A \leq \nu$. Hence, to prove the equality $\nu_A = \nu$ it is sufficient to show that for any $h \in C(X)$ there is $\mu \in A$ such that $\mu(h) = \nu(h)$.

Let h be a fixed element of $C(X)$. By $C = \{th + c_X : t, c \in \mathbb{R}\}$ we denote the subspace of $C(X)$ generated by h and constants.

We show that the following formula

$$\mu(th + c_X) = t\nu(h) + c, \quad t, c \in \mathbb{R}, \tag{4}$$

determines on C a positive linear functional such that

i) $\mu(h) = \nu(h)$;
ii) $\mu(g) \leq \nu(g)$, $g \in C$;
iii) $\mu(1_X) = 1$.

At first, let us check the correctness of definition of μ.

Case 1. $h = c' = \text{const}$. Let $g \in C$ and $g = t_1c' + c_1 = t_2c' + c_2$. We have $t_1\nu(h) + c_1 = t_1\nu(c'_X) + c_1 = t_1c' + c_1 = t_2c' + c_2 = t_2\nu(c'_X) + c_2 = t_2\nu(h) + c_2$. Hence, the value of μ on g doesn't depend on the decomposition of g.

Case 2. $h \neq \text{const}$. In this case an arbitrary $g \in C$ is uniquely decomposed in the following sum $g = th + c_X$. Indeed, if $t_1h + c_1 \equiv t_2h + c_2$ ($t_i, c_i \in \mathbb{R}$, $i = 1, 2$), then $(t_2 - t_1)h \equiv c_2 - c_1$. As $h \neq \text{const}$, $t_1 = t_2$, $c_1 = c_2$.

It is obvious that μ is linear on C.

It is clear that μ satisfies i), iii).

Let's show that ii) holds. Let $g = th + c$ be an element of C.

The case of $t = 0$ is trivial.

Consider the case of $t > 0$. In this case $\mu(g) = t\nu(h) + c = t(\nu(h) + \frac{c}{t}) = t\nu(h + \frac{c}{t}) = \nu(th + c) = \nu(g)$.

In the case of $t < 0$ we have $\mu(g) = t\nu(h) + c = t\left(\nu(h) + \frac{c}{t}\right) = t\nu\left(h + \frac{c}{t}\right) = -|t|\nu\left(h + \frac{c}{t}\right) = -\nu\left(|t|h + |t|\frac{c}{t}\right) = -\nu(-th - c) = -\nu(-g) = -(\nu(-g) + \nu(g)) + \nu(g) \leq -\nu(-g + g) + \nu(g) = -\nu(0_X) + \nu(g) = \nu(g)$.

Thus, we showed that on C there exists a functional μ satisfying the relations i), ii), iii). By the Hahn-Banach theorem the linear functional μ has a continuation on $C(X)$ (we denote it by μ, too) such that

$$\mu(h) = \nu(h);$$

$$\mu(g) \leq \nu(g), \quad g \in C(X);$$

$$\mu(1_X) = 1.$$

The inequality $\mu(g) \leq \nu(g)$ guarantees that $|\mu(f)| \leq 1$ for each $f \in C(X)$, $|f| \leq 1_X$. From here and taking into account that $\mu(1_X) = 1$ we obtain $\mu \in P(X)$. Again, from the inequality $\mu(g) \leq \nu(g)$ $(g \in C(X))$, we obtain $\mu \in A$.

The theorem is proved. □

The following result shows that the correspondence $A \leftrightarrow \nu_A$, where A is a non-empty convex compactum in $P(X)$, is a bijective mapping between the set $OS(X)$ and convex compact subsets of $P(X)$.

Theorem 3.4. *If A and B are non-empty convex compacta in $P(X)$, then $\nu_A = \nu_B$ if and only if $A = B$.*

Proof. First, let $A \neq B$. We can take that there will be found $\mu_0 \in B \setminus A$. As A is a non-empty convex compactum and $\mu_0 \notin A$, there exists an open neighborhood W of the functional μ_0 such that $W \cap A = \emptyset$. Moreover, we can take $W = \{\mu \in P(X) : |\mu(\varphi_i) - \mu_0(\varphi_i)| < \varepsilon, i = \overline{1,n}\}$ where $\varphi_i \in C(X)$, $i = \overline{1,n}$, $n \in \mathbb{N}$, $\varepsilon > 0$.

We define the mapping $\Phi : P(X) \to \mathbb{R}^n$ by the rule

$$\Phi(\mu) = (\mu(\varphi_1), \dots, \mu(\varphi_n)) \in \mathbb{R}^n, \quad \mu \in P(X).$$

It is clear that Φ is a continuous affine mapping. Therefore $\Phi(A)$ is a non-empty convex compactum in $\mathbb{R}^n$. $W \cap A = \emptyset$ implies $\Phi(\mu_0) \notin \Phi(A)$. Hence, by the Hahn-Banach theorem there is $(a_1, \dots, a_n) \in \mathbb{R}^n$ separating the point $\Phi(\mu_0)$ and the convex compactum $\Phi(A)$, i.e.,

$$\sup\left\{\sum_{i=1}^{n} a_i\mu(\varphi_i) : \mu \in A\right\} < \sum_{i=1}^{n} a_i\mu_0(\varphi_i). \tag{5}$$

Put $f = \sum\limits_{i=1}^{n} a_i\varphi_i$. We obtain from (5) that $\sup\{\mu(f) : \mu \in A\} < \mu_0(f)$. This means $\nu_A(f) < \mu_0(f)$. But $\mu_0 \in B$. Hence $\mu_0(f) \leq \nu_B(f)$. Therefore $\nu_A(f) < \nu_B(f)$. The last inequality shows that $\nu_A \neq \nu_B$.

Let now $\nu_A \neq \nu_B$. We can take that there exists $f \in C(X)$ such that $\nu_A(f) < \nu_B(f)$. Let $\varepsilon = \nu_B(f) - \nu_A(f)$. Take $\mu_0 \in B$ such that $\mu_0(f) > \nu_B(f) - \varepsilon/2$. Then $\mu_0(f) > \nu_B(f) - (\nu_B(f) - \nu_A(f))/2 = (\nu_B(f) + \nu_A(f))/2 > \nu_A(f)$, i.e., $\mu_0(f) > \nu_A(f)$. Hence, $\mu_0(f) > \lambda(f)$ for each $\lambda \in A$, which means $\mu_0 \notin A$, and therefore $A \neq B$.

The theorem is proved. □

Later, considering an element $\nu_A \in OS(X)$, we assume that A is a non-empty convex compactum in $P(X)$.

Corollary 3.5. *A functional $\nu_A \in OS(X)$ belongs to $P(X)$ if and only if A is a single-point set.*

Proposition 3.6. *Let A and B be convex compacta in $P(X)$, $0 < t < 1$, and $tA + (1 - t)B = \{t\lambda + (1 - t)\mu : \lambda \in A, \mu \in B\}$. Then*

$$t\nu_A + (1 - t)\nu_B = \nu_{tA+(1-t)B}.$$

Proof. Let $\varphi \in C(X)$. Take $\lambda \in A$ and $\mu \in B$. We have $(t\lambda + (1 - t)\mu)(\varphi) = t\lambda(\varphi) + (1 - t)\mu(\varphi) \leqslant t\nu_A(\varphi) + (1 - t)\nu_B(\varphi)$. Hence,

$$\nu_{tA+(1-t)B}(\varphi) \leq (t\nu_A + (1 - t)\nu_B)(\varphi). \tag{6}$$

On the other hand, by Theorem 3.3 there is $\lambda_0 \in A$ and $\mu_0 \in B$ such that $\lambda_0(\varphi) = \nu_A(\varphi)$, $\mu_0(\varphi) = \nu_B(\varphi)$. So, $\nu_{tA+(1-t)B}(\varphi) \geq t\lambda_0(\varphi) + (1 - t)\mu_0(\varphi) = (t\nu_A + (1 - t)\nu_B)(\varphi)$, i.e.

$$\nu_{tA+(1-t)B}(\varphi) \geq (t\nu_A + (1 - t)\nu_B)(\varphi). \tag{7}$$

Relations (6) and (7) imply $t\nu_A + (1 - t)\nu_B = \nu_{tA+(1-t)B}$.

The proposition is proved. □

Proposition 3.7. *The set $P(X)$ is a face in $OS(X)$, i.e., $t\nu_1 + (1-t)\nu_2 \in P(X)$, $\nu_1, \nu_2 \in OS(X)$, $0 < t < 1$, imply $\nu_1, \nu_2 \in P(X)$.*

Proof. Let ν_A, $\nu_B \in OS(X)$, $t \in (0, 1)$, be such that $t\nu_A + (1-t)\nu_B \in P(X)$. The latter is possible only when $tA + (1-t)B$ is a single-point set.

Let $\lambda \in A$, $\mu_i \in B$, $i = 1, 2$. Since $tA + (1-t)B$ is a single-point set, $t\lambda + (1-t)\mu_1 = t\lambda + (1-t)\mu_2$. Hence, $\mu_1 = \mu_2$, i.e., B consists of one point. Analogously, A also consists of one point. But then $\nu_A = \lambda$, $\nu_B = \mu$ belong to $P(X)$. This means that $P(X)$ is a face in $OS(X)$.

The proposition is proved. □

The convex set $P(X)$ is not a face in $OH(X)$ if $|X| \geq 2$. Indeed, take different $x_1, x_2 \in X$. Since

$$\frac{\delta_{x_1} \wedge \delta_{x_2} + \delta_{x_1} \vee \delta_{x_2}}{2} = \frac{\delta_{x_1} + \delta_{x_2}}{2},$$

we have that $\frac{\delta_{x_1} \wedge \delta_{x_2} + \delta_{x_1} \vee \delta_{x_2}}{2}$ belongs to $P(X)$. But $\delta_{x_1} \wedge \delta_{x_2}$ and $\delta_{x_1} \vee \delta_{x_2}$ don't belong to $P(X)$.

4. Categorical properties of the functor of semiadditive functionals

In this part we investigate categorical properties of the functor of semiadditive functionals and show that $OS : \mathrm{Comp} \to \mathrm{Comp}$ is a normal functor.

Let $\mathcal{F} : \mathrm{Comp} \to \mathrm{Comp}$ be a covariant functor.

Recall [1] that

1) $\mathcal{F}$ *preserves the weight* of compacts if $w(\mathcal{F}(X)) = w(X)$ for each infinite compactum X;
2) $\mathcal{F}$ is *monomorphic* if for any embedding i of a compactum X into a compactum Y, the mapping $\mathcal{F}(i) : \mathcal{F}(X) \to \mathcal{F}(Y)$ is also an embedding;
3) $\mathcal{F}$ is *epimorphic* if it preserves surjectivity of compacta;
4) $\mathcal{F}$ is *continuous* if for each inverse spectrum $\mathcal{P} = \{X_\alpha\ \pi^\alpha_\beta, I\}$, there is defined an inverse spectrum $\mathcal{F}(\mathcal{P}) = \{\mathcal{F}(X_\alpha), \mathcal{F}(\pi^\alpha_\beta), I\}$, and the limit
$$\pi : \mathcal{F}(\lim \mathcal{P}) \to \lim \mathcal{F}(\mathcal{P})$$
of the mappings $\mathcal{F}(\pi_\alpha) : \mathcal{F}(\lim \mathcal{P}) \to \mathcal{F}(X_\alpha)$, where $\pi_\alpha : \lim \mathcal{P} \to X_\alpha$ are through projections, is a homeomorphism;
5) $\mathcal{F}$ preserves *intersections* if for each family $\{B_\alpha : \alpha \in I\}$ of closed subsets of an arbitrary compactum we have $\bigcap\limits_{\alpha \in I} \mathcal{F}(B_\alpha) = \mathcal{F}(\bigcap\limits_{\alpha \in I} B_\alpha)$;
6) $\mathcal{F}$ preserves *preimages* if for each continuous mapping $f : X \to Y$ of a compactum X into a compactum Y and each closed subset $B \subset Y$, we have $\mathcal{F}(f^{-1}(B)) = \mathcal{F}(f)^{-1}(\mathcal{F}(B))$.

Definition 4.1. [2]. A functor $\mathcal{F} : \mathrm{Comp} \to \mathrm{Comp}$ is called *normal* if it is continuous, preserves intersections and preimages, is monomorphic and epimorphic, transfers the empty set into the empty set and a single-point set into a single-point one.

Recall that for any compact X there is defined a space $OH(X)$ consisting of all functionals $\mu : C(X) \to \mathbb{R}$ that satisfy the conditions 1), 2), 3) and 4) of Definition 2.1.

Let X and Y be compacts, $f : X \to Y$ be a continuous mapping. Then the mapping $OH(f) : OH(X) \to OH(Y)$ defined by

$$OH(f)(\mu)(\varphi) = \mu(\varphi \circ f)$$

is also continuous [13]. It is known [13] that the construction OH is a covariant functor in the category Comp of compacts and their continuous mappings.

Let now X and Y be compact spaces and $f : X \to Y$ be a continuous mapping between them. We define a mapping $OS(f) : OS(X) \to OS(Y)$ by the formula

$$OS(f)(\nu)(\varphi) = \nu(\varphi \circ f)$$

where $\nu \in OS(X)$ and $\varphi \in C(Y)$.

Proposition 4.2. *The operation OS gives a subfunctor of the functor OH :* Comp $\to$ Comp *.*

Proof. It is clear that for each X we have $OS(X) \subset OH(X)$.

We show that $OH(f)(OS(X)) \subset OS(Y)$ where $f : X \to Y$ is a continuous mapping. Let $\mu \in OS(X)$ and $\nu = OH(f)(\mu)$. We have, for $\varphi, \psi \in C(Y)$,

$$\begin{aligned}\nu(\varphi + \psi) &= OH(f)(\mu)(\varphi + \psi) = \mu((\varphi + \psi) \circ f) \leq \mu(\varphi \circ f) + \mu(\psi \circ f) \\ &= OH(f)(\mu)(\varphi) + OH(f)(\mu)(\psi) = \nu(\varphi) + \nu(\psi).\end{aligned}$$

This means that $\nu \in OS(Y)$. Therefore OS is a subfunctor of the functor OH.

The proposition is proved. □

Proposition 4.3. *Let X be an infinite compactum. Then*

$$w(X) = w(OS(X)).$$

Proof. Let X be an infinite compactum. Consider the functional $\delta_x : C(X) \to \mathbb{R}$ defined by the formula $\delta_x = \varphi(x)$, $\varphi \in C(X)$. It is clear that the mapping $\delta : X \to OS(X)$, defined by the formula $\delta(x) = \delta_x$, $x \in X$, is an embedding of the compactum X into $OS(X)$. That's why $w(X) \leq w(OS(X))$. We obtain from $OS(X) \subset OH(X)$ that $w(OS(X)) \leq w(OH(X))$, and we have from [9, Proposition 1] $w(X) = w(OH(X))$. Thus, $w(X) \leq w(OS(X)) \leq w(OH(X)) = w(X)$, i.e., $w(X) = w(OS(X))$.

The proposition is proved. □

Recall that P is the functor on probability measures. If $f : X \to Y$ is a continuous mapping, then the mapping $P(f) : P(X) \to P(Y)$ is defined by the rule

$$P(f)(\mu)(\varphi) = \mu(\varphi \circ f),$$

where $\mu \in P(X)$ and $\varphi \in C(Y)$.

The following result establishes a connection between $OS(f)$ and $P(f)$. This connection will play a key role in the investigation of categorical properties of the functor OS.

Proposition 4.4. *Let $f : X \to Y$ be a continuous mapping, $\nu_A \in OS(X)$. Then the following formula*

$$OS(f)(\nu_A) = \nu_{P(f)(A)} \tag{8}$$

is valid.

Proof. For $\varphi \in C(Y)$ we have $OS(f)(\nu_A)(\varphi) = \nu_A(\varphi \circ f) = \sup\{\lambda(\varphi \circ f) : \lambda \in A\} = \sup\{\mu(\varphi) : \mu \in P(f)(A)\} = \nu_{P(f)(A)}(\varphi)$. It shows that $OS(f)(\nu_A) = \nu_{P(f)(A)}$.

The proposition is proved. □

Further, we show using relation (8) that the functor OS is monomorphic, epimorphic, preserves preimages and intersections.

Proposition 4.5. *The functor OS :* Comp $\to$ Comp *is a monomorphic functor.*

Proof. Let $j : X \to Y$ be an embedding of a compactum X into a compactum Y. We show that $OS(j) : OS(X) \to OS(Y)$ is also an embedding. Let $\nu_A, \nu_B \in OS(X)$ be such that $\nu_A \neq \nu_B$. By Theorem 3.4 we have $A \neq B$. Since P is monomorphic, $P(j)$ is also an embedding and therefore $P(j)(A) \neq P(j)(B)$. We have from here and by virtue of (8) that $OS(j)(\nu_A) = \nu_{P(j)(A)} \neq \nu_{P(j)(B)} = OS(j)(\nu_B)$. The last inequality means that $OS(j)$ is an embedding as well, i.e., OS is monomorphic.

The proposition is proved. □

In works [9], [13], to prove epimorphity of the functors O and OH one has to prove, first, variants of the Hahn-Banach theorem for weakly additive functionals in the case of the functor O, and for positively-homogeneous functionals in the case of OH.

In our case we need not any propositions similar to the Hahn-Banach theorem. Formula (8) allows to prove epimorphity of the functor OS using epimorphity of the functor P.

Proposition 4.6. *The functor* $OS : \mathrm{Comp} \to \mathrm{Comp}$ *is epimorphic.*

Proof. Let $f : X \to Y$ be an epimorphism between compact spaces and $\nu_B \in OS(Y)$. As the functor P is epimorphic, $P(f)$ is an affine epimorphism. Therefore $A = P(f)^{-1}(B)$ is a non-empty convex compactum in $P(X)$. We obtain using (8) that $OS(f)(\nu_A) = \nu_{P(f)(A)} = \nu_B$, i.e., $OS(f)$ is an epimorphism.

The proposition is proved. □

The following result shows that the functor OS preserves preimages, but at the same time the functors O and OH don't possess this property (see [9], [13]).

Proposition 4.7. *The functor* $OS : \mathrm{Comp} \to \mathrm{Comp}$ *preserves preimages.*

Proof. Let $f : X \to Y$ be a continuous mapping and Z be a closed subset in Y. We show that $OS(f)^{-1}(OS(Z)) = OS(f^{-1}(Z))$. Take $\nu_A \in OS(f^{-1}(Z))$ where A is a convex compactum in $P(f^{-1}(Z))$. As P is normal, $P(f)(A) \subset P(f)(P(f^{-1}(Z))) = P(Z)$. That's why $OS(f)(\nu_A) = \nu_{P(f)(A)} \in OS(Z)$. This means that $\nu_A \in OS(f)^{-1}(OS(Z))$.

Now take $\nu_B \in OS(f)^{-1}(OS(Z))$. Let $\nu_A \in OS(Z)$ be such that $\nu_A = OS(f)(\nu_B)$. Again, we obtain from (8) that $\nu_A = OS(f)(\nu_B) = \nu_{P(f)(B)}$. Hence, $P(f)(B) = A \subset P(Z)$. Since P preserves preimages, $B \subset P(f)^{-1}(A) \subset P(f)^{-1}(P(Z)) = P(f^{-1}(Z))$, which means that $\nu_B \in OS(f^{-1}(Z))$.

The proposition is proved. □

Proposition 4.8. *The functor* $OS : \mathrm{Comp} \to \mathrm{Comp}$ *is continuous.*

Proof. Let $X = \lim \mathcal{P}$ where $\mathcal{P} = \{X_\alpha, \pi_\alpha^\beta, I\}$ is an inverse spectrum of compact spaces $X_\alpha, \alpha \in I$. Denote $Y = \lim OS(\mathcal{P})$ where $OS(\mathcal{P}) = \{OS(X_\alpha), OS(\pi_\alpha^\beta), I\}$.

Let $\pi : OS(X) \to Y$ be the limit of the mappings $OS(\pi_\alpha) : OS(X) \to OS(X_\alpha)$ where $\pi_\alpha : X \to X_\alpha$ are through projections.

As π is a continuous mapping, it is sufficient to prove that π is a bijection.

First, let $\nu_1, \nu_2 \in OS(X)$ be two different functionals. Then there exists $\varphi \in C(X)$ such that $|\nu_1(\varphi) - \nu_2(\varphi)| = \varepsilon > 0$. Since the set of functions $\psi_\alpha \circ \pi_\alpha$, where $\psi_\alpha \in C(X_\alpha)$, $\alpha \in I$, is dense in $C(X)$ (see, for the example [2]), there is $\alpha \in I$ and a function $\psi_\alpha \in C(X_\alpha)$ such that $|\varphi - \psi_\alpha \circ \pi_\alpha| < \varepsilon/3$. Since each weakly additive functional is a non-expanding mapping [9, Lemma 1], $|\nu_i(\varphi) - \nu_i(\psi_\alpha \circ \pi_\alpha)| < \varepsilon/3$. Further,

$$\varepsilon = |\nu_1(\varphi) - \nu_2(\varphi)| \le |\nu_1(\varphi) - \nu_1(\psi_\alpha \circ \pi_\alpha)| + |\nu_1(\psi_\alpha \circ \pi_\alpha) - \nu_2(\psi_\alpha \circ \pi_\alpha)| + |\nu_2(\psi_\alpha \circ \pi_\alpha) - \nu_2(\varphi)| \le 2\varepsilon/3 + |\nu_1(\psi_\alpha \circ \pi_\alpha) - \nu_2(\psi_\alpha \circ \pi_\alpha)|.$$

Hence, $\nu_1(\psi_\alpha \circ \pi_\alpha) \neq \nu_2(\psi_\alpha \circ \pi_\alpha)$ and therefore

$$OS(\pi_\alpha)(\nu_1)(\psi_\alpha) \neq OS(\pi_\alpha)(\nu_2)(\psi_\alpha).$$

Thus, $OS(\pi_\alpha)(\nu_1) \neq OS(\pi_\alpha)(\nu_2)$. As π is the limit of the mappings $OS(\pi_\alpha)$, we have $\pi(\nu_1) \neq \pi(\nu_2)$. Since the functor OS is epimorphic, π is a surjection.

The proposition is proved. $\square$

Proposition 4.9. *The functor OS preserves intersections of closed subsets of a compactum.*

Proof. It is sufficient to prove the assertion for the intersection of two closed subsets of the compactum X since OS is a continuous functor. It is clear that $OS(X_1 \cap X_2) \subset OS(X_1) \cap OS(X_2)$. Let's show the inverse inclusion. Let $\nu_A \in OS(X_1) \cap OS(X_2)$. Then A is a convex compact subset of $P(X_1) \cap P(X_2)$. Since P is normal, $A \subset P(X_1 \cap X_2)$. Therefore $\nu_A \in OS(X_1 \cap X_2)$.

The proposition is proved. $\square$

At last, the functor $OS : \mathrm{Comp} \to \mathrm{Comp}$ preserves a point and the empty set.

Propositions 4.2–4.9 imply the following

Theorem 4.10. *The functor* $OS : \mathrm{Comp} \to \mathrm{Comp}$ *is normal.*

Acknowledgments. The authors would like to express their sincere thanks to the referee for diligent reading of the manuscript and pointing out errors contained in the first version.

References

1. E. V. Shchepin, *Functors and uncountable degrees of compacta*, Uspekhi Mat. Nauk **36** (1981), no. 3, 3–62.
2. V. V. Fedorchuk, V. V. Filippov, *General Topology. Basic Constructions*, Moscow State University, Moscow, 1988.
3. V. V. Fedorchuk, *Probability measures in topology*, Uspekhi Mat. Nauk **46** (1991), no. 1, 41–80.
4. V. V. Fedorchuk, *Covariant functors in the category of compacts, absolute retracts, and Q-manifolds*, Uspekhi Mat. Nauk **36** (1981), no. 3, 177–190.
5. V. V. Fedorchuk, *Soft mappings, multivalued retractions, and functors*, Uspekhi Mat. Nauk **41** (1986), no. 6, 121–159.
6. V. V. Fedorchuk, *Some geometric properties of covariant functors*, Uspekhi Mat. Nauk **39** (1984), no. 5, 169–208.
7. V. N. Basmanov, *Covariant functors of finite degree on the category of Hausdorff compact spaces*, Fundam. Prikl. Mat. **2** (1996), no. 3, 637–654.
8. L. B. Shapiro, *On operators of the function continuation and normal functors*, Vestnik Moskov. Gos. Univ. Ser. I Mat. Mekh. (1992), no. 1, 35–42.
9. T. Radul, *On the functor of order-preserving functionals*, Comm. Math. Univ. Carol. **39** (1998), no. 3, 609–615.
10. A. A. Zaitov, *On categorical properties of order-preserving functionals*, Methods Funct. Anal. Topology **9** (2003), no. 4, 357–364.
11. R. B. Beshimov, *Some properties of the functor O_β*, Zapiski Nauchnykh Seminarov POMI **319** (2004), 131–134.
12. G. F. Djabbarov, *Description of extremal points of the space of weakly additive positively-homogeneous functionals of the two-point set*, Uzb. Math. J. (2005), no. 3, 17–25.
13. G. F. Djabbarov, *Categorical properties of the functor of weakly additive positively-homogeneous functionals*, Uzb. Math. J. (2006), no. 2, 20–28.

Institute of Mathematics, Uzbekistan Academy of Science, 29 F. Hodjaev, Tashkent, 700143, Uzbekistan

E-mail address: mathinst@uzsci.net

Institute of Mathematics, Uzbekistan Academy of Science, 29 F. Hodjaev, Tashkent, 700143, Uzbekistan

E-mail address: gayrat_77@bk.ru

Received 06/02/2007; Revised 20/03/2008