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ON CERTAIN RESOLVENT CONVERGENCE OF ONE NON-LOCAL PROBLEM TO A PROBLEM WITH SPECTRAL PARAMETER IN BOUNDARY CONDITION

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Dedicated to 100 anniversary of Mark Krein.

Abstract. A family of non-local problems with the same finite point spectrum is given. The resolvent convergence on a dense linear subspace which gives a problem with spectral parameter in the boundary condition is considered. The spectral eigenvalue decomposition of the last problem on the half line for Sturm-Liouville operator with trivial potential is given.

1. Introduction

There are many works concerned with problems with spectral parameters in the boundary condition. An approach which was developed in [1] and based on the fundamental notion of a spectral function contains various problems both with a parameter and without it in the boundary condition. Some references to the problems with a spectral parameter in the boundary condition (discrete spectrum etc.) can be found in [2].

A common approach in [1] to various problems indicates their "nearness". In this relation we want to point out one more type of the problem. Namely in this article it is shown that under the condition of conservation of point spectrum and the poles of analytic continuation of the resolvents too of some family of operators the family of such resolvents may be convergent to the resolvent of a problem with spectral parameter in the boundary condition. Note that resolvent convergence here takes place on a dense subspace only. Some information about resolvent convergence can be found in [3, ch. 8, § 1].

We consider a simple example of the Sturm-Liouville operator on the half line with trivial potential and a variable non-local boundary condition. The aim of this article is to prove that the limit problem has now a local boundary condition, but this condition contains a rational function of the spectral parameter. The poles of the analytical continuation of the resolvent are essential here, so we recall, for example, the work [4] which contains the physical meaning of such poles.

2. A non-local Sturm-Liouville problem with trivial potential

Let us consider the problem

$$\left\{\begin{array}{l} -v'' - \zeta v = u, \quad x > 0 \\ v(0) + (v, \eta)_{L^2(0,\infty)} = 0 \end{array}\right. \tag{2.1}$$

where $u(x)$, $\eta(x)$ are given functions from the space $L^2(0,\infty)$. Recall that the Sturm-Liouville operator L, generated by the expression $-v''$, $v(0) = 0$ is diagonalized by the

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transformation $\mathcal{F} : L^2(0,\infty) \to L^2_\rho(0,\infty)$, $\rho(\tau) = \frac{1}{\pi}\sqrt{\tau}$, namely,

$$\begin{aligned} \varphi(\tau) &= \mathcal{F}u(\tau) = \int_0^\infty u(x)\frac{\sin(x\sqrt{\tau})}{\sqrt{\tau}}\,dx, \\ u(x) &= \mathcal{F}^{-1}\varphi(x) = \frac{1}{\pi}\int_0^\infty \varphi(\tau)\sin(x\sqrt{\tau})\,d\tau. \end{aligned} \tag{2.2}$$

The scalar product in $L^2(0,\infty)$ and $L^2_\rho(0,\infty)$ is denoted by $(\cdot,\cdot)_{L^2(0,\infty)}$ and $(\cdot,\cdot)$. The integration by parts gives

$$\mathcal{F}(-v'')(\tau) = \tau\mathcal{F}v(\tau) - v(0). \tag{2.3}$$

If $v(0) = 0$, then (2.3) signifies the equality $L = \mathcal{F}^{-1}S\mathcal{F}$ where $S\varphi(\tau) \equiv \tau\varphi(\tau)$, $\tau > 0$. We introduce the operator $\widetilde{S} : L^2_\rho(0,\infty) \to L^2_\rho(0,\infty)$ as follows:

$$\left\{ \begin{array}{l} D(\widetilde{S}) = \left\{\psi \in L^2_\rho(0,\infty) \,|\, \exists c = c(\psi) : \int_0^\infty |\tau\psi(\tau) + c(\psi)|^2\rho(\tau)\,d\tau < \infty\right\} \\ \widetilde{S}\psi(\tau) = \tau\psi(\tau) + c(\psi) \end{array} \right\}. \tag{2.4}$$

If $v(0) \neq 0$ then (2.3) signifies the equality $L_{\max} = \mathcal{F}^{-1}\widetilde{S}\mathcal{F}$ where $L_{\max}$ is the corresponding maximal differential operator.

The values $c = c(\psi)$ defines a linear functional in the space $L^2_\rho(0,\infty)$ and due to (2.2)–(2.3), $c(\psi) = -v(0)$.

Let

$$\varphi = \mathcal{F}u, \quad \psi = \mathcal{F}v, \quad \gamma = \mathcal{F}\eta. \tag{2.5}$$

We introduce the operator $T : L^2_\rho(0,\infty) \to L^2_\rho(0,\infty)$ as follows:

$$\left\{ \begin{array}{l} D(T) = \{\psi \in L^2_\rho(0,\infty) : -c(\psi) + (\psi,\gamma) = 0\} \\ T\psi = \widetilde{S}\psi,\ \psi \in D(T). \end{array} \right. \tag{2.6}$$

Then the problem (2.1) takes the form

$$(T - \zeta)\psi = \varphi, \quad \psi \in L^2_\rho(0,\infty). \tag{2.7}$$

Taking the derivative of the second equality in (2.2) we obtain formally $u'(0) = (\varphi, 1)$. So, we need the operators S, T and the functionals

$$c(\psi) = -v(0), \quad (\psi, 1) = v'(0), \quad \psi = \mathcal{F}v. \tag{2.8}$$

The problem (2.1) due to (2.3), (2.5) takes the form

$$(\tau - \zeta)\psi(\tau) + (\psi,\gamma) = \varphi(\tau), \quad \tau > 0.$$

Let $S_\zeta = (S - \zeta)^{-1}$, $T_\zeta = (T - \zeta)^{-1}$, $E_\zeta(\tau) = 1/(\tau - \zeta)$, $\zeta \notin [0,\infty)$. Then

$$\psi + (\psi,\gamma)E_\zeta = S_\zeta\varphi.$$

Multiplying by γ we obtain $(\psi,\gamma)[1 + (E_\zeta,\gamma)] = (S_\zeta\varphi,\gamma)$.

We denote

$$\delta(\zeta) = 1 + (E_\zeta,\gamma) = 1 + \int_0^\infty \frac{\overline{\gamma(\tau)}}{\tau - \zeta}\rho(\tau)\,d\tau, \tag{2.9}$$

then

$$\psi = T_\zeta\varphi = S_\zeta\varphi - \frac{1}{\delta(\zeta)}(S_\zeta\varphi,\gamma)E_\zeta, \quad \zeta \notin [0,\infty), \quad \delta(\zeta) \neq 0. \tag{2.10}$$

The operator T_ζ is bounded, so T_ζ is the resolvent.

We need some limit values if $\zeta \to \sigma$, $Im\zeta \to \pm 0$, $\sigma > 0$, which we denote by $\delta_\pm(\sigma) = \lim_{\zeta\to\sigma}\delta(\zeta)$ and

$$(T_\sigma\varphi,\psi)_\pm = \lim_{\zeta\to\sigma}(T_\zeta\varphi,\psi), \quad \delta_\pm(\sigma) \neq 0. \tag{2.11}$$

These values exists if, for example, $\varphi, \psi, \gamma \in C^1[0,\infty)$.

Let

$$\left\{\begin{array}{l}(\varphi, b_\sigma) = \delta_-(\sigma)\varphi(\sigma) - (S_\sigma\varphi, \gamma)_- \\ (a_\sigma, \psi) = \dfrac{1}{\delta_+(\sigma)}(E_\sigma, \psi)_+ - \dfrac{1}{\delta_-(\sigma)}(E_\sigma, \psi)_-\end{array}\right. . \tag{2.12}$$

Lemma 2.1. *If $\gamma \in C^1[0,\infty) \cap L^2_\rho[0,\infty)$ then the resolvent of the operator T has the "jump" on the half line $(0,\infty)$,*

$$(T_\sigma\varphi, \psi)_+ - (T_\sigma\varphi, \psi)_- = (\varphi, b_\sigma)(a_\sigma, \psi), \tag{2.13}$$

where $\varphi, \psi \in C^1[0,\infty) \cap L^2_\rho[0,\infty)$ and $\delta_+(\sigma)\delta_-(\sigma) \neq 0$.

Proof. As

$$(S_\zeta\varphi, \psi) = \int_0^\infty \frac{\varphi(\tau)\overline{\psi(\tau)}}{\tau - \zeta}\rho(\tau)\, d\tau, \quad (E_\zeta, \psi) = \int_0^\infty \frac{\overline{\psi(\tau)}}{\tau - \zeta}\rho(\tau)\, d\tau,$$

we have

$$(S_\sigma\varphi, \psi)_+ - (S_\sigma\varphi, \psi)_- = 2\pi i\varphi(\sigma)\overline{\psi(\sigma)}, \quad (E_\sigma, \psi)_+ - (E_\sigma, \psi)_- = 2\pi i\overline{\psi(\sigma)}\rho(\sigma). \tag{2.14}$$

We have (see (2.10))

$$\begin{aligned} I &= (T_\sigma\varphi, \psi)_+ - (T_\sigma\varphi, \psi)_- = (S_\sigma\varphi, \psi)_+ - \frac{1}{\delta_+(\sigma)}(S_\sigma\varphi, \gamma)_+(E_\sigma, \psi)_+ \\ &- \left[(S_\sigma\varphi, \psi)_- - \frac{1}{\delta_-(\sigma)}(S_\sigma\varphi, \gamma)_-(E_\sigma, \psi)_-\right] = 2\pi i\varphi(\sigma)\overline{\psi(\sigma)}\rho(\sigma) \\ &- \frac{1}{\delta_+(\sigma)}\left[(S_\sigma\varphi, \gamma)_- + 2\pi i\varphi(\sigma)\overline{\gamma(\sigma)}\rho(\sigma)\right](E_\sigma, \gamma)_+ + \frac{1}{\delta_-(\sigma)}(S_\sigma\varphi, \gamma)_-(E_\sigma, \psi)_- \\ &= 2\pi i\varphi(\sigma)\rho(\sigma)\left[\overline{\psi(\sigma)} - \frac{\overline{\gamma(\sigma)}}{\delta_+(\sigma)}(E_\sigma, \psi)_+\right] \\ &- (S_\sigma\varphi, \gamma)_-\left[\frac{1}{\delta_+(\sigma)}(E_\sigma, \psi)_+ - \frac{1}{\delta_-(\sigma)}(E_\sigma, \psi)_-\right]. \end{aligned} \tag{2.15}$$

There exists the value $\Lambda(\sigma)$ such that

$$\begin{aligned} \overline{\psi(\sigma)} - \frac{\overline{\gamma(\sigma)}}{\delta_+(\sigma)}(E_\sigma, \psi)_+ &= \Lambda(\sigma)\left[\frac{1}{\delta_+(\sigma)}(E_\sigma, \psi)_+ - \frac{1}{\delta_-(\sigma)}(E_\sigma, \psi)_-\right] \\ &= \Lambda(\sigma)\left[\frac{1}{\delta_+(\sigma)}(E_\sigma, \psi)_+ - \frac{1}{\delta_-(\sigma)}(E_\sigma, \psi)_+ + \frac{2\pi i\overline{\psi(\sigma)}\rho(\sigma)}{\delta_-(\sigma)}\right]. \end{aligned} \tag{2.16}$$

Indeed, if $\Lambda(\sigma) = \dfrac{\delta_-(\sigma)}{2\pi i\rho(\sigma)}$ then (2.16) becomes

$$-\frac{\overline{\gamma(\sigma)}}{\delta_+(\sigma)} = \frac{\delta_-(\sigma)}{2\pi i\rho(\sigma)}\left[\frac{1}{\delta_+(\sigma)} - \frac{1}{\delta_-(\sigma)}\right],$$

i.e., the trivial equality $\delta_+(\sigma) - \delta_-(\sigma) = 2\pi i\overline{\gamma(\sigma)}\rho(\sigma)$. Substituting (2.16) into (2.15) we obtain

$$I = [2\pi i\varphi(\sigma)\rho(\sigma)\Lambda(\sigma) - (S_\sigma\psi, \gamma)_-](a_\sigma, \psi) = (\varphi, b_\sigma)(a_\sigma, \psi).$$

Lemma is proved. □

Obviously the function $E_\zeta(\tau)$ is the Fourier transform of the function

$$e_\zeta(x) = e^{i\sqrt{\zeta}x}, \quad \mathrm{Im}\sqrt{\zeta} > 0, \tag{2.17}$$

i.e., $\mathcal{F}(e_\zeta)(\tau) = \frac{1}{\tau-\zeta} = E_\zeta(\tau)$. Taking the derivative of the last equality with respect to ζ we obtain that Fourier transform $\gamma = \mathcal{F}\eta$ of finite sums

$$\eta(x) = \sum p_k(x) e^{\alpha_k x}, \quad \operatorname{Re}\alpha_k < 0,$$

where $p_k(x)$ are arbitrary polynomials, is a rational function $\gamma(\tau)$, bounded on $[0.\infty)$ and such that $\gamma(\tau) = O\left(\frac{1}{\tau}\right)$, $\tau \to \infty$. Due the elementary identity for the scalar product,

$$\left(\frac{1}{\tau-\zeta}, \frac{1}{\tau-\overline{\zeta}_1}\right) = \frac{i}{\sqrt{\zeta}+\sqrt{\zeta_1}}, \quad \operatorname{Im}\sqrt{\zeta} > 0, \quad \operatorname{Im}\sqrt{\zeta_1} > 0, \tag{2.18}$$

the function $\delta(\zeta)$ (see (2.9)), where $\gamma = \mathcal{F}\eta$, is a rational function on $\sqrt{\zeta}$, $\operatorname{Im}\sqrt{\zeta} > 0$. Later we consider only such a function $\delta(\zeta)$ and also we suppose that the functions $\varphi(\tau) = \mathcal{F}u(\tau)$, $\psi(\tau) = \mathcal{F}v(\tau)$ (see (2.1), (2.5)) are rational too.

Theorem 2.2. *Suppose that $\delta(\zeta) \neq 0$, $\zeta \notin [0,\infty)$ and $\delta_+(\sigma)\delta_-(\sigma) \neq 0$, $\sigma \geqslant 0$. Then*

$$(\varphi, \psi)_{L^2(0,\infty)} = \frac{1}{2\pi i}\int_0^\infty (\varphi, b_\sigma)(a_\sigma, \psi)\, d\sigma \tag{2.19}$$

where $\varphi(\tau)$, $\psi(\tau)$ are rational functions.

The proof is based on a well-known method of contour integration. Since $(T_\zeta(T-\zeta)\varphi,\psi) = (\varphi,\psi)$, $\varphi \in D(T)$, we have

$$\frac{(\varphi,\psi)}{\zeta} = -(T_\zeta\varphi,\psi) + \frac{1}{\zeta}(T_\zeta T\varphi,\psi). \tag{2.20}$$

Because $\varphi(\tau)$, $\psi(\tau)$ are rational functions (see (2.10), (2.19)),

$$\int_{|\zeta|=R} \frac{1}{\zeta}(T_\zeta T\varphi,\psi)\, d\zeta \to 0, \quad R \to \infty.$$

Integrating (2.20) we obtain

$$2\pi i(\varphi,\psi) = \int_0^\infty [(T_\sigma\varphi,\psi)_+ - (T_\sigma\varphi,\psi)_-]\, d\sigma,$$

then (2.19) follows from (2.13).

The theorem is proved. □

3. Stable finite point spectrum and the resolvent convergence on a dense subspace

According to (2.10), the spectrum of the operator T (see (2.6)) belongs to the union of the half line $[0,\infty)$ and the set of zeros of the function $\delta(\zeta)$, $\zeta \notin [0,\infty)$. If $\delta_+(\sigma)\delta_-(\sigma) \neq 0$ then, due to (2.13), the continuous spectrum of T coincides with $[0,\infty)$. The set of zeros of the function $\delta(\zeta)$, $\zeta \notin [0,\infty)$ (or $\delta_+(\sigma)\delta_-(\sigma)$, $\sigma \in [0,\infty)$) is the set of eigenvalues (or spectral singularities) of the operator T.

Let

$$\delta(\zeta) = \frac{(\sqrt{\zeta}-a_1)\dots(\sqrt{\zeta}-a_n)}{(\sqrt{\zeta}-\theta_1)\dots(\sqrt{\zeta}-\theta_n)}, \quad \operatorname{Im}\sqrt{\zeta} \geqslant 0, \tag{3.1}$$

where $\theta_i \neq \theta_j$, $i \neq j$ and $\operatorname{Im}\theta_j < 0$. The numbers $a_k \neq 0$ are arbitrary. If $\operatorname{Im} a_k > 0$, then $a_k^2 \notin [0,\infty)$ are eigenvalues of T and if $\operatorname{Im} a_k = 0$ then $a_k^2 \in (0,\infty)$ is a spectral singularity of T. If $a_k < 0$ then a_k^2 is not zero of the function $\delta(\zeta)$, but a_k^2 is a pole of the analytic continuation of the resolvent over continuous spectrum.

Further we suppose that $a_k = \text{const}$, $k = 1,\dots,n$, i.e., the point spectrum is stable.

Let $t > 0$, $t \to \infty$, be some parameter and

$$\theta_j = \alpha_j t \equiv -i\sqrt{\tau_j}\,t, \quad j = 1,\dots,n \tag{3.2}$$

where $\tau_j = \mathrm{const}$, $\tau_j > 0$, $\tau_j \neq \tau_k$, $j \neq k$. So,

$$\delta(\zeta, t) = \frac{(\sqrt{\zeta} - a_1) \dots (\sqrt{\zeta} - a_n)}{(\sqrt{\zeta} - \alpha_1 t) \dots (\sqrt{\zeta} - \alpha_n t)} \tag{3.3}$$

and, by analogy, we write $\gamma(\tau, t)$ instead of $\gamma(\tau)$. The corresponding operator is denoted by $T(t)$ and its resolvent by $T(t)_\zeta = (T(t) - \zeta)^{-1}$. The spectrum of the operator $T(t)$ is stable and we will study the limit values of $T(t)_\zeta \varphi$ if $t \to \infty$. According to (2.10) it remains to study the limit $\lim_{t\to\infty} \frac{\overline{\gamma(\tau,t)}}{\delta(\zeta,t)}$.

Denote

$$\begin{cases} m(s) = (s - a_1) \dots (s - a_n) \\ n(s) = (s - \theta_1) \dots (s - \theta_n). \end{cases} \tag{3.4}$$

Then $\lim_{t\to\infty} t^n \delta(\zeta, t) = Km(\sqrt{\zeta})$ and

$$\lim_{t\to\infty} \frac{\overline{\gamma(\tau, t)}}{\delta(\zeta, t)} = \frac{Z(\tau)}{Km(\sqrt{\zeta})}, \quad K = \frac{(-1)^n}{\alpha_1 \dots \alpha_n}, \tag{3.5}$$

where

$$Z(\tau) = \lim_{t\to\infty} t^n \overline{\gamma(\tau, t)}. \tag{3.6}$$

We need the following polynomial of degree $m - 1$:

$$p_{m-1}(\tau) = \frac{1}{2\sqrt{\tau}}(m(\sqrt{\tau}) - m(-\sqrt{\tau})), \quad \tau > 0, \quad n = 2m \quad \text{or} \quad n = 2m - 1. \tag{3.7}$$

Lemma 3.1. $\lim_{t\to\infty} t^n \overline{\gamma(\tau, t)} = -iKp_{m-1}(\tau)$.

Proof. Let us consider the decomposition of the rational function (see (3.2))

$$\delta(\zeta, t) = \frac{m(\sqrt{\zeta})}{n(\sqrt{\zeta})} = 1 + \frac{iA_1(t)}{\sqrt{\zeta} - \theta_1} + \dots + \frac{iA_n(t)}{\sqrt{\zeta} - \theta_n}. \tag{3.8}$$

Then (see (2.9))

$$\overline{\gamma(\tau, t)} = 1 + \frac{A_1(t)}{\tau + \tau_1 t^2} + \dots + \frac{A_n(t)}{\tau + \tau_n t^2} = 1 + \frac{A_1(t)}{\tau - \theta_1^2} + \dots + \frac{A_n(t)}{\tau - \theta_n^2}. \tag{3.9}$$

Indeed, $\theta_j^2 < 0$, $\overline{\theta_j^2} = \theta_j^2$ and the condition $\mathrm{Im}\sqrt{\theta_j^2} > 0$ gives $\sqrt{\theta_j^2} = \sqrt{-\tau_j t^2} = i\sqrt{\tau_j} t = -\theta_j$. So, according to (2.8),

$$\left(\frac{1}{\tau - \zeta}, \overline{\frac{A_j(t)}{\tau - \theta_j^2}} \right) = \frac{iA_j(t)}{\sqrt{\zeta} + \sqrt{\theta_j^2}} = \frac{iA_j(t)}{\sqrt{\zeta} - \theta_j}.$$

According to (3.8),

$$iA_j(t) = \frac{m(\theta_i)}{n'(\theta_j)} = O(t), \quad t \to \infty. \tag{3.10}$$

Later,

$$\begin{aligned} \frac{1}{\tau - \theta_j^2} &= -\sum_{k=1}^{p+1} \frac{\tau^{k-1}}{\theta_j^{2k}} + r_p(\tau, t),\ r_p(\tau, t) \\ &\equiv \frac{-\tau^{p+1}}{\theta_j^{2p+2}(\theta_j^2 - \tau)} = O\left(\frac{1}{t^{2p+4}}\right), \quad t \to \infty. \end{aligned} \tag{3.11}$$

The condition

$$t^n A_j(t) r_p(\tau, t) = o(1), \quad t \to \infty \tag{3.12}$$

demands that $2p+3>n$. So, we choose $p=m-1$ if $n=2m$ or $n=2m-1$. We have

$$\sum_{j=1}^{n}\frac{A_j(t)}{\tau-\theta_j^2}=-\sum_{j=1}^{n}A_j(t)\sum_{k=1}^{p+1}\frac{\tau^{k-1}}{\theta^{2k}}+O\left(\frac{1}{t^{2p+3}}\right)=\sum_{k=1}^{p+1}X_k(t)\tau^{k-1}+O\left(\frac{1}{t^{2p+3}}\right),$$

$$X_k(t)=-\sum_{j=1}^{n}\frac{A_j(t)}{\theta_j^{2k}}.$$

In view of (3.9), (3.11)–(3.12),

$$Z(\tau)=\lim_{t\to\infty}t^n\left(\sum_{k=1}^{p+1}X_k(t)\tau^{k-1}\right), \tag{3.13}$$

where (see (3.10))

$$i\lim_{t\to\infty}(t^nX_k(t))=-\lim_{t\to\infty}\left(t^n\sum_{j=1}^{n}\frac{m(\theta_j)}{n'(\theta_j)\theta_j^{2k}}\right). \tag{3.14}$$

Let (see (3.4))

$$m(s)=m_0+m_1s+\ldots+m_ns^n=\sum_{l=0}^{n}m_ls^l.$$

Then

$$\sum_{j=1}^{n}\frac{m(\theta_j)}{n'(\theta_j)\theta_j^{2k}}=\sum_{l=0}^{n}m_l\sum_{j=1}^{n}\frac{\theta_j^{l-2k}}{n'(\theta_j)}.$$

Since $\deg n'(s)=n-1$, we must calculate the limit (3.14) under the condition $l-2k\geqslant -1$ only, therefore,

$$i\lim_{t\to\infty}(t^nX_k(t))=-\sum_{l=2k-1}^{n}m_l\lim_{t\to\infty}\left(t^n\sum_{j=1}^{n}\frac{\theta_j^{l-2k}}{n'(\theta_j)}\right).$$

Here $-1\leqslant l-2k\leqslant n-2k\leqslant n-2$. Note, that

$$\sum_{j=1}^{n}\frac{z^\nu}{n'(z)}\bigg|_{z=\theta_j}=\sum_{j=1}^{n}\mathrm{Res}_{z=\theta_j}\frac{z^\nu}{n(z)}=0,\quad 0\leqslant\nu\leqslant n-2$$

and, in the case $\nu=-1$ (see (3.2), (3.4)),

$$\begin{aligned}\sum_{j=1}^{n}\frac{1}{zn'(z)}\bigg|_{z=\theta_j}&=\sum_{j=1}^{n}\mathrm{Res}_{z=\theta_j}\frac{1}{zn(z)}\\&=-\mathrm{Res}_{z=0}\frac{1}{zn(z)}=-\frac{1}{n(0)}=-\frac{(-1)^n}{\alpha_1\ldots\alpha_nt^n}=-\frac{K}{t^n}.\end{aligned}$$

Therefore, the nonzero limit exists in the case $l-2k=-1$ only, i.e., $l=2k-1$,

$$i\lim_{t\to\infty}(t^nX_k(t))=Km_{2k-1}.$$

Substituting in (3.13) we obtain $Z(\tau)=-iKp_{m-1}(\tau)$, where

$$p_{m-1}(\tau)=\sum_{k=1}^{p+1}m_{2k-1}\tau^{k-1}=\sum_{k=0}^{m-1}m_{2k+1}\tau^k \tag{3.15}$$

(recall that $p=m-1$). The representation (3.7) follows from the identity

$$m(s)-m(-s)=2s\sum_{k=0}^{m-1}m_{2k+1}s^{2k}=2sp_{m-1}(s^2).$$

The lemma is proved. □

As a corollary we obtain (see (3.5)) that

$$\lim_{t\to\infty}\frac{\overline{\gamma(\tau,t)}}{\delta(\zeta,t)}=\frac{1}{im(\sqrt{\zeta})}p_{m-1}(\tau). \tag{3.16}$$

Theorem 3.2. *On the dense subspace* $D(S^m)\subset L^2_\rho(0,\infty)$ *there exists the strong limit*

$$T(\infty)_\zeta\varphi\equiv s-\lim_{t\to\infty}T(t)_\zeta\varphi,\ \varphi\in D(S^m),$$

and

$$T(\infty)_\zeta\varphi=S_\zeta\varphi+\frac{i}{m(\sqrt{s})}(p_{m-1}\varphi,E_{\overline{\zeta}})E_\zeta. \tag{3.17}$$

Proof. Because $\theta_j^2<0$, if follows from (3.11) that

$$|r_p(\tau,t)|\leqslant\frac{\tau^{p+1}}{|\theta_j|^{2p+4}},\quad p+1=m.$$

Therefore if $\varphi\in D(S^m)$ then from point convergence (3.16) we have the strong convergence

$$\lim_{t\to\infty}\frac{\overline{\gamma(\cdot,t)}}{\delta(\zeta,t)}\varphi(\cdot)=\frac{1}{im(\sqrt{\zeta})}p_{m-1}(\cdot)\varphi(\cdot). \tag{3.18}$$

Due to continuity of the scalar product, the identity (3.17) follows from (2.10).

The theorem is proved. □

4. The problem with spectral parameter in the boundary condition

Obviously, $p_{m-1}\notin L^2_\rho(0,\infty)$, but we will denote the scalar product in (3.17) by the following formal symbol:

$$(S_\zeta\varphi,\overline{p}_{m-1})=(p_{m-1}\varphi,E_{\overline{\zeta}}).$$

We will use this convention in the other cases too.

So, the equality (3.17) becomes

$$T(\infty)_\zeta\varphi=S_\zeta\varphi+\frac{i}{m(\sqrt{\zeta})}(S_\zeta\varphi,\overline{p}_{m-1})E_\zeta. \tag{4.1}$$

Note that the element $T(\infty)_\zeta\varphi$ belongs to the direct sum $D(S^m)\dotplus\{E_\zeta\}$, $\zeta=\text{const}$. Every element of such a sum has a unique decomposition,

$$\psi=\psi_0+AE_\zeta,\quad \psi_0\in D(S^m),\quad A\in\mathbb{C} \tag{4.2}$$

where $A=-c(\psi)$.

Theorem 4.1. *The equation* $T(\infty)_\zeta\varphi=\psi$ *has a solution* $\varphi\in D(S^{m-1})$ *iff the element* ψ *has the form* (4.2) *and the condition*

$$\frac{i}{m(\sqrt{\zeta})}(\psi_0,\overline{p}_{m-1})=A \tag{4.3}$$

holds. Then

$$\varphi=(S-\zeta)\psi_0,\quad \psi_0\in D(S^m) \tag{4.4}$$

or

$$\varphi=(\widetilde{S}-\zeta)\psi. \tag{4.5}$$

Proof. Let φ be a solution of the equation $T(\infty)_\zeta\varphi=\psi$. The decomposition (4.2) is unique, therefore the equation

$$S_\zeta\varphi+\frac{i}{m(\sqrt{\zeta})}(S_\zeta\varphi,\overline{p}_{m-1})E_\zeta=\psi_0+AE_\zeta$$

is equivalent to the system

$$\begin{cases} S_\zeta\varphi = \psi_0 \\ \frac{i}{m(\sqrt{\zeta})}(S_\zeta\varphi, \overline{p}_{m-1}) = A. \end{cases}$$

Due to (4.3), this system reduces to one equation $S_\zeta\varphi = \psi_0$ from which we obtain (4.4). Since $(\widetilde{S}-\zeta)E_\zeta = 0$ and $\widetilde{S}|_{D(S)} = S|_{D(S)}$, we have $(\widetilde{S}-\zeta)\psi = (\widetilde{S}-\zeta)(\psi_0+AE_\zeta) = (\widetilde{S}-\zeta)\psi_0 = (S-\zeta)\psi_0 = \varphi$, which proves (4.5). Conversely, the equality $T(\infty)_\zeta\varphi = \psi$ results from (4.3)–(4.4).

The theorem is proved. □

Lemma 4.2. 1) *If the element $\psi_0 \in D(S^m)$ has the presentation*

$$\psi_0 = \psi + c(\psi)E_\zeta, \quad \psi \in D(\widetilde{S}), \tag{4.6}$$

then the form $(\psi_0, \overline{p}_{m-1})$ has following presentation in terms of the element $v = \mathcal{F}^{-1}\psi$:

$$(\psi_0, \overline{p}_{m-1}) = \sum_{k=0}^{m-1}(-1)^k m_{2k+1}v^{2k+1}(0) - \frac{i}{2}(m(\sqrt{\zeta}) - m(-\sqrt{\zeta}))v(0). \tag{4.7}$$

2) *If the function $u(x)$, where $\mathcal{F}u = \varphi = (\widetilde{S}-\zeta)\psi$ is such that $u^{(j)}(0) = 0$, $j = 0, 1, \ldots$, then the presentation* (4.7) *becomes*

$$(\psi_0, \overline{p}_{m-1}) = \frac{1}{2\sqrt{\zeta}}(m(\sqrt{\zeta}) - m(-\sqrt{\zeta}))v'(0) - \frac{i}{2}(m(\sqrt{\zeta}) - m(-\sqrt{\zeta}))v(0). \tag{4.8}$$

Proof. 1) Let $\psi_0 = \mathcal{F}y$. Then, due to (2.2),

$$y^{(2k+1)}(x) = \frac{1}{\pi}(-1)^k \int_0^\infty \psi_0(\tau)\tau^k \cos(x\sqrt{\tau})\sqrt{\tau}\, d\tau \tag{4.9}$$

from which, for $x = 0$, we obtain $(\psi_0, \tau^k) = (-1)^k y^{(2k+1)}(0)$, $k = 0, 1, \ldots, m-1$. Recall that $E_\zeta = \mathcal{F}(e_\zeta)$, $\psi = \mathcal{F}v$, and $c(\psi) = -v(0)$ (see (2.8)). So it follows from from (4.6) that $y(x) = v(x) - v(0)e^{i\sqrt{\zeta}x}$, $\mathrm{Im}\sqrt{\zeta} > 0$, therefore

$$y^{2k+1}(0) = v^{2k+1}(0) - (i\sqrt{\zeta})^{2k+1}v(0).$$

Finally,

$$\begin{aligned}(\psi_0, \overline{p}_{m-1}) &= \sum_{k=0}^{m-1} m_{2k+1}(\psi_0, \tau^k) = \sum_{k=0}^{m-1}(-1)^k m_{2k+1}y^{(2k+1)}(0) \\ &\equiv \sum_{k=0}^{m-1}(-1)^k m_{2k+1}v^{(2k+1)}(0) - r(\zeta)v(0)\end{aligned}$$

where the function $r(\zeta)$ is equal to

$$\begin{aligned} r(\zeta) &= \sum_{k=0}^{m-1}(-1)^k m_{2k+1}(i\sqrt{\zeta})^{2k+1} \\ &= i\sum_{k=0}^{m-1}(-1)^k m_{2k+1}(\sqrt{\zeta})^{2k+1} = \frac{i}{2}\left[m(\sqrt{\zeta}) - m(-\sqrt{\zeta})\right]. \end{aligned}$$

The statement 1) is proved.

2) If $\varphi = \mathcal{F}u$, $\psi = \mathcal{F}v$ then (4.5) signifies that $-v'' - \zeta v = u$, $x > 0$, i.e., $v'' = -\zeta v - u$. Further, $v^{IV} = -\zeta v'' - u'' = \zeta^2 v + \zeta u - u''$ and so on. Due to the condition $u^{(j)}(0) = 0$,

$j = 0, 1, \ldots$, we have $v^{(2k+1)}(0) = (-\zeta)^k v'(0)$. The first expression in (4.7) becomes

$$\sum_{k=0}^{m-1} (-1)^k m_{2k+1} v^{(2k+1)}(0) = \Bigl(\frac{1}{\sqrt{\zeta}} \sum_{k=0}^{m-1} m_{2k+1} (\sqrt{\zeta})^{(2k+1)})\Bigr) v'(0)$$
$$= \frac{1}{\sqrt{\zeta}} \Bigl(m(\sqrt{\zeta}) - m(-\sqrt{\zeta}) \Bigr).$$

Now the relation (4.8) follows from (4.7).

The lemma is proved. □

Theorem 4.3. *In the class of the functions $\varphi = \mathcal{F}u \in D(S^{m-1})$ such that $u^{(j)}(0) = 0$, $j = 0, 1, \ldots$, the equation $T(\infty)_\zeta \varphi = \psi$, $\psi = \mathcal{F}v$ is equivalent to the system*

$$-v'' - \zeta v = u, \quad x > 0 \tag{4.10}$$

$$\frac{1}{i\sqrt{\zeta}} \Bigl(m(\sqrt{\zeta}) - m(-\sqrt{\zeta}) \Bigr) v'(0) + \Bigl(m(\sqrt{\zeta}) + m(-\sqrt{\zeta}) \Bigr) v(0) = 0. \tag{4.11}$$

The proof follows from Theorem 4.1. Indeed, the equation (4.5) gives (4.10) and the condition (4.3) where $A = -c(\psi) = v(0)$ (see (2.8)) gives $i(\psi_0, \overline{p}_{m-1}) = m(\sqrt{\zeta}) v(0)$. Substituting here (4.8) we obtain (4.11).

The theorem is proved. □

5. Example: the eigenfunction expansion of the problem on the half line with spectral parameter in the boundary condition

As an example, we consider the operators $T(t)$ (see (2.6)) without point spectrum. It signifies that in (2.9), (3.3) we have $\operatorname{Im} a_k < 0$, $k = 1, \ldots, n$.

At the beginning, in order to ensure the convergence of the corresponding integrals, we suppose that the functions $u(x)$, $v(x)$ have rational Fourier transforms, $\varphi \in D(S^m)$, $\psi \in L^2_\rho(0, \infty)$.

Due to Lemma 3.1

$$\lim_{t\to\infty} t^n \overline{\gamma(\tau, t)} = -iKp_{m-1}(\tau).$$

According to (3.3),

$$\lim_{t\to\infty} t^n \delta(\zeta, t) = Km(\sqrt{\zeta}).$$

Therefore,

$$\begin{aligned}
&\lim_{t\to\infty} (\varphi, b_\sigma)(a_\sigma, \psi) \\
&\quad = \lim_{t\to\infty} \left\{ \left[t^n \left(\varphi(\sigma)\delta_-(\sigma) - (S_\sigma \varphi, \gamma)_- \right) \right] \left[\frac{1}{t^n} \left(\frac{1}{\delta_+(\sigma)} (E_\sigma, \psi)_+ - \frac{1}{\delta_-(\sigma)} (E_\sigma, \psi)_- \right) \right] \right\} \\
&\quad \equiv (\varphi, \hat{b}_\sigma)(\hat{a}_\sigma, \psi),
\end{aligned}$$

where

$$\left\{ \begin{array}{ll} (\varphi, \hat{b}_\sigma) & = \varphi(\sigma) m(-\sqrt{\sigma}) + i(S_\sigma \varphi, \overline{p}_{m-1})_- \\ (\hat{a}_\sigma, \psi) & = \dfrac{1}{m(\sqrt{\sigma})} (E_\sigma, \psi)_+ \dfrac{1}{m(-\sqrt{\sigma})} (E_\sigma, \psi)_- \end{array} \right. . \tag{5.1}$$

The equality (2.19) becomes

$$(\varphi, \psi) = \frac{1}{2\pi i} \int_0^\infty (\varphi, \hat{b}_\sigma)(\hat{a}_\sigma, \psi)\, d\sigma. \tag{5.2}$$

Let us introduce the rational function

$$R(\sigma) = -i\sqrt{\sigma} \frac{m(\sqrt{\sigma}) + m(-\sqrt{\sigma})}{m(\sqrt{\sigma}) - m(-\sqrt{\sigma})}. \tag{5.3}$$

Then the system (4.10)–(4.11) takes the form

$$\left\{\begin{array}{l} -v'' - \zeta v = u \\ v'(0) = R(\zeta)v(0) \end{array}\right. . \tag{5.4}$$

Let $s(\sigma) = \dfrac{m(-\sqrt{\sigma})}{m(\sqrt{\sigma})}$. Then

$$R(\sigma) = i\sqrt{\sigma}\frac{s(\sigma)+1}{s(\sigma)-1}, \quad s(\sigma) = \frac{R(\sigma)+i\sqrt{\sigma}}{R(\sigma)-i\sqrt{\sigma}}.$$

In view of (3.7),

$$\frac{p_{m-1}(\sigma)}{m(-\sqrt{\sigma})} = \frac{1}{2\sqrt{\sigma}}\left(\frac{1}{s(\sigma)} - 1\right) = -\frac{i}{R(\sigma)+i\sqrt{\sigma}}. \tag{5.5}$$

Let $d_k(\sigma)$, $k = 1, \ldots, m-2$, be polynomials defined by the identity

$$q_\sigma(\tau) \equiv (p_{m-1}(\tau) - p_{m-1}(\sigma))/(\tau - \sigma) \equiv \sum_{k=0}^{m-2} d_k(\sigma)\tau^k.$$

Then

$$(S_\sigma\varphi, \overline{p}_{m-1})_- = (\varphi, \overline{q}_\sigma) + p_{m-1}(\sigma)(\varphi, E_\sigma)_+.$$

Because of (4.9) we have $(\varphi, \tau^k) = (-1)^k u^{2k+1}(0)$, $\varphi = \mathcal{F}u$, therefore

$$(S_\sigma\varphi, \overline{p}_{m-1})_- = \sum_{k=0}^{m-2}(-1)^k u^{(2k+1)}(0)\overline{d_k(\sigma)} + p_{m-1}(\sigma)(\varphi, E_\sigma)_+.$$

If $u^{(j)}(0) = 0$, $j = 0, \ldots, n$, then

$$(S_\sigma\varphi, \overline{p}_{m-1})_- = p_{m-1}(\sigma)(\varphi, E_\sigma)_+. \tag{5.6}$$

Theorem 5.1. *Suppose that the polynomial $m(z)$ has zeros in the half plane* $\mathrm{Im} z < 0$. *Then the eigenfunction expansion of the problem*

$$\left\{\begin{array}{l} -v'' - \sigma v = 0,\ x > 0 \\ v'(0) = R(\sigma)v(0) \end{array}\right. \tag{5.7}$$

(*see* (5.3)) *is given by the relations*

$$\left\{\begin{array}{l} \Phi(\sigma) = \int_0^\infty u(x)\left[\cos\sqrt{\sigma}x + \frac{R(\sigma)}{\sqrt{\sigma}}\sin\sqrt{\sigma}x\right]dx \\ u(x) = \frac{1}{\pi}\int_0^\infty \Phi(\sigma)\left[\cos\sqrt{\sigma}x + \frac{R(\sigma)}{\sqrt{\sigma}}\sin\sqrt{\sigma}x\right]\frac{\sqrt{\sigma}}{R(\sigma)^2+\sigma}\,d\sigma \end{array}\right. \tag{5.8}$$

if $u^{(j)}(0) = 0$, $j = 0. \ldots, n$.

Proof. The relations (5.1)–(5.2) and the problem (5.4) have been obtained by passing to the limit. Therefore, we must prove that the function $\hat{a}(x)$ satisfies the equation and boundary condition in (5.7) and that (5.2) coincides with (5.8).

Note that the function

$$e(x, \sqrt{\sigma}) = \cos\sqrt{\sigma}x + \frac{R(\sigma)}{\sqrt{\sigma}}\sin\sqrt{\sigma}x$$

satisfies the equation and the condition (5.7).

In view of (5.1), (5.6) and (5.5),

$$
\begin{aligned}
&(\varphi,\hat b_\sigma)(\hat a_\sigma\psi)\\
&\quad=\left[\varphi(\sigma)m(-\sqrt{\sigma})+ip_{m-1}(\sigma)(\varphi,E_\sigma)_+\right]\left[\frac{1}{m(\sqrt{\sigma})}(E_\sigma,\psi)_+-\frac{1}{m(-\sqrt{\sigma})}(E_\sigma,\psi)_-\right]\\
&\quad=\left[\varphi(\sigma)+\frac{i}{2\sqrt{\sigma}}\left(\frac{1}{s(\sigma)}-1\right)(\varphi,E_\sigma)_+\right][s(\sigma)(E_\sigma,\psi)_+-(E_\sigma,\psi)_-]\\
&\quad\equiv(\varphi,B)(A,\psi).
\end{aligned}
$$

Further, (see (5.5))

$$
\overline{B(x)}=\frac{\sin\sqrt{\sigma}x}{\sqrt{\sigma}}+\frac{i}{2\sqrt{\sigma}}\frac{-2i\sqrt{\sigma}}{R(\sigma)+i\sqrt{\sigma}}e^{-i\sqrt{\sigma}x}=\frac{1}{R(\sigma)+i\sqrt{\sigma}}e(x,\sqrt{\sigma}),
$$

$$
A(x)=-\frac{R(\sigma)+i\sqrt{\sigma}}{R(\sigma)-i\sqrt{\sigma}}e^{i\sqrt{\sigma}x}+e^{-i\sqrt{\sigma}x}=\frac{2i\sqrt{\sigma}}{R(\sigma)-i\sqrt{\sigma}}e(x,\sqrt{\sigma}).
$$

Finally,

$$
(\varphi,\hat b_\sigma)(\hat a_\sigma\psi)=\frac{2i\sqrt{\sigma}}{R(\sigma)^2+\sigma}(u,\overline{e(\cdot,\sqrt{\sigma})})(e(\cdot,\sqrt{\sigma}),v)
$$

from which it follows (see (5.2)) that

$$
(u,v)_{L^2(0,\infty)}=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\sigma}}{R(\sigma)^2+\sigma}(u,\overline{e(\cdot,\sqrt{\sigma})})(e(\cdot,\sqrt{\sigma}),v)\,d\sigma,
$$

i.e., the relations (5.8).

The theorem is proved. □

Let $n=1$ and $\mathrm{Im}a_1<0$, for example $a_1=-i\theta$, $\theta>0$. Let $m(s)=s-a_1=s+i\theta$, then (see (5.3)) $R(\sigma)\equiv\theta=\mathrm{const}$. So, there is not a spectral parameter in the condition $v'(0)=\theta v(0)$. The expansion, in this case,

$$
\begin{cases}
\Phi(\lambda)=\displaystyle\int_0^\infty f(x)\left(\cos\sqrt{\lambda}x+\frac{\theta}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\right)dx\\
f(x)=\displaystyle\frac{1}{\pi}\int_0^\infty\Phi(\lambda)\left(\cos\sqrt{\lambda}x+\frac{\theta}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\right)\frac{\sqrt{\lambda}}{\lambda+\theta^2}\,d\lambda
\end{cases}
$$

is well known (see [5, p. 283]).

Let $n=2$, $m(s)=(s+i\theta_1)(s+i\theta_2)$, $\theta_1,\theta_2>0$, $\theta_1\neq\theta_2$, then $R(\sigma)=\dfrac{\theta_1\theta_2-\sigma}{\theta_1+\theta_2}$, it is a real valued function, the boundary condition is $(\theta_1+\theta_2)v'(0)=(\theta_1\theta_2-\sigma)v(0)$. If $u(0)=u'(0)=0$ then (5.8) defines a corresponding eigenfunction expansion. In other words, we have a simple decomposition (5.8) if the function $u(x)$ satisfies the boundary condition for all values of the spectral parameter σ.

6. Conclusion

The non-local Sturm-Liouville problem

$$
\begin{cases}
-v''-\zeta v=u, \quad x>0\\
v(0)+(v,\eta(t))_{L^2(0,\infty)}=0
\end{cases}
$$

where $\eta(t)$ is some variable element from the space $L^2(0,\infty)$, after passing to the limit for $t\to\infty$, becomes the problem

$$
\begin{cases}
-v''-\zeta v=u, \quad x>0\\
v'(0)=R(\zeta)v(0)
\end{cases}
$$

with a spectral parameter in the boundary condition.

The point spectrum of the Sturm-Liouville operator must be stable and this spectrum defines the rational function $R(\zeta)$. The expansion on eigenfunctions of second problem is generated by the expansion of the first one.

The calculus uses the language of Friedrichs' model. The maximal operator $\widetilde{S}$ was introduced in Friedrichs' model in [6]. Some applications of this notion and non-local problems in Friedrichs' model one be found in [7].

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