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G-FRAMES AND STABILITY OF G-FRAMES IN HILBERT SPACES

ABBAS NAJATI, M. H. FAROUGHI, AND ASGHAR RAHIMI

Abstract. Wenchang Sun in his paper [Wenchang Sun, *G-frames and g-Riesz bases,* J. Math. Anal. Appl. **322** (2006), 437–452] has introduced g-frames which are generalized frames and include ordinary frames and many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. In this paper we develop the g-frame theory for separable Hilbert spaces and give characterizations of g-frames and we show that g-frames share many useful properties with frames. We present a version of the Paley-Wiener Theorem for g-frames which is in spirit close to results for frames, due to Ole Christensen.

1. Introduction

There are some generalizations of frames, for example bounded quasi-projectors [8] and frames of subspaces [3]. The mean of g-frames has been presented by W. Sun in [15]. This is an extension of frames that include all of the previous extensions of frames.

Through this paper, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\{\mathcal{H}_i : i \in I\}$ is a sequence of Hilbert spaces, where I is a subset of $\mathbb{Z}$. $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_i$.

Note that for any sequence $\{\mathcal{H}_i : i \in I\}$, we can assume that there exists a Hilbert space $\mathcal{K}$ such that for all $i \in I, \mathcal{H}_i \subseteq \mathcal{K}$ (for example $\mathcal{K} = \bigoplus_{i\in I} \mathcal{H}_i$).

Definition 1.1. We call a sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a generalized frame, or simply a g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

We call A and B the lower and upper g-frame bounds, respectively.

We call $\{\Lambda_i : i \in I\}$ a tight g-frame if $A = B$ and a Parseval g-frame if $A = B = 1$.

We call $\{\Lambda_i : i \in I\}$ an exact g-frame if it ceases to be a g-frame whenever any of its elements is removed.

We say simply a g-frame for $\mathcal{H}$ whenever the space sequence $\{\mathcal{H}_i : i \in I\}$ is clear.

We say also a g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$ whenever $\mathcal{H}_i = \mathcal{K}$, for each $i \in I$.

We say $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame sequence, if it is a g-frame for $\overline{\operatorname{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in I}$.

We say $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence with bound B, if we only have the upper bound in (1.1).

Notation 1.2. *For each sequence $\{\mathcal{H}_i\}_{i\in I}$, we define the space $\Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2}$ by*

$$\Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2} = \Big\{ \{f_i\}_{i\in I} : f_i \in \mathcal{H}_i,\ i \in I \quad \textit{and} \quad \sum_{i\in I} \|f_i\|^2 < +\infty \Big\} \tag{1.2}$$

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with the inner product defined by

$$\langle \{f_i\}, \{g_i\} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle.$$

It is clear that $\left(\sum_{i \in I} \bigoplus \mathcal{H}_i \right)_{\ell_2}$ *is a Hilbert space with pointwise operations.*

2. Characterization of g-frames

In this section, we will try to characterize g-frames from the point of view of operator theory. We are starting with the definition of a synthesis operator for a g-frame. For this mean, we must show that the series appearing in the definition of a synthesis operator converges unconditionally. So we need the next lemma.

Lemma 2.1. *Let* $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *be a g-Bessel sequence for $\mathcal{H}$ with bound B. Then for each sequence* $\{f_i\}_{i \in I} \in \left(\sum_{i \in I} \bigoplus \mathcal{H}_i \right)_{\ell_2}$, *the series* $\sum_{i \in I} \Lambda_i^*(f_i)$ *converges unconditionally.*

Proof. Let $J \subseteq I$ with $|J| < \infty$, then

$$\begin{aligned}\Big\| \sum_{i \in J} \Lambda_i^*(f_i) \Big\| &= \sup_{\|g\|=1} \Big| \langle \sum_{i \in J} \Lambda_i^*(f_i), g \rangle \Big| \\ &\leq \Big(\sum_{i \in J} \|f_i\|^2 \Big)^{\frac{1}{2}} \sup_{\|g\|=1} \Big(\sum_{i \in J} \|\Lambda_i g\|^2 \Big)^{\frac{1}{2}} \leq \sqrt{B} \Big(\sum_{i \in J} \|f_i\|^2 \Big)^{\frac{1}{2}}.\end{aligned}$$

It follows that $\sum_{i \in I} \Lambda_i^*(f_i)$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\mathcal{H}$ (see [6], page 44, Theorems 6 and 8). $\square$

Definition 2.2. Let $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-frame for $\mathcal{H}$. Then the *synthesis operator* for $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is the operator

$$T : \Big(\sum_{i \in I} \bigoplus \mathcal{H}_i \Big)_{\ell_2} \longrightarrow \mathcal{H}$$

defined by

$$T(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(f_i).$$

We call the adjoint T^* of the synthesis operator the *analysis operator*.

The following proposition will provide a concrete formula for the analysis operator.

Proposition 2.3. *Let* $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *be a g-frame for $\mathcal{H}$. Then the analysis operator* $T^* : \mathcal{H} \longrightarrow \left(\sum_{i \in I} \bigoplus \mathcal{H}_i \right)_{\ell_2}$ *is given by*

$$T^*(f) = \{\Lambda_i f\}_{i \in I}.$$

Proof. For all $f \in \mathcal{H}$ and $\{g_i\}_{i \in I} \in \left(\sum_{i \in I} \bigoplus \mathcal{H}_i \right)_{\ell_2}$ we have

$$\begin{aligned}\Big\langle T^* f, \{g_i\}_{i \in I} \Big\rangle &= \Big\langle f, T\{g_i\}_{i \in I} \Big\rangle = \Big\langle f, \sum_{i \in I} \Lambda_i^*(g_i) \Big\rangle \\ &= \sum_{i \in I} \langle \Lambda_i f, g_i \rangle = \Big\langle \{\Lambda_i f\}_{i \in I}, \{g_i\}_{i \in I} \Big\rangle.\end{aligned}$$

So that $T^* f = \{\Lambda_i f\}_{i \in I}$. $\square$

The following proposition characterizes g-Bessel sequences.

Proposition 2.4. *$\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence for $\mathcal{H}$ with bound B, if and only if the operator*

$$T : \Big(\sum_{i \in I} \bigoplus \mathcal{H}_i \Big)_{\ell_2} \longrightarrow \mathcal{H}$$

defined by

$$T(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(f_i)$$

is a well-defined and bounded operator with $\|T\| \le \sqrt{B}$.

Proof. Let $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-Bessel sequence for $\mathcal{H}$ with bound B. Then by Lemma 2.1, T is well-defined and $\|T\| \le \sqrt{B}$.

Conversely, let T be a well-defined and bounded operator with $\|T\| \le \sqrt{B}$. Let $J \subseteq I$ with $|J| < +\infty$, then for each $f \in \mathcal{H}$,

$$\sum_{i \in J} \|\Lambda_i f\|^2 = \sum_{i \in J} \langle \Lambda_i^* \Lambda_i f, f \rangle = \Big\langle T\{g_i\}_{i \in I}, f \Big\rangle \le \|T\| \|\{g_i\}_{i \in I}\| \|f\|$$

where

$$g_i := \begin{cases} 0 & \text{if } i \in I \setminus J \\ \Lambda_i f & \text{if } i \in J \end{cases}.$$

Hence

$$\sum_{i \in J} \|\Lambda_i f\|^2 \le \|T\| \Big(\sum_{i \in J} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}} \|f\|$$

and so

$$\sum_{i \in J} \|\Lambda_i f\|^2 \le \|T\|^2 \|f\|^2.$$

Therefore

$$\sum_{i \in I} \|\Lambda_i f\|^2 \le \|T\|^2 \|f\|^2.$$

It follows that $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence for $\mathcal{H}$ with bound B. □

Definition 2.5. Let $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-frame for $\mathcal{H}$, the operator

$$S : \mathcal{H} \longrightarrow \mathcal{H}, \quad S = TT^*$$

is called the g-frame operator of $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$.

For any $f \in \mathcal{H}$ we have

$$Sf = \sum_{i \in I} \Lambda_i^* \Lambda_i f$$

and

$$\langle Sf, f \rangle = \Big\langle \sum_{i \in I} \Lambda_i^* \Lambda_i f, f \Big\rangle = \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle = \sum_{i \in I} \|\Lambda_i f\|^2.$$

Therefore

$$A\langle f, f \rangle \le \langle Sf, f \rangle \le B\langle f, f \rangle,$$

i.e.,

$$AI \le S \le BI.$$

Therefore S is a bounded, positive and invertible operator. In this case we have the reconstruction formula

$$f = \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1} f, \quad f \in \mathcal{H}.$$

The well-known relation between a frame and the associated synthesis operator also hold in g-frames.

Proposition 2.6. *A sequence* $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *is a g-frame for* $\mathcal{H}$ *if and only if*

$$T : \{f_i\}_{i\in I} \to \sum_{i\in I} \Lambda_i^*(f_i)$$

is a well-defined and bounded mapping from $\left(\sum_{i\in I} \bigoplus \mathcal{H}_i\right)_{\ell_2}$ *onto* $\mathcal{H}$.

Proof. If $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$, then $S = TT^*$ is invertible. So T is onto. Conversely, let T be well-defined, bounded and onto. Then by Proposition 2.4, $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence for $\mathcal{H}$. Therefore $T^*f = \{\Lambda_i f\}_{i\in I}$ for all $f \in \mathcal{H}$. Since T is onto, there exists an operator $T^\dagger : \mathcal{H} \longrightarrow \left(\sum_{i\in I} \bigoplus \mathcal{H}_i\right)_{\ell_2}$ such that $TT^\dagger = I_{\mathcal{H}}$. Hence $(T^\dagger)^*T^* = I_{\mathcal{H}}$. Then for all $f \in \mathcal{H}$,

$$\|f\|^2 \leq \|(T^\dagger)^*\|^2\|T^*f\|^2 = \|T^\dagger\|^2\|T^*f\|^2 = \|T^\dagger\|^2 \sum_{i\in I} \|\Lambda_i f\|^2.$$

It follows that $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$ with lower g-frame bound $\|T^\dagger\|^{-2}$ and upper g-frame bound $\|T\|^2$. □

Proposition 2.7. *A sequence* $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *is a g-frame for* $\mathcal{H}$ *if and only if*

$$S : f \to \sum_{i\in I} \Lambda_i^*\Lambda_i f$$

is a well-defined and bounded mapping from $\mathcal{H}$ *onto* $\mathcal{H}$.

Proof. If $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$, then S is the g-frame operator. Therefore S is well-defined, bounded and onto. Conversely, let S be a well-defined, bounded and onto operator. Since S is positive, (i.e., $\langle Sf, f\rangle \geq 0$ for all $f \in \mathcal{H}$) ${\mathcal{R}_S}^\perp = N(S)$. Then S is injective and therefore it is invertible. So $0 \notin \sigma(S)$. Let $C := \inf_{\|f\|=1}\langle Sf, f\rangle$. By Proposition 70.8 in [11], $C \in \sigma(S)$. Then $C > 0$. Hence for all $f \in \mathcal{H}$ we have

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \langle Sf, f\rangle \geq C\|f\|^2$$

and

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \langle Sf, f\rangle \leq \|S\|\|f\|^2.$$

Then $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$. □

Corollary 2.8. *If* $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *be a family of operators, then* T *is well-defined, bounded and onto if and only if* S *is well-defined, bounded and onto.*

Similarly to g-frames, we can define g-complete, g-Riesz bases and g-orthonormal bases.

Definition 2.9. (i) We say that $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is g-complete, if $\{f : \Lambda_i f = 0,\ i \in I \} = \{0\}$.

(ii) We say that $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-orthonormal basis for $\mathcal{H}$, if it satisfies the following

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{i,j}\langle g_i, g_j\rangle, \quad i, j \in I, \quad g_i \in \mathcal{H}_i, \quad g_j \in \mathcal{H}_j \tag{2.1}$$

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2, \quad f \in \mathcal{H}. \tag{2.2}$$

(*iii*) We say that $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g- Riesz basis for $\mathcal{H}$, if it is g-complete and there exist constants $0 < A \leq B < \infty$, such that for any finite subset $J \subseteq I$ and $g_i \in \mathcal{H}_i, i \in J$,

$$A \sum_{i \in J} \|g_i\|^2 \leq \Big\| \sum_{i \in J} \Lambda_i^* g_i \Big\|^2 \leq B \sum_{i \in J} \|g_i\|^2. \tag{2.3}$$

Proposition 2.10. *If $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$, then $\overline{\operatorname{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}$.*

Proof. Let $f \in \mathcal{H}$ and $f \perp \operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$. Then for all $i \in I$ and $g \in \mathcal{H}_i$,

$$\langle \Lambda_i f, g \rangle = \langle f, \Lambda_i^* g \rangle = 0.$$

Therefore $\Lambda_i f = 0$, for all $i \in I$. Since $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$, then $f = 0$. $\square$

The following proposition gives an equivalent condition for g-completeness of $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$.

Proposition 2.11. $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ *is g-complete if and only if*

$$\overline{\operatorname{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}.$$

Proof. Let $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be g-complete. Since $\overline{\operatorname{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} \subseteq \mathcal{H}$, it is enough to prove that if $f \in \mathcal{H}$ and $f \perp \operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$, then $f = 0$. Let $f \in \mathcal{H}$ and $f \perp \operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$. Since for any $i \in I$, $f \perp \Lambda_i^* \Lambda_i(f)$, then for all $i \in I$,

$$\|\Lambda_i f\|^2 = \langle f, \Lambda_i^* \Lambda_i(f) \rangle = 0.$$

Therefore $f = 0$, by g-completeness of $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$. Conversely, let $\overline{\operatorname{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}$. Let $f \in \mathcal{H}$ and suppose that $\Lambda_i f = 0$ for all $i \in I$. Then for each $g \in \mathcal{H}_i$

$$\langle \Lambda_i f, g \rangle = \langle f, \Lambda_i^* g \rangle = 0,$$

hence $f \perp \operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$. Therefore $f \perp \overline{\operatorname{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}$. It shows that $f = 0$. Hence $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is g-complete. $\square$

The following proposition shows that we can remove the second equality in 2.9.

Proposition 2.12. *Let $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-frame for $\mathcal{H}$ and suppose that (2.1) holds. Then $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-orthogonal basis for $\mathcal{H}$.*

Proof. Let S be the g-frame operator of $\{\Lambda_i\}$ and let $\mathcal{K} = \{ f \in \mathcal{H} : Sf = f \}$. It is clear that $\mathcal{K}$ is a nonempty closed subspace of $\mathcal{H}$. We show that $\mathcal{K}^\perp = \{0\}$. Let $f \in \mathcal{H}$ and $j \in I$, then $\Lambda_j^* \Lambda_j f \in \mathcal{K}$ since for $i \neq j$ we have

$$\begin{aligned} &\Lambda_i^* \Lambda_i \Lambda_j^* \Lambda_j f = 0, \\ &\Lambda_j^* \Lambda_j \Lambda_j^* \Lambda_j f = \Lambda_j^* \Lambda_j f \end{aligned}$$

so

$$\sum_{i \in I} \Lambda_i^* \Lambda_i \Lambda_j^* \Lambda_j f = \Lambda_j^* \Lambda_j f.$$

Hence for $f \in \mathcal{K}^\perp$ and $g \in \mathcal{K}$ we have

$$\langle \Lambda_j^* \Lambda_j f, g \rangle = \langle f, \Lambda_j^* \Lambda_j g \rangle = 0, \quad j \in I.$$

Therefore

$$\Lambda_i^* \Lambda_i f = 0, \quad i \in I$$

and so $\|\Lambda_i f\| = 0$, for each $i \in I$. By definition of g-frame we conclude $f = 0$ and therefore $\mathcal{H} = \mathcal{K}$. So $S = I$ and hence for every $f \in \mathcal{H}$

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \langle Sf, f\rangle = \langle f, f\rangle = \|f\|^2.$$

□

Corollary 2.13. *$\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-orthonormal basis for $\mathcal{H}$ if and only if $\{\Lambda_i\}$ is a g-frame and for all $i, j \in I$ and $g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j$*

$$\langle \Lambda_i^* g_i, \Lambda_i^* g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle.$$

Proposition 2.14. *$\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-orthonormal basis for $\mathcal{H}$ if and only if*

(1) *Λ_i^* is isometric for all $i \in I$;*
(2) *$\bigoplus_{i\in I} \Lambda_i^*(\mathcal{H}_i) = \mathcal{H}$.*

Proof. First assume that $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-orthonormal basis for $\mathcal{H}$. Then for each $i \in I$ and $g \in \mathcal{H}_i$, we obtain from (2.1)

$$\langle \Lambda_i^* g, \Lambda_i^* g\rangle = \delta_{i,i}\langle g, g\rangle = \langle g, g\rangle.$$

Then Λ_i^* is isometric for all $i \in I$. Therefor $\Lambda_i^*(\mathcal{H}_i)$ is a closed subspace of $\mathcal{H}$, for all $i \in I$. From (2.2), we obtain

$$\langle f, f\rangle = \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle = \sum_{i\in I}\langle \Lambda_i^*\Lambda_i f, f\rangle = \Big\langle \sum_{i\in I} \Lambda_i^*\Lambda_i f, f\Big\rangle, \quad f \in \mathcal{H}.$$

Hence

$$f = \sum_{i\in I} \Lambda_i^*\Lambda_i f, \quad f \in \mathcal{H}.$$

By letting $g_i = \Lambda_i f$, we have $f = \sum_{i\in I} \Lambda_i^* g_i$ and

$$\sum_{i\in I} \|\Lambda_i^* g_i\|^2 = \sum_{i\in I} \|g_i\|^2 = \sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2.$$

Then $\mathcal{H} = \bigoplus_{i\in I} \Lambda_i^*(\mathcal{H}_i)$. Conversely, if (1) and (2) are satisfied, then it is clear that (2.1) holds, since for all $i \neq j$, $\Lambda_i^*\mathcal{H}_i \perp \Lambda_j^*\mathcal{H}_j$ and Λ_i^* is isometric for all $i \in I$. Let $f \in \mathcal{H}$, then we obtain from (2),

$$f = \sum_{i\in I} \Lambda_i^* g_i, \quad g_i \in \mathcal{H}_i, \quad i \in I,$$

and

$$\|f\|^2 = \sum_{i\in I} \|\Lambda_i^* g_i\|^2 = \sum_{i\in I} \|g_i\|^2.$$

Let $m \in I$, then for each $h \in \mathcal{H}$, we have

$$\langle \Lambda_m f, h\rangle = \Big\langle \sum_{i\in I} \Lambda_m\Lambda_i^* g_i, h\Big\rangle = \sum_{i\in I}\langle \Lambda_i^* g_i, \Lambda_m^* h\rangle = \langle \Lambda_m^* g_m, \Lambda_m^* h\rangle = \langle g_m, h\rangle.$$

Hence $g_m = \Lambda_m f$, for all $m \in I$. Therefore $f = \sum_{i\in I} \Lambda_i^*\Lambda_i f$, for all $f \in \mathcal{H}$. It follows that

$$\|f\|^2 = \sum_{i\in I} \|\Lambda_i f\|^2, \quad f \in \mathcal{H}.$$

□

Corollary 2.15. *Every g-orthonormal basis for $\mathcal{H}$ is a g-Riesz basis for $\mathcal{H}$ with bounds $A = B = 1$.*

In the following proposition we show that g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ are characterized as the families $\{\Theta_i V^*\}_{i\in I}$, where $\{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ and $V : \mathcal{H} \to \mathcal{H}$ is bounded and onto.

Proposition 2.16. *Let $\{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ and $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$. Then there is a bounded and onto operator $V : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\Lambda_i = \Theta_i V^*$, for all $i \in I$.*

Proof. Let

$$V : \mathcal{H} \longrightarrow \mathcal{H}, \quad Vf = \sum_{i\in I} \Lambda_i^* \Theta_i f,$$

then V is well defined, since for any finite $J \subseteq I$ and $f \in \mathcal{H}$,

$$\Big\| \sum_{i\in J} \Lambda_i^* \Theta_i f \Big\| = \sup_{\|h\|=1} \Big| \Big\langle \sum_{i\in J} \Lambda_i^* \Theta_i f, h \Big\rangle \Big| \le \sqrt{B} \Big(\sum_{i\in J} \|\Theta_i f\|^2 \Big)^{\frac{1}{2}},$$

where B is an upper g-frame bound for $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$. Since for all $f \in \mathcal{H}$, $\sum_{i\in I} \|\Theta_i f\|^2 = \|f\|^2$, it follows that $\sum_{i\in I} \Lambda_i^* \Theta_i f$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\mathcal{H}$ (see[6], Page 44, Theorems 6 and 8). Also it is clear that $\|V\| \le \sqrt{B}$. Since $\{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ is a g-orthonormal basis, then $\Theta_i \Theta_j^* g = \delta_{ij} g$ and

$$V\Theta_j^* g = \sum_{i\in I} \Lambda_i^* \Theta_i \Theta_j^* g = \Lambda_j^* \Theta_j \Theta_j^* g = \Lambda_j^* g,$$

for all $g \in \mathcal{H}_j, j \in I$. Therefore

$$\Theta_j V^* = \Lambda_j, \quad j \in I.$$

Now we show that V is onto. Let T_Λ and T_Θ be the synthesis operators for $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ and $\{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ respectively. Let $f \in \mathcal{H}$. Then there exists $\{g_i\}_{i\in I} \in \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2}$ such that $\sum_{i\in I} \Lambda_i^* g_i = f$, since T_Λ is onto. Let $g = T_\Theta(\{g_i\}_{i\in I})$. Then

$$Vg = \sum_{i\in I} V\Theta_i^* g_i = \sum_{i\in I} \Lambda_i^* g_i = f.$$

Therefore V is onto. □

Corollary 2.17. *If $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ is a Parseval g-frame for $\mathcal{H}$, then V is co-isometric.*

Corollary 2.18. *If $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ is a g-Riesz basis for $\mathcal{H}$, then V is invertible.*

Proof. Let $f \in \ker V$. Then $T_\Theta^* f = 0$, since $V = T_\Lambda T_\Theta^*$ and T_Λ is one to one. Therefore $\|f\|^2 = \|T_\Theta^* f\|^2 = 0$, so $f = 0$. □

Corollary 2.19. *If $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ is a g-orthonormal basis for $\mathcal{H}$, then V is unitary.*

Proof. By Corollaries 2.15 and 2.18, V is invertible. For each $f \in \mathcal{H}$ we have

$$\|f\|^2 = \sum_{i\in I} \|\Lambda_i f\|^2 = \sum_{i\in I} \|\Theta_i V^* f\|^2 = \|V^* f\|^2.$$

Therefore $VV^* = I_{\mathcal{H}}$. It follows that V is unitary. □

Han and Larson in [10] have proved that for any Parseval frame $\{f_i\}_{i\in I}$ for a Hilbert space $\mathcal{H}$ there is a Hilbert space $\mathcal{H} \subseteq \mathcal{K}$ and an orthonormal basis $\{e_i\}_{i\in I}$ for $\mathcal{K}$ such that $Pe_i = f_i$, for all $i \in I$, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Asgari and Khosravi in [1] have extended this result for frame of subspaces and the following theorem is its extension to g-frames.

Theorem 2.20. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a Parseval g-frame for $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis of subspaces $\{\mathcal{K}_i\}_{i\in I}$ for $\mathcal{K}$ such that for each $i \in I$, $P(\mathcal{K}_i) = \Lambda_i^*(\mathcal{H}_i)$, where P is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$.*

Proof. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$ and for each $i \in I, j \in J_i$, let E_{ij} be an element of $\left(\sum_{i\in I} \bigoplus \mathcal{H}_i\right)_{\ell_2}$ defined by

$$(E_{ij})_k = \begin{cases} e_{ij} & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}.$$

It is clear that $\{E_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\left(\sum_{i\in I} \bigoplus \mathcal{H}_i\right)_{\ell_2}$. For each $i \in I$, let $\mathcal{K}_i = \overline{\text{span}}\{E_{ij}\}_{j\in J_i}$, then

$$\bigoplus_{i\in I} \mathcal{K}_i = \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2}.$$

Let $\mathcal{K} = \bigoplus_{i\in I} \mathcal{K}_i$ and let Θ be the analysis operator of $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$. Since $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a Parseval g-frame, then Θ is an isometry. So we can embed $\mathcal{H}$ into $\mathcal{K}$ by identifying $\mathcal{H}$ with $\Theta(\mathcal{H})$. Let $P : \mathcal{K} \longrightarrow \Theta(\mathcal{H})$ be the orthogonal projection. Then for each $i \in I, j \in J_i$ and $f \in \mathcal{H}$ we have

$$\langle \Theta f, P(E_{ij})\rangle = \langle \Theta f, E_{ij}\rangle = \Big\langle \{\Lambda_i f\}_{i\in I}, E_{ij}\Big\rangle = \langle \Lambda_i f, e_{ij}\rangle = \langle f, \Lambda_i^* e_{ij}\rangle = \langle \Theta f, \Theta\Lambda_i^* e_{ij}\rangle.$$

Thus $P(E_{ij}) - \Theta\Lambda_i^* e_{ij} \perp \Theta(\mathcal{H})$. Since $P(E_{ij}) - \Theta\Lambda_i^* e_{ij} \in \Theta(\mathcal{H})$, then $P(E_{ij}) = \Theta\Lambda_i^* e_{ij}$. Thus $P(\mathcal{K}_i) = \Theta\Lambda_i^*(\mathcal{H}_i)$. □

The next result will turn out to be useful for finding a lower bound for $\sum_{i\in I} \|\Lambda_i f\|^2$.

Proposition 2.21. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two g-Bessel sequences for $\mathcal{H}$ with bounds A and B respectively and let T_Λ and T_Θ be their analysis operators such that $T_\Theta T_\Lambda^* = I_\mathcal{H}$. Then both $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ are g-frames.*

Proof. For any $f \in \mathcal{H}$ we have

$$\begin{aligned}\|f\|^4 = \langle f, f\rangle^2 = \langle T_\Lambda^* f, T_\Theta^* f\rangle^2 &\leq \|T_\Lambda^* f\|^2 \|T_\Theta^* f\|^2 = \|\{\Lambda_i f\}_{i\in I}\|^2 \|\{\Theta_i f\}_{i\in I}\|^2 \\ &= \Big(\sum_{i\in I} \|\Lambda_i f\|^2\Big)\Big(\sum_{i\in I} \|\Theta_i f\|^2\Big) \leq \Big(\sum_{i\in I} \|\Lambda_i f\|^2\Big) B\|f\|^2.\end{aligned}$$

So $\frac{1}{B}\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2$. Therefore $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame. Similarly $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame. □

The next proposition shows that with a given g-frame, we can produce another g-frame by using its g-frame operator.

Proposition 2.22. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with frame operator S and frame bounds A, B. Then for each $\alpha \in \mathbb{R}$, $\{\Lambda_i S^\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ and its frame operator is $S^{2\alpha+1}$.*

Proof. Since $\{ \Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$, then

$$A\langle S^{2\alpha} f, f\rangle = A\|S^\alpha f\|^2 \le \sum_i \|\Lambda_i S^\alpha f\|^2 \le B\|S^\alpha f\|^2 = B\langle S^{2\alpha} f, f\rangle.$$

For $\alpha \ge 0$,

$$A\|S^\alpha f\|^2 = A\langle S^{2\alpha} f, f\rangle \ge AA^{2\alpha}\langle f, f\rangle = A^{2\alpha+1}\|f\|^2,$$
$$B\|S^\alpha f\|^2 = B\langle S^{2\alpha} f, f\rangle \le BB^{2\alpha}\langle f, f\rangle = B^{2\alpha+1}\|f\|^2.$$

Therefore

$$A^{2\alpha+1}\|f\|^2 \le \sum_i \|\Lambda_i S^\alpha f\|^2 \le B^{2\alpha+1}\|f\|^2.$$

Similarly for $\alpha < 0$,

$$B^{2\alpha+1}\|f\|^2 \le \sum_i \|\Lambda_i S^\alpha f\|^2 \le A^{2\alpha+1}\|f\|^2.$$

Now, let $S^\diamond$ be the frame operator of $\{\Lambda_i S^\alpha \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I \}$, then for any $f \in \mathcal{H}$,

$$\begin{aligned} S^\diamond(f) &= \sum_i (\Lambda_i S^\alpha)^*(\Lambda_i S^\alpha) f = \sum_i S^\alpha \Lambda_i^* \Lambda_i S^\alpha f \\ &= S^\alpha \sum_i \Lambda_i^* \Lambda_i S^\alpha f = S^\alpha S S^\alpha f = S^{2\alpha+1} f. \end{aligned}$$

So $S^\diamond = S^{2\alpha+1}$. □

Corollary 2.23. *By assumptions of Proposition 2.22,* $\{\Lambda_i S^{-\frac{1}{2}} \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I \}$ *is a Parseval g-frame.*

Proposition 2.24. *Let* $\{ \Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I \}$ *be a g-frame for* $\mathcal{H}$ *with frame operator* S *and* $U : \mathcal{H} \longrightarrow \mathcal{K}$ *be a bounded and onto operator. Then* $\{\Lambda_i U^* \in \mathcal{L}(\mathcal{K},\mathcal{H}_i) : i \in I \}$ *is a g-frame for* $\mathcal{K}$ *and its frame operator is* USU^*.

Proof. Let A, B be the frame bounds for $\{ \Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I \}$. Since U is bounded and onto, there exists bounded operator $U^\dagger : \mathcal{K} \longrightarrow \mathcal{H}$ such that $UU^\dagger = I_{\mathcal{K}}$. Thus $(U^\dagger)^* U^* = I_{\mathcal{K}}$. Hence for any $f \in \mathcal{K}$, we have

$$A\|(U^\dagger)^*\|^{-2}\|f\|^2 \le A\|U^* f\|^2 \le \sum_{i\in I} \|\Lambda_i U^* f\|^2 \le B\|U^* f\|^2 \le B\|U^*\|^2\|f\|^2.$$

Therefore $\{\Lambda_i U^* \in \mathcal{L}(\mathcal{K},\mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{K}$ with frame bounds $A\|U^\dagger\|^{-2}$ and $B\|U\|^2$. Now, let S_U be the frame operator for$\{\Lambda_i U^* \in \mathcal{L}(\mathcal{K},\mathcal{H}_i) : i \in I \}$, then for each $f \in \mathcal{K}$,

$$S_U f = \sum_{i\in I} (\Lambda_i U^*)^*(\Lambda_i U^*) f = U \sum_{i\in I} \Lambda_i^* \Lambda_i U^* f = USU^* f.$$

Hence $S_U = USU^*$. □

Corollary 2.25. *Let* $\{ \Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I \}$ *be a g-frame for* $\mathcal{H}$ *and let* $U : \mathcal{H} \longrightarrow \mathcal{K}$ *be a bounded operator with closed range. Then* $\{\Lambda_i U^* \in \mathcal{L}(\mathcal{K},\mathcal{H}_i) : i \in I \}$ *is a g-frame for* $\mathcal{R}_U$.

Corollary 2.26. *Let* $\{ \Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in I \}$ *be a g-frame for* $\mathcal{H}$ *and let* $U : \mathcal{H} \longrightarrow \mathcal{K}$ *be a bounded and invertible operator. Then* $\{\Lambda_i U^{-1} \in \mathcal{L}(\mathcal{K},\mathcal{H}_i) : i \in I \}$ *is a g-frame for* $\mathcal{K}$.

Corollary 2.27. *Every g-Riesz basis for* $\mathcal{H}$ *is a g-frame for* $\mathcal{H}$.

3. Perturbation of g-frames

Perturbation of frames has been discussed in [2]. However there are similar theorems about perturbation of frames for measurable spaces (see [9], [13]). Stability of g-frames and their duals has been investigated by W. Sun [16]. In this manuscript we give other perturbations of g-frames. At first we need the following lemma which is proved in [2].

Lemma 3.1. *Let U be a linear operator on a Banach space X and assume that there exist $\lambda_1, \lambda_2 \in [0,1)$ such that*

$$\|x - Ux\| \leq \lambda_1 \|x\| + \lambda_2 \|Ux\|$$

for all $x \in X$. Then U is bounded and invertible. Moreover

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \leq \|Ux\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|x\|$$

and

$$\frac{1-\lambda_2}{1+\lambda_1}\|x\| \leq \|U^{-1}x\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|x\|$$

for all $x \in X$.

Theorem 3.2. *Let $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ be a g-frame for $\mathcal{H}$ with bounds A, B and $\{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ be a sequence of operators such that for any finite subset $J \subseteq I$ and for each $f \in \mathcal{H}$,*

$$\begin{aligned} &\Big\| \sum_{i \in J} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f) \Big\| \\ &\leq \lambda \Big\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \Big\| + \mu \Big\| \sum_{i \in J} \Theta_i^* \Theta_i f \Big\| + \gamma \Big(\sum_{i \in J} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}}, \end{aligned} \tag{3.1}$$

where $0 \leq \max\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\} < 1$. Then $\{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ is a g-frame for $\mathcal{H}$ with frame bounds

$$A \frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1+\mu} \quad \text{and} \quad B \frac{1 + \lambda + \frac{\gamma}{\sqrt{B}}}{1-\mu}. \tag{3.2}$$

Proof. Assume that $J \subseteq I$ and $|J| < +\infty$. For each $f \in \mathcal{H}$, we have

$$\begin{aligned} \Big\| \sum_{i \in J} \Theta_i^* \Theta_i f \Big\| &\leq \Big\| \sum_{i \in J} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f) \Big\| + \Big\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \Big\| \\ &\leq (1+\lambda) \Big\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \Big\| + \mu \Big\| \sum_{i \in J} \Theta_i^* \Theta_i f \Big\| + \gamma \Big(\sum_{i \in J} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\Big\| \sum_{i \in J} \Theta_i^* \Theta_i f \Big\| \leq \frac{1+\lambda}{1-\mu} \Big\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \Big\| + \frac{\gamma}{1-\mu} \Big(\sum_{i \in J} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}}.$$

Also

$$\begin{aligned} \Big\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \Big\| &= \sup_{\|g\|=1} \Big| \langle \sum_{i \in J} \Lambda_i^* \Lambda_i f, g \rangle \Big| = \sup_{\|g\|=1} \Big| \langle \sum_{i \in J} \Lambda_i f, \Lambda_i g \rangle \Big| \\ &\leq \Big(\sum_{i \in J} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}} \sup_{\|g\|=1} \Big(\sum_{i \in J} \|\Lambda_i g\|^2 \Big)^{\frac{1}{2}} \leq \Big(\sum_{i \in J} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}} \sqrt{B} \leq B\|f\|. \end{aligned}$$

Therefore for all $f \in \mathcal{H}$

$$\Big\| \sum_{i \in J} \Theta_i^* \Theta_i f \Big\| \leq \Big(\frac{1+\lambda}{1-\mu}\sqrt{B} + \frac{\gamma}{1-\mu}\Big) \Big(\sum_{i \in J} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}},$$

so $\sum_{i\in I}\Theta_i^*\Theta_i f$ is unconditionally convergent. Let

$$G:\mathcal{H}\longrightarrow\mathcal{H},\quad G(f)=\sum_{i\in I}\Theta_i^*\Theta_i f,\quad f\in\mathcal{H}.$$

Then G is well-defined and bounded operator with

$$\|G\|\le\frac{1+\lambda}{1-\mu}B+\frac{\gamma\sqrt{B}}{1-\mu}$$

and for each $f\in\mathcal{H}$, we have

$$\sum_{i\in I}\|\Theta_i f\|^2=\langle Gf,f\rangle\le\|G\|\|f\|^2.$$

It follows that $\{\,\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ is a g-Bessel sequence for $\mathcal{H}$.

Let S be the g-frame operator of $\{\,\Lambda_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ then we obtain from (3.1)

$$\|Sf-Gf\|\le\lambda\|Sf\|+\mu\|Gf\|+\gamma\Big(\sum_{i\in I}\|\Lambda_i f\|^2\Big)^{\frac{1}{2}},\quad f\in\mathcal{H}.$$

Therefore

$$\begin{aligned}\|f-GS^{-1}f\|&\le\lambda\|f\|+\mu\|GS^{-1}f\|+\gamma\Big(\sum_{i\in I}\|\Lambda_i S^{-1}f\|^2\Big)^{\frac{1}{2}}\\&\le(\lambda+\frac{\gamma}{\sqrt{A}})\|f\|+\mu\|GS^{-1}f\|,\end{aligned}$$

since $0\le\max\{\lambda+\frac{\gamma}{\sqrt{A}},\mu\}<1$, then by Lemma 3.1, GS^{-1} and consequently G is invertible and

$$\|G^{-1}\|\le\|S^{-1}\|\|SG^{-1}\|\le\frac{1+\mu}{A\Big(1-(\lambda+\frac{\gamma}{\sqrt{A}})\Big)}.$$

Hence by Proposition 2.7, $\{\,\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ is a g-frame for $\mathcal{H}$. It is clear that the optimal lower bound and optimal upper bound of $\{\,\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ are $\|G^{-1}\|^{-1}$ and $\|G\|$, respectively. Then we can obtain the required frame bounds in (3.2). □

Corollary 3.3. *Let $\{\,\Lambda_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ be a g-frame for $\mathcal{H}$ with bounds A,B and let $\{\,\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ be a family of operators. If there exists a constant $0<R<A$ such that*

$$\sum_{i\in I}\|\Lambda_i^*\Lambda_i f-\Theta_i^*\Theta_i f\|\le R\|f\|$$

for all $f\in\mathcal{H}$, then $\{\,\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ is a g-frame with g-frame bounds $A-R$ and $\min\Big\{1+R\sqrt{\frac{B}{A}},R+B\Big\}$.

Proof. It is clear that $\sum_{i\in I}\Theta_i^*\Theta_i f$ is converges for each $f\in\mathcal{H}$. So we have

$$\Big\|\sum_{i\in I}\Lambda_i^*\Lambda_i f-\Theta_i^*\Theta_i f\Big\|\le R\|f\|\le\frac{R}{\sqrt{A}}\Big(\sum_{i\in I}\|\Lambda_i f\|^2\Big)^{\frac{1}{2}},\quad f\in\mathcal{H}.$$

By letting, $\lambda=\mu=0$ and $\gamma=R/\sqrt{A}$ in Theorem 3.2, $\{\,\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ will be a g-frame for $\mathcal{H}$ with lower bound $A-R$ and upper bound $1+R\sqrt{\frac{B}{A}}$. Also we have

$$\Big\|\sum_{i\in I}\Theta_i^*\Theta_i f\Big\|\le R\|f\|+\Big\|\sum_{i\in I}\Lambda_i^*\Lambda_i f\Big\|\le(R+B)\|f\|$$

for all $f\in\mathcal{H}$. Hence the upper g-frame bound for $\{\,\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in I\,\}$ is

$$\min\Big\{1+R\sqrt{\frac{B}{A}},R+B\Big\}.\quad\square$$

Corollary 3.4. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with frame bounds A, B and let $\{g_i\}_{i\in I}$ be a sequence of $\mathcal{H}$. Suppose that there exist constants λ, μ and γ with $0 \le \max\{\mu, \lambda+\frac{\gamma}{\sqrt{A}}\} < 1$ such that for each $f \in \mathcal{H}$ and every finite subset $J \subseteq I$,*

$$\begin{aligned}\Big\|\sum_{i\in J}\langle f, f_i\rangle f_i - \langle f, g_i\rangle g_i\Big\| \\ \le \lambda\Big\|\sum_{i\in J}\langle f, f_i\rangle f_i\Big\| + \mu\Big\|\sum_{i\in J}\langle f, g_i\rangle g_i\Big\| + \gamma\Big(\sum_{i\in J}|\langle f, f_i\rangle|\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.3}$$

Then $\{g_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with frame bonds

$$A\frac{1-(\lambda+\frac{\gamma}{\sqrt{A}})}{1+\mu} \quad \textit{and} \quad B\frac{1+\lambda+\frac{\gamma}{\sqrt{B}}}{1-\mu}.$$

Proof. For each $i \in I$, let

$$\Lambda_i : \mathcal{H} \longrightarrow \mathbb{C}, \quad \Lambda_i(f) = \langle f, f_i\rangle$$

and

$$\Theta_i : \mathcal{H} \longrightarrow \mathbb{C}, \quad \Theta_i(f) = \langle f, g_i\rangle.$$

Then $\{\, \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathbb{C}) : i \in I \,\}$ is a g-frame for $\mathcal{H}$ and (3.3) means (3.1). Therefore the result follows from Theorem 3.2. □

The following Theorem is another version of perturbation of g-frames.

Theorem 3.5. *Let $\{\, \Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I \,\}$ be a g-frame for $\mathcal{H}$ with bounds A, B and let $\{\, \Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I \,\}$ be a family of operators such that for any $J \subseteq I$ with $|J| < +\infty$,*

$$\begin{aligned}\Big\|\sum_{i\in J}(\Lambda_i^* f_i - \Theta_i^* f_i)\Big\| \\ \le \lambda\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| + \mu\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| + \gamma\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}, \quad (f_i \in \mathcal{H}_i)\end{aligned} \tag{3.4}$$

where $0 \le \max\{\lambda + \frac{\gamma}{A}, \mu\} < 1$. Then $\{\, \Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I \,\}$ is a g-frame for $\mathcal{H}$ with g-frame bounds

$$A\Big(\frac{1-(\lambda+\frac{\gamma}{\sqrt{A}})}{1+\mu}\Big)^2 \quad \textit{and} \quad B\Big(\frac{1+\lambda+\frac{\gamma}{\sqrt{B}}}{1-\mu}\Big)^2.$$

Proof. It is clear that if $J \subseteq I$ with $|J| < +\infty$, then for all $i \in J$ and $f_i \in \mathcal{H}_i$, we have

$$\begin{aligned}\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| &\le \Big\|\sum_{i\in J}(\Lambda_i^* f_i - \Theta_i^* f_i)\Big\| + \Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| \\ &\le (\lambda+1)\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| + \mu\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| + \gamma\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

So we have

$$\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| \le \frac{\lambda+1}{1-\mu}\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| + \frac{\gamma}{1-\mu}\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}.$$

Since

$$\begin{aligned}\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| &= \sup_{\|g\|=1}\Big|\Big\langle \sum_{i\in J}\Lambda_i^* f_i, g\Big\rangle\Big| \\ &\le \Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}\sup_{\|g\|=1}\Big(\sum_{i\in J}\|\Lambda_i g\|^2\Big)^{\frac{1}{2}} \le \sqrt{B}\Big(\sum_{i\in J}\|f_i\|^2)^{\frac{1}{2}},\end{aligned}$$

then

$$\Big\| \sum_{i\in J} \Theta_i^* f_i \Big\| \leq \Big(\frac{\lambda+1}{1-\mu}\sqrt{B} + \frac{\gamma}{1-\mu} \Big) \Big(\sum_{i\in J} \|f_i\|^2 \Big)^{\frac{1}{2}}. \tag{3.5}$$

Let

$$T : \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i \Big)_{\ell_2} \longrightarrow \mathcal{H}, \quad T\Big(\{f_i\}_{i\in I} \Big) = \sum_{i\in I} \Theta_i^* f_i.$$

By (3.5), T is well-defined and bounded with

$$\|T\| \leq \frac{\lambda+1}{1-\mu}\sqrt{B} + \frac{\gamma}{1-\mu}.$$

Then by Proposition 2.4, $\{ \Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence for $\mathcal{H}$ and

$$T^* : \mathcal{H} \longrightarrow \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i \Big)_{\ell_2}, \quad T^*(f) = \{\Theta_i f\}_{i\in I}.$$

Let U and S be the synthesis operator and g-frame operator for $\{ \Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I \}$, respectively. Let $G = TU^*S^{-1}$. Then we have from (3.4)

$$\begin{aligned} &\Big\| \sum_{i\in I} (\Lambda_i^*\Lambda_i S^{-1} f - \Theta_i^*\Lambda_i S^{-1} f) \Big\| \\ &\quad \leq \lambda \Big\| \sum_{i\in I} \Lambda_i^*\Lambda_i S^{-1} f \Big\| + \mu \Big\| \sum_{i\in I} \Theta_i^*\Lambda_i S^{-1} f \Big\| + \gamma \Big(\sum_{i\in I} \|\Lambda_i S^{-1} f\|^2 \Big)^{\frac{1}{2}}, \quad f \in \mathcal{H}. \end{aligned}$$

Since for all $f \in \mathcal{H}$, $f = \sum_{i\in I} \Lambda_i^*\Lambda_i S^{-1} f$, so we have

$$\begin{aligned} \|f - Gf\| &\leq \lambda\|f\| + \mu\|Gf\| + \gamma \Big(\sum_{i\in I} \|\Lambda_i S^{-1} f\|^2 \Big)^{\frac{1}{2}} \\ &\leq \lambda\|f\| + \mu\|Gf\| + \frac{\gamma}{\sqrt{A}}\|f\| \leq (\lambda + \frac{\gamma}{\sqrt{A}})\|f\| + \mu\|Gf\|. \end{aligned}$$

Since $0 \leq \max\{\lambda + \frac{\gamma}{A}, \mu\} < 1$, by Lemma 3.1, $G = TU^*S^{-1}$ and consequently TU^* is invertible and

$$\|G^{-1}\| \leq \frac{1+\mu}{1-(\lambda+\frac{\gamma}{\sqrt{A}})}.$$

Let$f \in \mathcal{H}$, then $f = GG^{-1}f = \sum_{i\in I} \Theta_i^*\Lambda_i S^{-1}G^{-1}f$. Hence for each $f \in \mathcal{H}$, we have

$$\begin{aligned} \|f\|^4 = |\langle f, f\rangle|^2 &= \Big| \Big\langle \sum_{i\in I} \Theta_i^*\Lambda_i S^{-1}G^{-1}f, f \Big\rangle \Big|^2 \leq \sum_{i\in I} \|\Lambda_i S^{-1}G^{-1}f\|^2 \sum_{i\in I} \|\Theta_i f\|^2 \\ &\leq \frac{1}{A}\|G^{-1}f\|^2 \sum_{i\in I} \|\Theta_i f\|^2 \leq \frac{1}{A}\Big(\frac{1+\mu}{1-(\lambda+\frac{\gamma}{\sqrt{A}})} \Big)^2 \|f\|^2 \sum_{i\in I} \|\Theta_i f\|^2. \end{aligned}$$

Therefore

$$\sum_{i\in I} \|\Theta_i f\|^2 \geq A\Big(\frac{1-(\lambda+\frac{\gamma}{\sqrt{A}})}{1+\mu} \Big)^2 \|f\|^2, \quad f \in \mathcal{H}.$$

Then $\{ \Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$ with the required g-frame bounds. $\square$

The following corollary has been proved in [2].

Corollary 3.6. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with bounds A, B. Let $\{g_i\}_{i\in I}$ be a sequence in $\mathcal{H}$ and assume that there exist non-negative constants λ, μ and γ such that*

$$\max\Big\{ \mu, \lambda + \frac{\gamma}{\sqrt{A}} \Big\} < 1$$

and

$$\Big\|\sum_i c_i f_i - c_i g_i\Big\| \le \lambda\Big\|\sum_i c_i f_i\Big\| + \mu\Big\|\sum_i c_i g_i\Big\| + \gamma\Big(\sum_i |c_i|^2\Big)^{\frac{1}{2}} \tag{3.6}$$

for all finite scalar sequences $\{c_i\}_i$. *Then* $\{g_i\}_{i\in I}$ *is a frame for* $\mathcal{H}$ *with frame bounds*

$$A\Big(\frac{1-(\lambda+\frac{\gamma}{\sqrt{A}})}{1+\mu}\Big)^2 \quad and \quad B\Big(\frac{1+\lambda+\frac{\gamma}{\sqrt{B}}}{1-\mu}\Big)^2.$$

Proof. For each $i \in I$, let

$$\Lambda_i : \mathcal{H} \longrightarrow \mathbb{C}, \quad \Lambda_i(f) = \langle f, f_i\rangle$$

and

$$\Theta_i : \mathcal{H} \longrightarrow \mathbb{C}, \quad \Theta_i(f) = \langle f, g_i\rangle.$$

Then $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathbb{C}) : i \in I\,\}$ is a g-frame for $\mathcal{H}$ and (3.6) means (3.4). Therefore the result follows from Theorem 3.5. $\square$

Proposition 3.7. *Let* $\{\,\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ *be a g-frame for* $\mathcal{H}$ *with bounds* A, B *and let* $\{\,\Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ *be a family of operators. If there exists a* $0 < R < A$ *such that*

$$\sum_{i\in I} \|\Lambda_i f - \Theta_i f\|^2 \le R\|f\|^2$$

for all $f \in \mathcal{H}$. *Then* $\{\,\Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ *is a g-frame for* $\mathcal{H}$ *with bounds* $(\sqrt{A}-\sqrt{R})^2$ *and* $(\sqrt{B}+\sqrt{R})^2$.

Proof. By the triangle inequality in $\Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2}$, we have

$$\Big\|\{\Theta_i f\}_{i\in I}\Big\|_{\ell_2} \le \Big\|\{\Theta_i f - \Lambda_i f\}_{i\in I}\Big\|_{\ell_2} + \Big\|\{\Lambda_i f\}_{i\in I}\Big\|_{\ell_2}, \quad f \in \mathcal{H}$$

therefore

$$\sum_{i\in I} \|\Theta_i f\|^2 \le (\sqrt{B}+\sqrt{R})^2\|f\|^2, \quad f \in \mathcal{H}.$$

Also for each $f \in \mathcal{H}$

$$\Big\|\{\Theta_i f\}_{i\in I}\Big\|_{\ell_2} \ge \Big\|\{\Lambda_i f\}_{i\in I}\Big\|_{\ell_2} - \Big\|\{\Theta_i f - \Lambda_i f\}_{i\in I}\Big\|_{\ell_2}$$

thus

$$\sum_{i\in I} \|\Theta_i f\|^2 \ge (\sqrt{A}-\sqrt{R})^2\|f\|^2, \quad f \in \mathcal{H}.$$

$\square$

The following theorem is a generalization of a result in [5].

Theorem 3.8. *Let* $\{\,\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ *be a g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in I\}$ *and* $\{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ *be a family of operators. If*

$$K : \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2} \longrightarrow \mathcal{H}, \quad K\Big(\{f_i\}_{i\in I}\Big) = \sum_{i\in I} (\Lambda_i^* - \Theta_i^*) f_i$$

is a well-defined and compact operator, then $\{\,\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\,\}$ *is a g-frame sequence.*

Proof. Since $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame for $\mathcal{H}$ and the perturbation operator K is bounded, then the synthesis g-frame operator U for $\{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is well-defined and bounded. Therefore by Proposition 2.4, $\{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence for $\mathcal{H}$. Let T and S be synthesis operator and g-frame operator for $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$, respectively. Then $U = T - K$ and

$$UU^* = (T-K)(T^* - K^*) = S\Big(I + S^{-1}(-TK^* - KT^* + KK^*)\Big).$$

Since K is compact operator, then $S^{-1}(-TK^* - KT^* + KK^*)$ is a compact operator and by Theorem 4.23 of [14] the operator $I + S^{-1}(-TK^* - KT^* + KK^*)$ has closed range. Composing this with S, we see that UU^* also has closed range. Hence $\mathcal{R}_U$ is closed. So $\mathcal{R}_U = \overline{\text{span}}\{\Theta_i^*(\mathcal{H}_i)\}_{i\in I}$. Therefore the result follows from Proposition 2.6. □

Corollary 3.9. *Let $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-frame for $\mathcal{H}$. Let J be a finite subset of I such that for each $j \in J$, $\dim \mathcal{H}_j < \infty$. Then*

$$\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \setminus J \}$$

is a g-frame sequence.

Proof. Let $\{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a sequence of operators such that $\Theta_i = 0$, if $i \in J$ and $\Theta_i = \Lambda_i$, if $i \notin J$. Then the operator

$$K : \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2} \longrightarrow \mathcal{H}, \quad K(\{f_i\}_{i\in I}) = \sum_{i\in I}(\Lambda_i^* - \Theta_i^*)f_i = \sum_{i\in J} \Lambda_i^* f_i$$

has finite dimensional range and hence is compact. Then by Theorem 3.10, $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \setminus J \}$ is a g-frame sequence. □

Theorem 3.10. *Let $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a g-frame for $\mathcal{H}$ and let $\{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ be a family of operators. If*

$$K : \mathcal{H} \longrightarrow \mathcal{H}, \quad Kf = \sum_{i\in I}(\Lambda_i^*\Lambda_i f - \Theta_i^*\Theta_i f)$$

is a well-defined and compact operator, then $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-frame sequence.

Proof. Let S be the g-frame operator of $\{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$, then $S^\circ = S - K$ is a bounded operator. Therefore for each $f \in \mathcal{H}$

$$\sum_{i\in I} \|\Theta_i f\|^2 = \langle S^\circ f, f\rangle \le \|S^\circ\| \|f\|^2.$$

Then $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a g-Bessel sequence. Let

$$T : \Big(\sum_{i\in I} \bigoplus \mathcal{H}_i\Big)_{\ell_2} \longrightarrow \mathcal{H}, \quad T(\{f_i\}_{i\in I}) = \sum_{i\in I} \Theta_i^* f_i.$$

By Proposition 2.4, T is bounded and $S^\circ = TT^*$. Since $S^{-1}K$ is compact, so S° has closed range. Therefore $\mathcal{R}_{S^\circ} = \mathcal{R}_T = \overline{\text{span}}\{\Theta_i^*(\mathcal{H}_i)\}_{i\in I}$. Hence the result follows from Proposition 2.7. □

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Faculty of Sciences, Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Islamic Republic of Iran
E-mail address: a.nejati@@yahoo.com

Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Islamic Republic of Iran
E-mail address: mhfaroughi@@yahoo.com

Department of Mathematics, University of Maragheh, Maragheh, Islamic Republic of Iran
E-mail address: asgharrahimi@@yahoo.com

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