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THE INVOLUTIVE AUTOMORPHISMS OF τ-COMPACT OPERATORS AFFILIATED WITH A TYPE I VON NEUMANN ALGEBRA

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Abstract. Let M be a type I von Neumann algebra with a center Z, and a faithful normal semi-finite trace τ. Consider the algebra $L(M,\tau)$ of all τ-measurable operators with respect to M and let $S_0(M,\tau)$ be the subalgebra of τ-compact operators in $L(M,\tau)$. We prove that any Z-linear involutive automorphisms of $S_0(M,\tau)$ is inner.

1. Introduction

The present paper is devoted to an investigation of $*$-automorphisms of τ-compact operators affiliated with a type I von Neumann algebra.

It is well known [5] that, if M is a type I von Neumann algebra and $\Phi : M \to M$ is an $*$-automorphism such that $\Phi(zx) = z\Phi(x)$ for all central elements z in M, then Φ is inner, i.e., $\Phi(x) = uxu^*$ for some unitary element $u \in M$. Some results of such a kind for unbounded operator algebras were obtained in [9]. Namely, it was proved that any $*$-automorphism of the maximal O^*-algebra is inner.

One of important classes of unbounded operator algebras are algebras of τ-compact operators affiliated with a von Neumann algebra.

In the present paper, using the description of the algebra of τ-measurable operators affiliated with a type I von Neumann algebra obtained in [1] and also the description of automorphisms of standard subalgebras of the algebra of bounded linear operators acting in Banach-Kantorovich modula from [2], we prove that any Z-linear $*$-automorphism of the algebra of τ-compact operators affiliated with a type I von Neumann algebra is inner.

2. Preliminaries

Let (Ω, Σ, μ) be a measurable space with a σ-finite measure μ, i. e., there is family $\{\Omega_i\}_{i\in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exists a countable subset $J_0 \subset J$ and a set B with zero measure such that $A = \bigcup\limits_{i\in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^0 = L^0(\Omega, \Sigma, \mu)$ the algebra of all (classes of) complex measurable functions on (Ω, Σ, μ) equipped with the topology of convergence in measure. Then L^0 is a complete metrizable commutative regular algebra with the unit $\mathbf{1}$ given by $\mathbf{1}(\omega) = 1$, $\omega \in \Omega$.

Denote by ∇ the complete Boolean algebra of all idempotents from L^0, i.e., $\nabla = \{\chi_A : A \in \Sigma\}$, where χ_A is the characteristic function of the set A.

A complex linear space E is said to be normed by L^0 if there is a map $\|\cdot\| : E \longrightarrow L^0$ such that for any $x, y \in E, \lambda \in \mathbb{C}$, the following conditions are fulfilled:

$$\|x\| \geq 0, \quad \|x\| = 0 \Longleftrightarrow x = 0, \quad \|\lambda x\| = |\lambda|\|x\|, \quad \|x + y\| \leq \|x\| + \|y\|.$$

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The pair $(E, \|\cdot\|)$ is called a lattice-normed space over L^0. A lattice-normed space E is called d-decomposable, if for any $x \in E$ with $\|x\| = \lambda_1 + \lambda_2$, where $\lambda_1, \lambda_2 \in L^0$, $\lambda_1\lambda_2 = 0$, there exists $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_i\| = \lambda_i$, $i = 1, 2$. A net (x_α) in E is (bo)-converging to $x \in E$, if $\|x_\alpha - x\| \to 0$ μ-almost everywhere in L^0. A lattice-normed space E which is d-decomposable and complete with respect to the (bo)-convergence is called a *Banach-Kantorovich space*.

It is known that every Banach-Kantorovich space E over L^0 is a module over L^0 and $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in L^0$, $x \in E$ (see [6]).

A module F over L^0 is said to be finite-generated, if there are $x_1, x_2, ..., x_n$ in F for any $x \in F$ there exists $\lambda_i \in L^0\,(i = \overline{1,n})$ such that $x = \lambda_1 x_1 + ... + \lambda_n x_n$. The elements $x_1, x_2, ..., x_n$ are called generators of F. We denote by $d(F)$ the minimal number of generators of F.

A finite-generated module F over L^0 is called homogeneous of type n, if for every nonzero $e \in \nabla$ we have $n = d(eF)$.

Let $\mathcal{K}$ be a module over L^0. A map $\langle\cdot,\cdot\rangle : \mathcal{K} \times \mathcal{K} \to L^0$ is called an L^0-valued inner product, if for all $x, y, z \in \mathcal{K}$, $\lambda \in L^0$, the following conditions are fulfilled: $\langle x, x\rangle \geq 0$; $\langle x, x\rangle = 0 \Leftrightarrow x = 0$; $\langle x, y\rangle = \overline{\langle y, x\rangle}$; $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$; $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

If $\langle\cdot,\cdot\rangle : \mathcal{K} \times \mathcal{K} \to L^0$ is an L^0-valued inner product, then $\|x\| = \sqrt{\langle x, x\rangle}$ defines an L^0-valued norm on $\mathcal{K}$. The pair $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ is called a *Kaplansky-Hilbert module* over L^0, if $(\mathcal{K}, \|\cdot\|)$ is a Banach-Kantorovich space over L^0 (see [6]).

Let X be a Banach space. A map $s : \Omega \to X$ is called a simple, if

$$s(\omega) = \sum_{k=1}^{n} \chi_{A_k}(\omega) c_k,$$

where $A_k \in \Sigma, A_i \cap A_j = \emptyset$, $i \neq j$, $c_k \in X$, $k = \overline{1,n}$, $n \in \mathbb{N}$. A map $u : \Omega \to X$ is said to be measurable, if there is a sequence (s_n) of simple maps such that $\|s_n(\omega) - u(\omega)\| \to 0$ almost everywhere on any $A \in \sum$ with $\mu(A) < \infty$.

Let $\mathcal{L}(\Omega, X)$ be the set of all measurable maps from Ω into X, and let $L^0(\Omega, X)$ denote the factorization of this set with respect to equality almost everywhere. Denote by $\hat{u}$ the equivalence class from $L^0(\Omega, X)$ which contains the measurable map $u \in \mathcal{L}(\Omega, X)$. Further we shall identity the element $u \in \mathcal{L}(\Omega, X)$ with the class $\hat{u}$. Note that the function $\omega \to \|u(\omega)\|$ is measurable for any $u \in \mathcal{L}(\Omega, X)$. The equivalence class containing the function $\|u(\omega)\|$ is denoted by $\|\hat{u}\|$. For $\hat{u}, \hat{v} \in L^0(\Omega, X), \lambda \in L^0$ put $\hat{u} + \hat{v} = \widehat{u(\omega) + v(\omega)}, \lambda\hat{u} = \widehat{\lambda(\omega)u(\omega)}$.

It is known [6] that $(L^0(\Omega, X), \|\cdot\|)$ is a Banach-Kantorovich space over L^0.

Put $L^\infty(\Omega, X) = \{x \in L^0(\Omega, X) : \|x\| \in L^\infty(\Omega)\}$. Then $L^\infty(\Omega, X)$ is a Banach space with respect the norm $\|x\|_\infty = \|\|x\|\|_{L^\infty(\Omega)}$.

If H is a Hilbert space, then $L^0(\Omega, H)$ can be equipped with an L^0-valued inner product $\langle x, y\rangle = \widehat{(x(\omega), y(\omega))}$, where $(\cdot,\cdot)$ is the inner product on H.

Then $(L^0(\Omega, H), \langle\cdot,\cdot\rangle)$ is a Kaplansky-Hilbert module over L^0.

Let E be a Banach-Kantorovich space over L^0. An operator $T : E \to E$ is called L^0-linear if $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ for all $\lambda_1, \lambda_2 \in L^0, x_1, x_2 \in E$. An L^0-linear operator $T : E \to E$ is called L^0-bounded, if there exists an element $c \in L^0$ such that $\|T(x)\| \leq c\|x\|$ for any $x \in E$. For an L^0-bounded linear operator T we put $\|T\| = \sup\{\|T(x)\| : \|x\| \leq \mathbf{1}\}$.

An L^0-linear operator $T : E \to E$ is called finite-generated (homogeneous of type n) if $T(E) = \{T(x) : x \in E\}$ is a finite-generated (respectively homogeneous of type n) submodule in E.

We denote by $B(E)$ the algebra of all L^0-linear L^0-bounded operators on E and $\mathcal{F}(E)$ be the set of all finite-generated L^0-linear L^0-bounded operators on E.

An algebra $\mathcal{U} \subset B(E)$ is called *standard* over L^0, if $\mathcal{U}$ is a submodule in $B(E)$ and $\mathcal{F}(E) \subset \mathcal{U}$.

Recall that bijective linear operator $\Phi : \mathcal{U} \to \mathcal{U}$ is said an automorphism, if $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathcal{U}$

Theorem 2.1. [2]. *Let $\mathcal{U}$ be a standard algebra in $B(L^0(\Omega, H))$ and let $\Phi : \mathcal{U} \to \mathcal{U}$ be an L^0 – linear automorphism of the algebra $\mathcal{U}$. Then there is an invertible element $a \in B(L^0(\Omega, H))$ such that*

$$\Phi(A) = axa^{-1}$$

for all $x \in \mathcal{U}$.

3. The main result

A linear subspace D in H is said to be affiliated with M (denotes as $D\eta M$), if $u(D) \subset D$ for any unitary operator u from the commutant

$$M' = \{y' \in B(H) : xy' = y'x, \forall x \in M\}$$

of the algebra M.

A linear operator x on H with domain $D(x)$ is said to be affiliated with M (denoted as $x\eta M$) if $u(D(x)) \subset D(x)$ and $ux(\xi) = xu(\xi)$ for all $u \in M'$, $\xi \in D(x)$.

A linear subspace D in H is called τ-dense, if

1) $D\eta M$;

2) given any $\varepsilon > 0$ there exists a projection $p \in \mathcal{P}(M)$ such that $p(H) \subset D$ and $\tau(p^{\perp}) \leq \varepsilon$.

A closed linear operator x is said to be τ-measurable (or totally measurable) with respect to the von Neumann algebra M, if $x\eta M$ and $D(x)$ is τ-dense in H.

We will denote by $L(M, \tau)$ the set of all τ-measurable operators affiliated with M. Let $\|\cdot\|_M$ stand for the uniform norm in M. The *measure topology*, t_τ, in $L(M, \tau)$ is the one given by the following system of neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in L(M, \tau) : \exists e \in \mathcal{P}(M), \tau(e^{\perp}) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where $\varepsilon > 0, \delta > 0$.

It is known [8] that $L(M, \tau)$ equipped with the measure topology is a complete metrizable topological $*$-algebra.

In the algebra $L(M, \tau)$ consider the subset $S_0(M, \tau)$ of all operators x such that given any $\varepsilon > 0$ there is a projection $p \in \mathcal{P}(M)$ with $\tau(p^{\perp}) < \infty$, $xp \in M$ and $\|xp\| < \varepsilon$. Following [10] let us call the elements of $S_0(M, \tau)$ *τ-compact operators* affiliated with M. It is known [12], [7] that $S_0(M, \tau)$ is a $*$-subalgebra in $L(M, \tau)$ and an M-bimodule, i. e. $ax, xa \in S_0(M, \tau)$ for all $x \in S_0(M, \tau)$ and $a \in M$. It is clear that if the trace τ is finite then $S_0(M, \tau) = L(M, \tau)$.

The following properties of the algebra $S_0(M, \tau)$ of τ-compact operators are known [10], [3].

Let M be a von Neumann algebra with a faithful normal semi-finite trace τ. Then

1) $L(M, \tau) = M + S_0(M, \tau)$;

2) $S_0(M, \tau)$ is an ideal in $L(M, \tau)$.

Let $L^{\infty}(\Omega)\bar{\otimes}B(H)$ be the tensor product of von Neumann algebra $L^{\infty}(\Omega)$ and $B(H)$, with the trace $\tau = \mu \otimes \mathrm{Tr}$, where Tr is the canonical trace for operators in $B(H)$ (with its natural domain).

Denote by $L^0(\Omega, B(H))$ the space of equivalence classes of measurable maps from Ω into $B(H)$. Given $\hat{u}, \hat{v} \in L^0(\Omega, B(H))$ put $\hat{u}\hat{v} = \widehat{u(\omega)v(\omega)}$, $\hat{u}^* = \widehat{u(\omega)^*}$.

Define

$$L^{\infty}(\Omega, B(H)) = \{x \in L^0(\Omega, B(H)) : \|x\| \in L^{\infty}(\Omega)\}.$$

The space $(L^{\infty}(\Omega, B(H)), \|\cdot\|_{\infty})$ is a Banach $*$-algebra.

It is known [11] that the algebra $L^\infty(\Omega)\bar{\otimes}B(H)$ is $*$-isomorphic to the algebra $L^\infty(\Omega, B(H))$.

Note also that

$$\tau(x) = \int\limits_\Omega \operatorname{Tr}(x(\omega))\, d\mu(\omega).$$

Further we shall identity the algebra $L^\infty(\Omega)\bar{\otimes}B(H)$ with the algebra $L^\infty(\Omega, B(H))$.

Denote by $B(L^0(\Omega, H))$ (resp. $B(L^\infty(\Omega, H))$) the algebra of all L^0-linear and L^0-bounded (resp. $L^\infty(\Omega)$-linear and $L^\infty(\Omega)$-bounded) operators on $L^0(\Omega, H)$ (resp. $L^\infty(\Omega, H)$).

Given any $f \in L^\infty(\Omega, B(H))$ consider the element $\Psi(f)$ from $B(L^\infty(\Omega, H))$ defined by

$$\Psi(f)(x) = \widehat{f(\omega)(x(\omega))}, \quad x \in L^\infty(\Omega, H).$$

Then the correspondence $f \to \Psi(f)$ gives an isometric $*$-isomorphism between the algebras $L^\infty(\Omega, B(H))$ and $B(L^\infty(\Omega, H))$ (see [6]).

It is known [1], that the algebra $L(L^\infty(\Omega)\bar{\otimes}B(H), \tau)$ of all τ-measurable operators affiliated with the von Neumann algebra $L^\infty(\Omega)\bar{\otimes}B(H)$ is L^0-linear $*$-isomorphic with the algebra $B(L^0(\Omega, H))$.

Therefore one has the following relations for the algebras mentioned above:

$$L(L^\infty(\Omega)\bar{\otimes}B(H)), \tau) \cong L^0(\Omega, B(H)) \cong B(L^0(\Omega, H)).$$

Let $\operatorname{mix}(S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau))$ be the cyclic hull of the set $S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau)$, i. e. it consists of all elements of the form $x = (bo) - \sum\limits_\alpha \pi_\alpha x_\alpha$, where (π_α) is a partition of the unit in ∇, $(x_\alpha) \subset S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau)$.

Since $S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau)$ is a module over $L^\infty(\Omega)$ and $L^0 = \operatorname{mix}(L^\infty(\Omega))$, we have that $\operatorname{mix}(S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau))$ is a module over L^0.

Proposition 3.1. $\operatorname{mix}(S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau))$ *is a standard algebra in* $L(L^\infty(\Omega)\bar{\otimes} B(H), \tau)$.

Proof. First suppose that the measure μ is finite. Consider a finite-generated operator x from the algebra $L(L^\infty(\Omega)\bar{\otimes}B(H), \tau)$. Let p be the orthogonal projection onto the image of x and n be the number of its generators. By ([4], Theorem 2), $\operatorname{Tr}(p(\omega)) = \dim p(\omega) \le n$ for almost all $\omega \in \Omega$. Therefore $\tau(p) = \int\limits_\Omega \operatorname{Tr}(p(\omega))\, d\mu(\omega) \le n\mu(\Omega)$, i. e. $\tau(p) < \infty$.

It is clear that $xp^\perp = 0$. Thus $\tau(p) < \infty$ and $xp^\perp = 0$, i. e. $x \in S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau)$.

Now suppose that μ is σ-finite and x is a finite-generated operator from $L(L^\infty(\Omega)\bar{\otimes} B(H), \tau)$. Since the measure μ is σ-finite, there exists a partition of the unit (e_α) in ∇ such that $e_\alpha = \chi_{A_\alpha}, A_\alpha \in \Sigma, \mu(A_\alpha) < \infty$. From the above it follows $e_\alpha x \in e_\alpha S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau)$ and therefore $x = (bo) - \sum\limits_\alpha e_\alpha x$ belongs to $\operatorname{mix}(S_0(L^\infty(\Omega)\bar{\otimes} B(H), \tau))$. Thus $\operatorname{mix}(S_0(L^\infty(\Omega)\bar{\otimes}B(H), \tau))$ is a standard algebra. The proof is complete. $\square$

Proposition 3.2. *Let* Φ *be an* $*$*-automorphism of the algebra* $S_0(L^\infty(\Omega, B(H)), \tau)$*. Then there exists an unitary element* $u \in L^\infty(\Omega, B(H))$ *such that* $\Phi(x) = uxu^*$ *for all* $x \in S_0(L^\infty(\Omega, B(H)), \tau)$.

Proof. First show that $*$-isomorphism Φ is continued till $\operatorname{mix}(S_0(L^\infty(\Omega, B(H)), \tau))$.

Put

$$\widetilde{\Phi}(x) = (bo) - \sum_\alpha e_\alpha \Phi(x_\alpha), \quad x \in \operatorname{mix}(S_0(L^\infty(\Omega, B(H)), \tau)).$$

In the standard way, one can proved that $\widetilde{\Phi}$ is defined correctly and it is isomorphism.

Now, show that $\widetilde{\Phi}$ is L^0-linear.

Let $\lambda \in L^0$, $x \in \operatorname{mix}(S_0(L^\infty(\Omega, B(H)), \tau))$. Take a unit decomposition (e_α) in ∇ such that $e_\alpha\lambda \in L^\infty(\Omega)$, $e_\alpha x \in S_0(L^\infty(\Omega, B(H)), \tau)$ for all α. Since Φ is $L^\infty(\Omega)$-linear, then $\Phi(e_\alpha \lambda x) = \Phi(e_\alpha \lambda e_\alpha x) = e_\alpha \lambda \Phi(e_\alpha x)$. Therefore $\widetilde{\Phi}(\lambda x) = (bo) - \sum\limits_\alpha e_\alpha \Phi(e_\alpha \lambda x) = \sum\limits_\alpha e_\alpha \lambda \Phi(e_\alpha x) = \lambda \widetilde{\Phi}(x)$, i.e. $\widetilde{\Phi}(\lambda x) = \lambda \widetilde{\Phi}(x)$.

Since $\operatorname{mix}(S_0(L^\infty(\Omega, B(H)), \tau))$ is a standard algebra, then by Theorem 2.1, there exists an invertible operator $a \in B(L^0(\Omega, H))$ such that $\widetilde{\Phi}(x) = axa^{-1}$, $x \in \operatorname{mix}(S_0(L^\infty(\Omega, B(H)), \tau))$.

Show that a^*a is a central element. Since Φ is $*$-automorphism, then $\Phi(x^*) = \Phi(x)^*$. Hence $ax^*a^{-1} = (axa^{-1})^* = (a^{-1})^* x^* a^* = (a^*)^{-1} x^* a^*$, i.e. $ax^*a^{-1} = (a^*)^{-1}x^*a^*$. That's why $a^*ax^* = x^*a^*a$. If we change x to x^*, then we have $a^*ax = xa^*a$ for all $x \in \operatorname{mix}(S_0(L^\infty(\Omega, B(H)), \tau))$. Therefore $ea^*aexe = exea^*ae$ for all $x \in eL^0(\Omega, B(H))$ where e is a projector with the finite trace. That's why ea^*ae is a central element in $eL^0(\Omega, B(H))$. Hence a^*a is a central element in $L^0(\Omega, B(H))$.

Since the center of $L^0(\Omega, B(H))$ is isomorphic to L^0, then $a^*a = \lambda e$ for any $\lambda \in L^0$. Since a is an invertible operator, then λ is also an invertible element in L^0. Put $u = \lambda^{-\frac{1}{2}}a$. Then $u^*u = e$. Hence u is an unitary element. Moreover, $\Phi(x) = uxu^*$ for all $x \in \operatorname{mix}(S_0(L^\infty(\Omega, B(H)), \tau))$, particularly, for all $x \in S_0(L^\infty(\Omega, B(H)), \tau)$. The proof is complete. $\square$

Recall that a von Neumann algebra M is an algebra of *type I* if it is isomorphic to a von Neumann algebra with an Abelian commutant.

It is well-known [11] that if M is a type I von Neumann algebra then there is a unique (cardinal-indexed) orthogonal family of projections $(q_\alpha)_{\alpha \in I} \subset \mathcal{P}(M)$ with $\sum\limits_{\alpha \in I} q_\alpha = \mathbf{1}$ such that $q_\alpha M$ is isomorphic to the tensor product of an Abelian von Neumann algebra $L^\infty(\Omega_\alpha, \mu_\alpha)$ and $B(H_\alpha)$ with $\dim H_\alpha = \alpha$, i. e.

$$q_\alpha M \cong \sum_\alpha^{\oplus} L^\infty(\Omega_\alpha, \mu_\alpha) \bar{\otimes} B(H_\alpha).$$

Consider the faithful normal semi-finite trace τ on M, defined as

$$\tau(x) = \sum_\alpha \tau_\alpha(x_\alpha), \quad x = (x_\alpha) \in M, \quad x \geq 0,$$

where $\tau_\alpha = \mu_\alpha \otimes \mathrm{Tr}_\alpha$.

Let

$$\prod_\alpha S_0(L^\infty(\Omega_\alpha, \mu_\alpha) \bar{\otimes} B(H_\alpha), \tau_\alpha)$$

be the topological (Tychonoff) product of the spaces $S_0(L^\infty(\Omega_\alpha, \mu_\alpha) \bar{\otimes} B(H_\alpha), \tau_\alpha)$.

Then (see [7]) we have the topological embedding

$$S_0(M, \tau) \subset \prod_\alpha S_0(L^\infty(\Omega_\alpha, \mu_\alpha) \bar{\otimes} B(H_\alpha), \tau_\alpha).$$

Theorem 3.3. *If M is a Type I von Neumann Algebra, then any Z-linear *-automorphism of the algebra $S_0(M, \tau)$ is inner.*

Proof. Let q_α is a central projector in M, such that $q_\alpha M \cong L^\infty(\Omega_\alpha, B(H_\alpha))$. Then $q_\alpha S_0(M, \tau) \cong S_0(L^\infty(\Omega_\alpha, B(H_\alpha)), \tau_\alpha)$ for all α. Since Φ is Z-linearly then $\Phi(q_\alpha x) = q_\alpha \Phi(x)$ for all α. Hence Φ maps any algebra $q_\alpha S_0(M, \tau)$ into itself. By Proposition 3.2 there exists unitary elements $u_\alpha \in q_\alpha M$ such that

$$\Phi(q_\alpha x_\alpha) = u_\alpha x_\alpha u_\alpha$$

for all $x_\alpha \in q_\alpha S_0(M, \tau)$.

Put $u = (u_\alpha)$. Then u is an unitary element in M. For $x \in S^0(M, \tau)$ we have $q_\alpha \Phi(x) = \Phi(q_\alpha x) = u_\alpha q_\alpha x u_\alpha^* = q_\alpha u x q_\alpha u^* = q_\alpha(uxu^*)$, i.e. $q_\alpha \Phi(x) = q_\alpha(uxu^*)$ in all α. Therefore $\Phi(x) = uxu^*$. The proof is complete. □

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