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INVERSE PROBLEM FOR STIELTJES STRING DAMPED AT ONE END

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Abstract. Small transversal vibrations of the Stieltjes string, i.e., an elastic thread bearing point masses is considered for the case of one end being fixed and the other end moving with viscous friction in the direction orthogonal to the equilibrium position of the string. The inverse problem of recovering the masses, the lengths of subintervals and the coefficient of damping by the spectrum of vibrations of such a string and its total length is solved.

1. Introduction

The equation

$$\frac{\partial^2 u}{\partial M(s)\partial s} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{1.1}$$

describes small transversal vibrations of a stretched inhomogeneous string. Here t stands for the time, s for the longitudinal coordinate, $u(s,t)$ is the transverse displacement, $M(s)$ is a nonnegative nondecreasing function on $[0,l]$ describing the mass distribution. Substituting $u(s,t) = y(\lambda,s)e^{i\lambda t}$ into (1.1) we obtain the following equation for the amplitude function $y(\lambda,s)$:

$$\frac{dy'}{dM(s)} + \lambda^2 y = 0. \tag{1.2}$$

The operation $\frac{d}{dM(s)}$ has the following meaning, see [1]. If $M(s)$ is an absolutely continuous function and $M'(s) > 0$ almost everywhere on $[0,l]$ then the operation acts on absolutely continuous functions which have absolutely continuous first order derivatives, and the operation is given by the equation

$$\frac{dy'}{dM(s)} \overset{\text{a.e.}}{=} \frac{y''}{M'(s)}. \tag{1.3}$$

Here a.e. means almost everywhere. In the general case the operation is defined only on the so-called prolonged functions $u[s]$ which are obtained from usual functions $u(s)$ $(0 < s < l)$ by attaching two arbitrary numbers $u'_-(0)$ and $u'_+(l)$ which are called the left derivative at $s = 0$ and the right derivative at $s = l$, respectively. Then $u[s] = \{u(s), u'_-(0), u'_+(l)\}$. Under the domain of the operation we mean the set D_M of all complex-valued functions $u[s]$ of the form

$$u(s) = a + bs - \int_0^s (s-p)g(p)dM(p) \quad (0 < s < l), \tag{1.4}$$

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$$u'_-(0) = b, \quad u'_+(l) = b - \int_0^l g(p)dM(p), \tag{1.5}$$

where a and b are complex numbers, $g(p)$ is a complex-valued function M-summable on $[0,l]$. The operation is defined by the equality

$$-\frac{du'}{dM(s)} = g(s).$$

We assume $M(s+0) = M(s)$ for all $s \in [0,l)$ by definition. If $M(s_0 - 0) < M(s_0)$ then there is a point (concentrated) mass $(M(s_0) - M(s_0 - 0))$ at $s = s_0 \in (0,l]$. If $M(0) > 0$ then there is a point mass $M(0)$ at $s = 0$. The horizontal intervals ($M(s) = const$) correspond to massless intervals (where the string appears to be a thread). In [1] the notion of a regular string was introduced: if $l < \infty$ and $M(l) < \infty$, then the string is said to be regular. A more general class of strings was introduced in [2], [3]. The right end l of the string is said to be regular if $l < \infty$ and in some neighborhood $(l-\epsilon, l)$ the mass $\int_{l-\epsilon}^l dM(s)$ is finite. A (possibly half-infinite) string with the right end regular and with finite momentum $\int_{l_1}^l (l-s)dM(s)$ is said to be an S-string. Here $l_1 \geq -\infty$. It is easy to see that such a string has finite mass $\int_{l_1}^l dM(s)$ but the length of it can be infinite. General properties of such strings were investigated in [4].

Let us suppose that the left end of the string described by equation (1.1) is fixed. Then we have the boundary condition at the left

$$u(0,t) = 0.$$

The right end is supposed to be able to move with viscous friction in the direction orthogonal to the equilibrium position of the string. Then we have

$$\left.\frac{\partial u(s,t)}{\partial s}\right|_{s=l+0} + \nu \left.\frac{\partial u(s,t)}{\partial t}\right|_{s=l+0} = 0.$$

Here $\nu > 0$ is the coefficient of damping. Substituting $u(s,t) = e^{i\lambda t}y(\lambda,s)$ into these boundary conditions we obtain taking into account (1.2) that

$$\frac{dy'}{dM(s)} + \lambda^2 y = 0, \tag{1.6}$$

$$y'_+(\lambda,l) + i\lambda\nu y(\lambda,l) = 0, \tag{1.7}$$

$$y(\lambda,0) = 0. \tag{1.8}$$

As usual, by eigenfunction we mean a function $u[s]$ defined by (1.4), (1.5) which is not equal to 0 almost everywhere and satisfies (1.6)–(1.8). The corresponding value of λ is said to be the corresponding eigenvalue. Such a problem but with the left end free was considered in [5], [2], [3]. In [5] necessary and sufficient conditions were given for a function to be a characteristic function, i.e., the function the set of zeros of which coincides with the spectrum of the problem, for a regular string with the left end free and the right one damped. In [2], [3] for a class of S-strings, necessary and sufficient conditions were given in an explicit form for a sequence of complex numbers to be spectrum of the problem. However, the method of recovering the density of the string was given in [5] only for a certain subclass of regular strings.

If the density of a string satisfies the conditions $\rho(s) \overset{\text{def}}{=} M'(s) \geq \epsilon > 0$ and $\rho(s) \in W_2^2(0,l)$, then we apply the Liouville transformation [6, p. 292],

$$x(s) = \int_0^s \rho^{\frac{1}{2}}(s')ds', \tag{1.9}$$

$$v(\lambda,x) = \rho^{\frac{1}{4}}(s(x))y(\lambda,s(x)) \tag{1.10}$$

to reduce the problem (1.6)–(1.8) to the following one:

$$v'' + \lambda^2 v - q(x)v = 0, \tag{1.11}$$

$$v(\lambda, 0) = 0, \tag{1.12}$$

$$v'(\lambda, a) + (i\alpha\lambda + \beta)\, y(\lambda, a) = 0, \tag{1.13}$$

where $'$ means derivative with respect to the new variable x and

$$a = \int_0^l \rho^{\frac{1}{2}}(s)ds > 0, \tag{1.14}$$

$$\rho[x] = \rho(s(x)), \tag{1.15}$$

$$q(x) = \rho^{-\frac{1}{4}}[x]\frac{d^2}{dx^2}\rho^{\frac{1}{4}}[x], \tag{1.16}$$

$$\beta = -\frac{1}{4}\rho^{-1}[a]\frac{d\rho[x]}{dx}\bigg|_{x=a-0}, \tag{1.17}$$

$$\alpha = \rho^{-\frac{1}{2}}[a]\mu > 0. \tag{1.18}$$

From (1.16) it follows that $q(x) \in L_2(0, a)$. Since the spectrum of problem (1.6)–(1.8) is invariant under transformation (1.9)–(1.10), one can investigate the spectrum of problem (1.11)–(1.13). This problem has appeared in many papers [7], [8], [9], [10] and others. The case of a string bearing a point mass at the right end was considered in [11] and the case of a string with a massless interval at the right end was considered in [12]. By inverse problem we mean recovering the string parameters using the spectrum of its vibrations and some additional information. In [9], [10] the inverse problem was solved for smooth strings described by (1.11)–(1.13).

The opposite case of extremely nonsmooth so-called Stieltjes string, i.e., a thread bearing point masses, was considered in [13] and [14] assuming absence of damping. The inverse problem for the Stieltjes string with the left end free and the right end damped is a particular case of the inverse problem solved in [5]. It should be mentioned that the case of Stieltjes strings of finite number of masses with point-wise (i.e. one dimensional) damping at the right end can be reduced also to the problem of damped oscillators considered in [15], [16]. Another approach to inverse problem for a damped finite dimensional system was developed in [17] where the given data included not only eigenvalues but also the so-called Jordan pairs.

In the present paper we consider the inverse problem for the Stieltjes string with the left end fixed and the right end damped. Throughout the paper we assume the total length of the string as well as the number of the point masses to be finite. In Section 2 we describe the spectrum of a damped Stieltjes string in terms of continued fractions using the method of [13] based on the results of [18]. In Section 3 we give a solution of the corresponding inverse problem. By inverse problem here we mean recovering the values of point masses, the lengths of the subintervals between them and the coefficient of damping by the spectrum of vibrations and the total length of the string.

2. Direct problem for a damped Stieltjes string

Like in [13] we suppose the string to be a thread (i.e. a string of zero density) bearing a finite number of point masses. Let l_k $(k = 0, 1, \ldots, n)$ be the lengths of the intervals of zero density and let m_k $(k = 1, 2, \ldots, n+1)$ be values of the masses separating the intervals (l_k lies between m_k and m_{k+1}, the last mass has only one thread at the left). We assume $m_k > 0$ for $k = 1, 2, \ldots, n$ and $m_{n+1} \geq 0$. Let us denote by $\alpha > 0$ the coefficient of damping (viscous friction) of the point mass m_{n+1}. Denote by $v_k(t)$ the transversal displacements of the point masses at the time t. We assume the thread to be stretched by the force equal to 1. Taking into account that on the intervals of zero density the general solution of (1.1) is a linear function of s multiplied by a function of t we obtain the following recurrences:

$$\frac{v_k(t)-v_{k+1}(t)}{l_k}+\frac{v_k(t)-v_{k-1}(t)}{l_{k-1}}+m_k v_k''(t)=0 \quad (k=1,2,\ldots,n). \tag{2.1}$$

We impose the Dirichlet boundary condition at the left:

$$v_0(t)=0, \tag{2.2}$$

which means that the left end is fixed. We assume that the right end bearing the mass m_{n+1} can move with damping (viscous friction) in the direction orthogonal to the equilibrium position of the string:

$$\frac{v_{n+1}(t)-v_n(t)}{l_n}+m_{n+1}v_{n+1}''(t)+\alpha v_{n+1}'=0, \tag{2.3}$$

where α is a positive constant proportional to the coefficient of damping.

Substituting $v_k(t)=u_k e^{i\lambda t}$ into (2.1)–(2.3) we obtain

$$\frac{u_k-u_{k+1}}{l_k}+\frac{u_k-u_{k-1}}{l_{k-1}}-m_k\lambda^2 u_k=0 \quad (k=1,2,\ldots,n), \tag{2.4}$$

$$u_0=0, \tag{2.5}$$

$$\frac{u_{n+1}-u_n}{l_n}+(-m_{n+1}\lambda^2+i\lambda\alpha)u_{n+1}=0. \tag{2.6}$$

The relation between u_k and u_1 involves only even powers of λ, therefore we can write following [13] that

$$u_k=R_{2k-2}(\lambda^2)u_1 \quad (k=1,2,\ldots,n),$$

where $R_{2k-2}(\lambda^2)$ is a polynomial of degree $2k-2$ obtained by (2.4). Also we define the corresponding polynomials of odd index

$$R_{2k-1}(\lambda^2)=\frac{R_{2k}(\lambda^2)-R_{2k-2}(\lambda^2)}{l_k}.$$

Due to (2.4) the polynomials R_k satisfy the recurrence relations

$$R_{2k-1}(\lambda^2)=-\lambda^2 m_k R_{2k-2}(\lambda^2)+R_{2k-3}(\lambda^2),$$
$$R_{2k}(\lambda^2)=l_k R_{2k-1}(\lambda^2)+R_{2k-2}(\lambda^2) \quad (k=1,2,\ldots,n),$$
$$R_{-1}(\lambda^2)=\frac{1}{l_0}, \quad R_0(\lambda^2)=1.$$

Using boundary condition (2.6) we obtain

$$R_{2n-1}(\lambda^2)+(-m_{n+1}\lambda^2+i\lambda\alpha)R_{2n}(\lambda^2)=0. \tag{2.7}$$

It is shown in [13] that the polynomials $R_{2n}(\lambda^2)$ and $R_{2n-1}(\lambda^2)$ have no common zeros and the ratio $\frac{R_{2n}(\lambda^2)}{R_{2n-1}(\lambda^2)}$ can be expanded into a continued fraction,

$$\frac{R_{2n}(\lambda^2)}{R_{2n-1}(\lambda^2)}=l_n+\cfrac{1}{-m_n\lambda^2+\cfrac{1}{l_{n-1}+\cfrac{1}{-m_{n-1}\lambda^2+\ldots+\cfrac{1}{l_1+\cfrac{1}{-m_1\lambda^2+\frac{1}{l_0}}}}}}. \tag{2.8}$$

The eigenvalues of problem (2.4)–(2.6) are nothing else but the zeros of the polynomial

$$\phi(\lambda)=R_{2n-1}(\lambda^2)+(-m_{n+1}\lambda^2+i\lambda\alpha)R_{2n}(\lambda^2). \tag{2.9}$$

Let us introduce the following even polynomials

$$P(\lambda^2)=\frac{\phi(\lambda)+\phi(-\lambda)}{2}=R_{2n-1}(\lambda^2)-m_{n+1}\lambda^2R_{2n}(\lambda^2), \tag{2.10}$$

$$Q(\lambda^2)=\frac{\phi(\lambda)-\phi(-\lambda)}{2i\lambda}=\alpha R_{2n}(\lambda^2). \tag{2.11}$$

We can consider these polynomials as functions of $z=\lambda^2$.

Definition 2.1. The function $\omega(\lambda)$ is said to be a Nevanlinna function (or an R-function in terms of [1]) if the following is verified:
1) it is analytic in the half-planes $\mathrm{Im}\lambda>0$ and $\mathrm{Im}\lambda<0$;
2) $\omega(\overline{\lambda})=\overline{\omega(\lambda)}$ ($\mathrm{Im}\lambda\neq 0$);
3) $\mathrm{Im}\lambda\ \mathrm{Im}\ \omega(\lambda)\geq 0$ for $\mathrm{Im}\lambda\neq 0$.

Definition 2.2. (see [1]). The Nevanlinna function $\omega(\lambda)$ is said to be an S-function if $\omega(\lambda)\geq 0$ for $\lambda<0$.

Lemma 2.1. *The function $\frac{Q(z)}{P(z)}$ is an S-function.*

Proof. It is clear that

$$\frac{Q(z)}{P(z)}=\alpha\Big(-m_{n+1}z+\Big(\frac{R_{2n}(z)}{R_{2n-1}(z)}\Big)^{-1}\Big)^{-1}. \tag{2.12}$$

It is known, see [13, p. 334], that $R_{2n}(z)$ and $R_{2n-1}(z)$ are of the form

$$R_{2n}(z)=d_1\prod_{k=1}^{n}(-z+\nu_k^2),\quad R_{2n-1}(z)=d_2\prod_{k=1}^{n}(-z+\mu_k^2)$$

where

$$0<\mu_1^2<\nu_1^2<\mu_2^2<\nu_2^2<\cdots<\mu_n^2<\nu_n^2$$

and $d_1>0$, $d_2>0$. That means [21] that $\frac{R_{2n}(z)}{R_{2n-1}(z)}$ is an S-function and, consequently,

$$\mathrm{Im}\Big(-\mathrm{m_{n+1}z}+\Big(\frac{\mathrm{R_{2n}(z)}}{\mathrm{R_{2n-1}(z)}}\Big)^{-1}\Big)<0\quad\text{for}\quad \mathrm{Im\ z}>0$$

and

$$\mathrm{Im}\Big(-\mathrm{m_{n+1}z}+\Big(\frac{\mathrm{R_{2n}(z)}}{\mathrm{R_{2n-1}(z)}}\Big)^{-1}\Big)^{-1}>0\quad\text{for}\quad \mathrm{Im\ z}>0.$$

It is also clear that

$$-m_{n+1}z+\Big(\frac{R_{2n}(z)}{R_{2n-1}(z)}\Big)^{-1}>0\quad\text{for}\quad \mathrm{z}<0.$$

The assertion of the lemma follows. □

Corollary 2.1. $\lambda\frac{Q(\lambda^2)}{P(\lambda^2)}$ *is a Nevanlinna function.*

Proof. According to Lemma 5.1 in [1] the assertion of the corollary is due to Lemma 2.1. □

Definition 2.3. A polynomial is said to be real if it takes real values on the real axis.

Definition 2.4. A polynomial is said to be Hermite-Biehler (HB) if all its zeros lie in the open upper half-plane.

It should be mentioned that the transformation $\lambda\to i\lambda$ transfers a HB polynomial into a so-called Hurwitz polynomial [19].

Theorem 2.1. (Hermite-Biehler theorem, see [20], [21]). *In order that the polynomial*

$$\omega(\lambda)=P(\lambda)+iQ(\lambda)$$

where $P(\lambda)$ and $Q(\lambda)$ are real polynomials, not have any zeros in the closed lower half-plane $\mathrm{Im}\,\lambda \le 0$, *i.e be HB, it is necessary and sufficient that the following conditions be satisfied*:

1) *the polynomials $P(\lambda)$ and $Q(\lambda)$ have only simple real zeros, while these zeros separate one another, i.e., between two successive zeros of one of these polynomials there lies exactly one zero of the other one*;

2) *at some point λ_0 of the real axis,*

$$Q'(\lambda_0)P(\lambda_0) - Q(\lambda_0)P'(\lambda_0) > 0. \tag{2.13}$$

The fact that the two polynomials satisfy condition (1) will be expressed by saying that "the zeros of the polynomials $P(\lambda)$ and $Q(\lambda)$ are interlaced".

Now Lemma 2.1 and Theorem 2.1 imply the following result.

Corollary 2.2. *The polynomial $P(\lambda^2) + i\lambda Q(\lambda^2)$ belongs to the Hermite-Biehler class.*

Definition 2.5. The polynomial $\omega(\lambda)$ is said to be symmetric if $\omega(-\overline{\lambda}) = \overline{\omega(\lambda)}$ for all $\lambda \in C$. The polynomial $\omega(\lambda)$ is said to belong to the SHB class if it is symmetric and belongs to the Hermite-Biehler class.

Corollary 2.3. *The polynomial $P(\lambda^2) + i\lambda Q(\lambda^2)$ belongs to the SHB class.*

Corollary 2.4. *If $m_{n+1} > 0$, then the eigenvalues of problem* (2.4)–(2.6) *satisfy the following conditions:*

1) $\mathrm{Im}\lambda_k > 0$ *for* $k = \pm 1, \pm 2, \dots, \pm(n+1)$;

2) $\lambda_{-k} = -\overline{\lambda_k}$ *for not pure imaginary λ_{-k} and the multiplicities of symmetrically located eigenvalues are equal.*

If $m_{n+1} = 0$, then the eigenvalues of problem (2.4)–(2.6) *satisfy the conditions:*

1) $\mathrm{Im}\lambda_k > 0$ *for* $k = 0, \pm 1, \pm 2, \dots, \pm n$;

2) $\lambda_{-k} = -\overline{\lambda_k}$ *for not pure imaginary λ_{-k} and the multiplicities of symmetrically located eigenvalues are equal.*

3. Inverse problem for a Stieltjes string damped at the right end

By inverse problem we mean recovering the parameters of the problems of small vibrations of a Stieltjes string with the left end fixed and the right end damped, i.e. problem (2.4)–(2.6). The parameters to be recovered are $\{m_k\}$, $(k = 1, 2, \dots, n+1)$, $\{l_k\}$ $(k = 0, 1, \dots, n)$ and α. As the given data we use the spectrum $\{\lambda_k\}$ ($k = \pm 1, \pm 2, \dots, \pm(n+1)$ if $m_{n+1} \neq 0$ and $k = 0, \pm 1, \pm 2, \dots, \pm n$ if $m_{n+1} = 0$) of problem (2.4)–(2.6) and the total length of the string $l = l_0 + l_1 + \cdots + l_n$.

Theorem 3.1. *Let $l > 0$ be given together with the set of complex numbers $\{\lambda_k\}$ ($k = \pm 1, \pm 2, \dots, \pm(n+1)$) which satisfy the conditions*

1) $\mathrm{Im}\lambda_k > 0$ for $k = \pm 1, \pm 2, \dots, \pm(n+1)$;

2) $\lambda_{-k} = -\overline{\lambda_k}$ *for not pure imaginary λ_{-k} and the multiplicities of symmetrically located numbers are equal.*

Then there exists a unique Stieltjes string, i.e., a unique set of intervals $l_k > 0$ ($k = 0, 1, \dots, n$) of the total length $\sum_{k=0}^{n} l_k = l$, a unique set of masses $m_k > 0$ ($k = 1, 2, \dots, n+1$) and a unique positive number α which generate problem (2.4)–(2.6) *with the spectrum coinciding with the set* $\{\lambda_k\}$.

Proof. Let us construct the polynomial

$$\Phi(\lambda) = \prod_{-n-1,\ k\neq 0}^{n+1} \left(1 - \frac{\lambda}{\lambda_k}\right). \tag{3.1}$$

Due to the symmetry of the zeros of this polynomial the following even polynomials are real,

$$P(\lambda^2) = \frac{\Phi(\lambda) + \Phi(-\lambda)}{2} \tag{3.2}$$

and

$$Q(\lambda^2) = \frac{\Phi(\lambda) - \Phi(-\lambda)}{2i\lambda}. \tag{3.3}$$

Set

$$\alpha = \frac{Q(0)}{l}. \tag{3.4}$$

Using (3.3) and (3.1) we obtain

$$\alpha = \frac{i}{l} \sum_{k=-n-1, k\neq 0}^{n+1} \frac{1}{\lambda_k} \tag{3.5}$$

and because of the symmetry in location of the zeros of the polynomial $\Phi(\lambda)$ and conditions 1) and 2) we conclude that $\alpha > 0$. Set

$$m_{n+1} = -\alpha \lim_{|\lambda|\to\infty} \frac{P(\lambda^2)}{\lambda^2 Q(\lambda^2)}. \tag{3.6}$$

The limit in the right-hand side of (3.6) exists because the degree of $P(\lambda^2)$ is $2n+2$ and the degree of $Q(\lambda^2)$ is $2n$. Moreover,

$$\frac{P(\lambda^2)}{\lambda^2 Q(\lambda^2)} \underset{|\lambda|\to\infty}{=} \left(i \sum_{k=-n-1, k\neq 0}^{n+1} \lambda_k \right)^{-1} + o(1).$$

Conditions 1) and 2) imply

$$i \sum_{k=-n-1, k\neq 0}^{n+1} \lambda_k < 0,$$

and according to (3.6) $m_{n+1} > 0$.

Since $P(\lambda^2) + i\lambda Q(\lambda^2) = \Phi(\lambda)$ is SHB, $P(\lambda^2) + i\lambda\alpha^{-1}Q(\lambda^2)$ is also an SHB polynomial. We consider the polynomial $\phi(\lambda, m_{n+1}) := P(\lambda^2) + m_{n+1}\alpha^{-1}\lambda^2 Q(\lambda^2) + i\lambda\alpha^{-1}Q(\lambda^2)$ as a perturbation of $P(\lambda^2) + i\lambda\alpha^{-1}Q(\lambda^2)$. Since $\phi(\lambda, \eta)$ is a polynomial with respect to the variables λ and η, the zeros of it in λ-plane are piece-wise analytic and continuous in η [22]. The zeros do not cross the real axis when η changes from 0 to m_{n+1}. Otherwise, we would have $P(\lambda^2) = \lambda Q(\lambda^2) = 0$ for some $\eta > 0$ and some real λ and, therefore, $\Phi(\lambda) = 0$ for this real λ, which contradicts condition 1). Therefore, $\phi(\lambda, m_{n+1}) \in SHB$ for each $m_{n+1} > 0$. This implies [21, p. 308] that

$$\frac{\alpha^{-1}\lambda Q(\lambda^2)}{P(\lambda^2) + m_{n+1}\lambda^2\alpha^{-1}Q(\lambda^2)}$$

is a Nevanlinna function. That means that

$$0 < \tau_1^2 < \nu_1^2 < \tau_2^2 < \cdots < \nu_n^2,$$

where ν_k are the zeros of $Q(\lambda^2)$ and τ_k are the zeros of $P(\lambda^2) + m_{n+1}\lambda^2\alpha^{-1}Q(\lambda^2)$. Therefore,

$$\frac{\alpha^{-1}Q(\lambda)}{P(\lambda) + m_{n+1}\lambda\alpha^{-1}Q(\lambda)}. \tag{3.7}$$

is a Nevanlinna function also. Now Lemma 5.1 in [1] implies that

$$\frac{\alpha^{-1}Q(\lambda)}{P(\lambda) + m_{n+1}\lambda\alpha^{-1}Q(\lambda)} \tag{3.8}$$

in an S-function. Then according to [13]

$$\frac{\alpha^{-1}Q(\lambda)}{P(\lambda)+m_{n+1}\lambda\alpha^{-1}Q(\lambda)} = a_n + \cfrac{1}{-b_n\lambda + \cfrac{1}{a_{n-1}+\cfrac{1}{-b_{n-1}\lambda+\ldots+\cfrac{1}{a_1+\cfrac{1}{-b_1\lambda+\cfrac{1}{a_0}}}}}} \tag{3.9}$$

with $a_k>0$ and $b_k>0$ for each k.

We identify a_k as the length of k-th interval and b_k as the k-th mass of a Stieltjes string, i.e.,

$$\frac{\alpha^{-1}Q(\lambda^2)}{P(\lambda^2)+m_{n+1}\lambda^2\alpha^{-1}Q(\lambda^2)} = \frac{R_{2n}(\lambda^2)}{R_{2n-1}(\lambda^2)}, \tag{3.10}$$

where $R_{2n}(\lambda^2)$ and $R_{2n-1}(\lambda^2)$ are the corresponding polynomials for this Stieltjes string. Consequently,

$$\alpha^{-1}Q(\lambda^2) = TR_{2n}(\lambda^2),$$
$$P(\lambda^2)+m_{n+1}\lambda^2\alpha^{-1}Q(\lambda^2) = TR_{2n-1}(\lambda^2),$$

where T is a positive constant. Therefore,

$$\Phi(\lambda) = P(\lambda^2)+i\lambda Q(\lambda^2) = T(R_{2n-1}(\lambda^2)+(-m_{n+1}\lambda^2+i\lambda\alpha)R_{2n}(\lambda^2)).$$

This means according to (2.7) that the set $\{\lambda_k\}$ is the spectrum of the problem (2.4)–(2.6) with the masses b_k $(k=1,2,\ldots,n)$ and m_{n+1} and the lengths a_k $(k=0,1,\ldots,n)$ damped at the right with the coefficient of damping α. The length of this string according to [13] is equal to $\frac{R_{2n}(0)}{R_{2n-1}(0)}$. From (3.10) we obtain

$$\frac{R_{2n}(0)}{R_{2n-1}(0)} = \frac{Q(0)}{\alpha P(0)} = \frac{Q(0)}{\alpha}. \tag{3.11}$$

Due to (3.4), the right-hand side of (3.11) is equal to l.

To prove uniqueness suppose there exists another Stieltjes string of the same length l and with the masses $\{\tilde{m}_k\}_{k=1}^{n+1}$, the intervals $\{\tilde{l}_k\}_{k=0}^{n}$ $(\sum\limits_{k=0}^{n}\tilde{l}_k = l)$, with the coefficient $\tilde{\alpha}>0$ and with the same spectrum $\{\lambda_k\}_{k=-(n+1),k\neq 0}^{k=n+1}$. Then solving the corresponding direct problem we obtain the following analogue of (2.8):

$$\frac{\tilde{R}_{2n}(\lambda^2)}{\tilde{R}_{2n-1}(\lambda^2)} = \tilde{l}_n + \cfrac{1}{-\tilde{m}_n\lambda^2 + \cfrac{1}{\tilde{l}_{n-1}+\cfrac{1}{-\tilde{m}_{n-1}\lambda^2+\ldots+\cfrac{1}{\tilde{l}_1+\cfrac{1}{-\tilde{m}_1\lambda^2+\cfrac{1}{\tilde{l}_0}}}}}}. \tag{3.12}$$

Here $\tilde{R}_{2n}(\lambda^2)$ and $\tilde{R}_{2n-1}(\lambda^2)$ are the polynomials described in Section 2 for the problem (2.4)–(2.6) generated by the sets $\{\tilde{m}_k\}_{k=1}^{n+1}$, $\{\tilde{l}_k\}_{k=0}^{n}$ and the coefficient $\tilde{\alpha}$. The spectrum of problem (2.4)–(2.6) generated by $\{\tilde{m}_k\}_{k=1}^{n+1}$, $\{\tilde{l}_k\}_{k=0}^{n}$, $\tilde{\alpha}$ is the set of zeros of the polynomial

$$\tilde{\phi}(\lambda) = \tilde{R}_{2n-1}(\lambda^2)+(-\tilde{m}_{n+1}\lambda^2+i\lambda\tilde{\alpha})\tilde{R}_{2n}(\lambda^2). \tag{3.13}$$

The set of zeros of this polynomial coincides with $\{\lambda_k\}_{k=-(n+1),k\neq 0}^{k=n+1}$ and, therefore,

$$\tilde{\phi}(\lambda) = C\phi(\lambda). \tag{3.14}$$

The constant C is positive because $\tilde{\phi}(0)=\tilde{R}_{2n-1}(0)>0$ and $\phi(0)=R_{2n-1}(0)>0$. From (3.14) we obtain

$$\tilde{\alpha}\tilde{R}_{2n}(\lambda^2) = C\alpha R_{2n}(\lambda^2) \tag{3.15}$$

and

$$\tilde{R}_{2n-1}(\lambda^2)-\tilde{m}_{n+1}\lambda^2\tilde{R}_{2n}(\lambda^2) = CR_{2n-1}(\lambda^2)-Cm_{n+1}\lambda^2R_{2n}(\lambda^2). \tag{3.16}$$

Equations (3.15) and (3.16) imply

$$\frac{\tilde{R}_{2n-1}(\lambda^2)}{\tilde{\alpha}\tilde{R}_{2n}(\lambda^2)} - \frac{\tilde{m}_{n+1}\lambda^2}{\tilde{\alpha}} = \frac{R_{2n-1}(\lambda^2)}{\alpha R_{2n}(\lambda^2)} - \frac{m_{n+1}\lambda^2}{\alpha}. \tag{3.17}$$

Since the polynomials $\tilde{R}_{2n}(\lambda^2)$, $\tilde{R}_{2n-1}(\lambda^2)$, $R_{2n}(\lambda^2)$ and $R_{2n-1}(\lambda^2)$ are of the same degree, (3.17) implies

$$\frac{\tilde{m}_{n+1}}{\tilde{\alpha}} = \frac{m_{n+1}}{\alpha} \tag{3.18}$$

and

$$\frac{\tilde{R}_{2n-1}(\lambda^2)}{\tilde{\alpha}\tilde{R}_{2n}(\lambda^2)} = \frac{R_{2n-1}(\lambda^2)}{\alpha R_{2n}(\lambda^2)}. \tag{3.19}$$

By conditions the total length of the string is l, therefore,

$$\frac{\tilde{R}_{2n-1}(0)}{\tilde{R}_{2n}(0)} = \frac{R_{2n-1}(0)}{R_{2n}(0)} = \frac{1}{l}. \tag{3.20}$$

Using (3.20) we obtain from (3.19) that $\tilde{\alpha} = \alpha$. Then (3.18) implies $\tilde{m}_{n+1} = m_{n+1}$ and (3.19) implies

$$\frac{\tilde{R}_{2n-1}(\lambda^2)}{\tilde{R}_{2n}(\lambda^2)} = \frac{R_{2n-1}(\lambda^2)}{R_{2n}(\lambda^2)}. \tag{3.21}$$

Since the left-hand sides of (2.8) and of (3.12) are the same, we conclude that the right-hand sides coincide also. Theorem 3.1 is proved. □

In case of $m_{n+1} = 0$ using similar arguments we obtain the following analogue of Corollary 2.4 and Theorem 3.1.

Theorem 3.2. *In order for the set* $\{\lambda_k\}$ $(k = 0, \pm 1, \pm 2, \dots, \pm n)$ *to be the spectrum of problem* (2.4)–(2.6) *with* $m_{n+1} = 0$ *and given total length* $l > 0$ *it is necessary and sufficient that the following conditions be satisfied:*

1) $\mathrm{Im}\lambda_k > 0$ *for* $k = 0, \pm 1, \pm 2, \dots, \pm n$;

2) $\lambda_{-k} = -\overline{\lambda_k}$ *for not pure imaginary* λ_{-k} *and the multiplicities of symmetrically located elements of the set are equal.*

If these conditions are satisfied then there exists an unique set $\{m_k\}_1^n$, *a unique set* $\{l_k\}_0^n$ *with* $\sum_{k=0}^{n} l_k = l$ *and unique* $\alpha > 0$ *which generate problem* (2.4)–(2.6) *on the interval* l *with* $m_{n+1} = 0$ *and with the spectrum* $\{\lambda_k\}$.

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References

1. I. S. Kac, M. G. Krein, *On spectral functions of a string*, in F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Mir, Moscow, 1968, pp. 648–737 (Supplement II added to the Russian translation). (Russian); I. S. Kac, M. G. Krein, *On the Spectral Function of the String*, Amer. Math. Soc. Transl., Ser. 2, Vol. 103, 1974, 19–102.
2. M. G. Krein, A. A. Nudelman, *On some spectral properties of an inhomogeneous string with dissipative boundary condition*, J. Operator Theory **22** (1989), 369–395. (Russian)
3. M. G. Krein, A. A. Nudelman, *On direct and inverse problems for frequencies of boundary dissipation of inhomogeneous string*, Dokl. Akad. Nauk SSSR **247** (1979), no. 5, 1046–1049. (Russian)
4. H. Dym, H. P. McKean, *Gaussian Processes, Function Theory and the Inverse Spectral Problem*, Academic Press, New York—San Francisco—London, 1976.

5. D. Z. Arov, *Realization of a canonical system with a dissipative boundary condition at one end of the segment in terms of the coefficient of dynamical compliance*, Sib. Mat. Zh. **16** (1975), no. 3, 440–463. (Russian)
6. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience Publishers, New York—London, 1953.
7. C. Cox, E. Zuazua, *The rate at which energy decays in a string damped at one end*, Indiana Univ. Math. J. **44** (1995), no. 2, 545–573.
8. M. A. Shubov, *Asymptotics of resonances and geometry of resonance states in the problem of scattering of acoustic waves by a spherically symmetric inhomogeneity of the density*, Differential and Integral Equations **8** (1995), no. 5, 1073–1115.
9. G. Gubreev, V. Pivovarchik, *Spectral analysis of T. Regge problem with parameter*, Funktsional. Anal. i Prilozhen. **31** (1997), no. 1, 70–74. (Russian)
10. V. Pivovarchik, C. van der Mee, *The inverse generalized Regge problem*, Inverse Problems **17** (2001), no. 6, 1831–1845.
11. V. Pivovarchik, *Inverse problem for a smooth string damped with damping at one end*, J. Operator Theory **38** (1997), no. 2, 243–263.
12. R. Mennicken, V. Pivovarchik, *An inverse problem for an inhomogeneous string with an interval of zero density*, Math. Nachr. **259** (2003), 51–65.
13. F. R. Gantmakher, M. G. Krein, *Oscillating Matrices and Kernels and Small Vibrations of Mechanical Systems*, GITTL, Moscow—Leningrad, 1950. (Russian); German transl.: Akademie Verlag, Berlin, 1960.
14. M. G. Krein, *Some new problems in the theory of oscillations of Sturm systems*, Prikladnaya matematika i mekhanika **16** (1952), no. 5, 555–568. (Russian)
15. K. Veselić, *On linear vibrational systems with one dimensional damping*, Applicable Analysis **29** (1988), 1–18.
16. K. Veselić, *On linear vibrational systems with one dimensional damping* II, Integral Equations and Operator Theory **13** (1990), 883–897.
17. P. Lancaster, J. Maroulas, *Inverse eigenvalue problems for damped vibrating systems*, J. Math. Anal. Appl. **123** (1987), 238–261.
18. T.-L. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse **8** (1894), 1–122; **9** (1895), 1–47.
19. F. R. Gantmaher, *Theory of Matrices*, Nauka, Moscow, 1988. (Russian)
20. C. Hermite, *Extract d'une lettre de M. Ch. Hermite de Paris a' Mr. Borchardt de Berlin sur le nombre des racines d'une e'quation alge'brique comprises entre des limites donne'es*, J. Reine Angew. Math. **52** (1856), 39–51; reprinted in his Oeuvres, Vol. 1, Gauthier-Villars, Paris, 1905, pp. 397–414.
21. B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Transl. Math. Monographs, Vol. 5, Amer. Math. Soc., Providence, R. I., 1980.
22. A. I. Markushevich, *Theory of Analytic Functions*, Vol. 1, Nauka, Moscow, 1968. (Russian)

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