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DEVELOPMENT OF THE MARKOV MOMENT PROBLEM APPROACH IN THE OPTIMAL CONTROL THEORY

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Dedicated to 100 anniversary of Mark Krein.

Abstract. The paper is a survey of the main ideas and results on using of the Markov moment problem method in the optimal control theory. It contains a version of the presentation of the Markov moment approach to the time-optimal control theory, linear and nonlinear.

1. Introduction

The idea of using of the moment problem in the optimal control theory was proposed by N. N. Krasovskii [1, 2]. He initiated the application of the Krein moment L-problem [3, 4, 5] in the linear optimal control problems where the cost function was interpreted as a norm. This became a basis for various numerical methods [6]. Numerous examples of using of the moment approach in the optimal control problems for PDE systems are contained in [7].

In the case of geometric constraints on the control the Markov moment problem [8] can be applied [5, p. 372]. A new fundamental progress in this field was connected with the development of the Markov moment problem method [8] with a view to the analytical solving of the linear time-optimal control problem. This approach was proposed and worked out by V. I. Korobov and G. M. Sklyar in the 80-th–90-th of the last century [9, 10, 11, 12, 13, 14, 15, 16, 17]. The main idea is to interpret the linear time-optimal problem as the Markov moment problem on a nonconstant (namely, the minimal possible) interval. One of the most important results in this way [9, 10] is the analytic solution of the problem of Pontryagin *et al* [18] on the time-optimal control for the canonical system of an arbitrary dimension.

The new advance in the application of the Markov moment problem approach was its extension to nonlinear case carried out in the works of G. M. Sklyar and S. Yu. Ignatovich during the last decade [19, 20, 21, 22, 23, 24, 25, 26]. It turns out that the study of the nonlinear power Markov moment problem is connected in a natural way with properties of certain structures in free associative algebras. The basic results of this direction find their application in the homogeneous approximation problem of nonlinear control systems.

The present paper is a survey of the main ideas and results of the Markov moment problem method in the optimal control theory. The paper is split into two parts. The first part (Section 2) contains a version of presentation of the approach in the case of linear systems. The second part (Section 3) is devoted to the nonlinear case.

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2. Linear time optimality and Markov moment min-problem

Consider a linear control system of the form

$$\dot{x} = A(t)x + b(t)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \tag{1}$$

where $A(t)$ and $b(t)$ are $n \times n$ matrix and n-dimensional vector respectively, with continuous entries. Suppose a control $u = u(t)$ steers the system from some initial state $x(0) = x^0$ to the final state $x(T) = 0$. Denote by $\phi(t)$ the fundamental matrix of the equation $\dot{x} = A(t)x$ such that $\phi(0) = I$. Then, due to the Cauchy formula,

$$x(T) = 0 = \phi(T)x^0 + \phi(T)\int_0^T \phi^{-1}(t)b(t)u(t)\,dt.$$

Hence, the function $u(t)$ satisfies the following moment equalities

$$x_k^0 = \int_0^T g_k(t)u(t)\,dt, \quad k = 1,\dots,n, \tag{2}$$

where $g(t) = (g_1(t),\dots,g_n(t)) = -\phi^{-1}(t)b(t)$.

For the complete statement of the controllability problem, the description of admissible controls is required. If the set of admissible controls is a ball in some functional space then we get *the abstract Krein moment L-problem* [3, 4, 5]. Then the conditions of controllability are reduced to the conditions of solvability of the moment problem.

In particular, if in the control problem geometric constraints $|u(t)| \le 1$ are adopted then we get the Krein moment problem in the space $L_1[0,T]$, that is, the Markov moment $(-1,1)$-problem [8]

$$s_k = \int_0^T g_k(t)u(t)\,dt, \quad k = 1,\dots,n, \quad u(t) \in [-1,1]. \tag{3}$$

Thus, the controllability problem from the point x^0 to the origin on the time interval $[0,T]$ by use of the controls satisfying the constraint $|u(t)| \le 1$ for system (1) is reduced to the Markov moment problem (3) with $s = x^0$ and $g(t) = -\phi^{-1}(t)b(t)$. From the point of view of the functional analysis, this problem corresponds to the extension of the functional defined on the finite-dimensional subspace $\mathrm{Lin}\{g_1(t),\dots,g_n(t)\} \subset L_1[0,T]$ to the whole space $L_1[0,T]$ with preservation of its norm.

As it is well-known, the Markov moment problem has a unique solution $u(t) = u^0(t)$ if and only if T is the smallest positive number such that (3) holds. Moreover, if the functions $g_1(t),\dots,g_n(t)$ form a Tchebycheff system on the interval $(0,T)$ then this function $u^0(t)$ takes the values ± 1 and has no more than $n-1$ points of discontinuity. This means that the moment problem, in essence, is reduced to the finding of n independent parameters (the value of T and points of discontinuity of $u^0(t)$).

On the other hand, from the point of view of the control theory, the minimal possible T satisfying (3) is interpreted as *the minimal time* in which it is possible to steer system (1) from the initial state to the origin by use of the controls satisfying the condition $|u(t)| \le 1$. Hence, in this case T is the optimal time and the function $u^0(t)$ is the time-optimal control in the linear problem of time optimality.

This leads to the following statement of *the Markov moment problem on the smallest possible interval* (Markov moment min-problem) [12]: *for a given sequence of functions $\{g_k(t)\}_{k=1}^n$, $t \in [0,T]$, and a vector $s \in \mathbb{R}^n$, to find the smallest possible interval $[0,\theta_s] \subset [0,T]$ such that for $\theta = \theta_s$ the following representation holds*

$$s_k = \int_0^\theta g_k(t)u(t)\,dt, \quad k = 1,\dots,n, \quad |u(t)| \le 1, \tag{4}$$

and to find a function $u(t) = u_s(t)$ corresponding to this representation. The pair $(\theta_s, u_s(t))$ is called a solution of Markov moment min-problem (4). If the sequence

$\{g_k(t)\}_{k=1}^n$ form a Tchebycheff system then the solution is unique, the function $u_s(t)$ equals ± 1, and has no more than $n-1$ points of discontinuity. As it follows from the above, the solution of the moment min-problem solves the time-optimal control problem; the mentioned properties of the optimal control follow also from the Pontryagin maximum principle.

Thus, below we identify the time-optimal control problem for system (1) to the origin under the constraint $|u(t)| \le 1$ and the Markov moment min-problem (4) where $g(t) = -\phi^{-1}(t)b(t)$.

Let $M(T) \subset \mathbb{R}^n$ denote the solvability set for the Markov moment problem (3). Then θ_s solves the Markov moment min-problem (4) if and only if s belongs to the boundary of the solvability set $M(\theta_s)$. This leads to the problem of description of the solvability set $M(\theta)$ as a function of θ what is a typical problem in the optimal control theory.

2.1. **Power Markov moment problem.** Now, let us consider the particular case of the Markov moment min-problem (4) where $g_k(t) = t^{k-1}$, $k = 1, \dots, n$, i.e. the power Markov moment min-problem

$$s_k = \int_0^\theta t^{k-1} u(t)\, dt, \quad k = 1, \dots, n, \quad |u(t)| \le 1, \quad \theta \to \min . \tag{5}$$

This case corresponds to the time-optimal control problem for the "canonical" control system

$$\dot{x}_1 = u,\ \dot{x}_k = x_{k-1},\ k = 2, \dots, n, \quad x(0) = x^0,\ x(\theta) = 0, \quad |u(t)| \le 1,\ \theta \to \min, \tag{6}$$

where $x_k^0 = \frac{(-1)^k s_k}{(k-1)!}$, $k = 1, \dots, n$.

As it is well known, the power Markov moment problem can be interpreted from the point of view of the function theory as the problem of finding of a function from some class satisfying certain conditions on its first Laurent series coefficients [5]. We briefly explain this approach for the case of min-problem following Korobov and Sklyar [9, 14].

Suppose $(\theta_s, u_s(t))$ is the solution of (5). Then $u_s(t)$ takes the values ± 1 and has no more than $n-1$ points of discontinuity. For simplicity, suppose now that $u_s(t)$ has exactly $n-1$ points of discontinuity; denote them by $0 < t_1 < \cdots < t_{n-1} < \theta_s$. Then (5) gives

$$\sum_{j=1}^{n-1} (-1)^{n-j+1} t_j^k = c_k^{\pm}(\theta_s, s), \quad k = 1, \dots, n,$$

where $c_k^{\pm}(\theta, s) = \frac{1}{2}(\theta^k \mp k s_k)$ and the sign in the upper index of c_k corresponds to the sign of $u_s(t)$ on the last interval (that is the value $u_s(\theta_s - 0)$).

First consider the case $n = 2m+1$. Introduce the rational function

$$R(z) = \frac{\prod\limits_{j=1}^{m} (z - t_{2j})}{\prod\limits_{j=1}^{m} (z - t_{2j-1})} = 1 - \sum_{k=1}^{\infty} \frac{\gamma_k}{z^k} \tag{7}$$

which is analytic when $|z|$ is rather large. The rational function $R(z)$ has m roots and m poles on the interval $(0, \theta_s)$ which alternate. This implies the following property of the coefficients γ_k: if we denote by $\Gamma_{p,q}$ the determinant of the Hankel matrix

$$\Gamma_{p,q} = \det \begin{pmatrix} \gamma_p & \gamma_{p+1} & \cdots & \gamma_q \\ \gamma_{p+1} & \gamma_{p+2} & \cdots & \gamma_{q+1} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_q & \gamma_{q+1} & \cdots & \gamma_{2q-p} \end{pmatrix}$$

then $\Gamma_{1,m+1} = 0$ and, moreover, $\Gamma_{1,p} > 0$, $\Gamma_{2,p+1} > 0$ for $p = 1, \dots, m$.

This can be proved directly by manipulation with rational functions. However, one can also use the deep connection with the Hausdorff moment problem. In fact, since the roots and poles of $R(z)$ alternate then $R(z)$ can be represented as $R(z) = 1 - \int_0^{\theta_s} \frac{1}{z-t} d\mu(t)$, where $\mu(t)$ is a monotonically nondecreasing piecewise constant function having m jumps. Hence, the function $\mu(t)$ solves the moment problem of the form $\gamma_k = \int_0^{\theta} t^{k-1} d\mu(t)$, $k = 1, \ldots, n$. The solvability conditions for this moment problem are expressed via Hankel determinants of γ_k. In order to solve the Markov moment problem, one can observe that the function $-R(z)$ belongs to the Nevanlinna class what allows to apply the well-known additive and multiplicative representations for such functions.

We briefly discuss the direct way to pass to the Markov moment min-problem. Notice that

$$\ln R(z) = \sum_{j=1}^{m} \ln\left(1 - \frac{t_{2j}}{z}\right) - \sum_{j=1}^{m} \ln\left(1 - \frac{t_{2j-1}}{z}\right) = -\sum_{k=1}^{\infty} \frac{1}{kz^k} \sum_{j=1}^{2m} (-1)^j t_j^k.$$

Hence,

$$\ln R(z) = -\frac{c_1^{\pm}(\theta_s, s)}{z} - \cdots - \frac{c_n^{\pm}(\theta_s, s)}{nz^n} + O(\frac{1}{z^{n+1}}).$$

This means that one can find $\gamma_1, \ldots, \gamma_n$ using $c_k = c_k^{\pm}(\theta_s, s)$, $k = 1, \ldots, n$. Namely, the following formula can be proved [16]

$$\gamma_k = \frac{(-1)^{k-1}}{k!} \det \begin{pmatrix} c_1 & 1 & 0 & \ldots & 0 \\ c_2 & c_1 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ c_{k-1} & c_{k-2} & \ldots & c_1 & k-1 \\ c_k & c_{k-1} & \ldots & c_2 & c_1 \end{pmatrix}. \tag{8}$$

Thus, we get the following plan for the solving of the moment min-problem (for the case $n = 2m + 1$).

(a) Substituting the given vector s, to find $c_k^{\pm}(\theta) = c_k^{\pm}(\theta, s)$, $k = 1, \ldots, n$, as *polynomials* of the unknown variable θ (the power of $c_k^{\pm}(\theta)$ equals k).

(b) Using formula (8), to find $\gamma_k^{\pm} = \gamma_k^{\pm}(\theta)$ which are polynomials of θ (the power of $\gamma_k^{\pm}(\theta)$ equals k).

(c) To consider the relation $\Gamma_{1,m+1}^{\pm} = \Gamma_{1,m+1}^{\pm}(\theta) = 0$ as the equation for determining of θ. In fact, $\Gamma_{1,m+1}^{\pm}(\theta)$ is the polynomial of power $\frac{1}{2}n(n+1)$. If the conditions assumed above are satisfied then $\theta = \theta_s$ is a root of one of two polynomials $\Gamma_{1,m+1}^{\pm}(\theta)$. The additional conditions $\Gamma_{1,p}^{\pm}(\theta) > 0$, $\Gamma_{2,p+1}^{\pm}(\theta) > 0$, $p = 1, \ldots, m$ allow to choose the proper root.

The case when $n = 2m$ can be analyzed by considering the function

$$R(z) = \frac{\prod_{j=1}^{m} (z - t_{2j-1})}{z \prod_{j=1}^{m-1} (z - t_{2j})} = 1 - \sum_{k=1}^{\infty} \frac{\gamma_k}{z^k};$$

this leads to the equation $\Gamma_{2,m+1}^{\pm} = \Gamma_{2,m+1}^{\pm}(\theta) = 0$. The analogous relations hold if the control $u_s(t)$ has less than $n-1$ points of discontinuity.

The complete solution of the power Markov moment min-problem was constructed by Korobov and Sklyar in [9]. To formulate this result, let us introduce some additional notations. Denote

$$\Delta_p^{\pm} = \begin{cases} \Gamma_{1,k+1}^{\pm} & \text{as} \quad p = 2k+1, \\ \Gamma_{2,k+1}^{\pm} & \text{as} \quad p = 2k, \end{cases} \quad , \quad p = 1, \ldots, n.$$

Consider also the expansion $\dfrac{1}{R(z)} = 1 - \sum_{k=1}^{\infty} \dfrac{\bar\gamma_k}{z^k}$, then

$$\bar\gamma_k = \frac{(-1)^{k-1}}{k!} \det \begin{pmatrix} -c_1 & 1 & 0 & \dots & 0 \\ -c_2 & -c_1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ -c_{k-1} & -c_{k-2} & \dots & -c_1 & k-1 \\ -c_k & -c_{k-1} & \dots & -c_2 & -c_1 \end{pmatrix}.$$

Theorem 2.1. (Korobov–Sklyar [9]**)** *For any $s \in \mathbb{R}^n$, the solution $(\theta_s, u_s(t))$ of power Markov moment min-problem (5) can be found as follows:*

(i) θ_s equals the maximal (real) root of the equation

$$\Delta_n^+(\theta) \cdot \Delta_n^-(\theta) = 0;$$

(ii) the number of points of discontinuity $q-1$ of the function $u_s(t)$ is uniquely defined from the conditions

$$\Delta_q^+(\theta_s) \cdot \Delta_q^-(\theta_s) = 0, \quad (\Delta_q^+(\theta_s))^2 + (\Delta_q^-(\theta_s))^2 \neq 0;$$

(iii) $u_s(\theta_s - 0) = 1$ if in the previous relation $\Delta_q^+(\theta_s) = 0$ and $u_s(\theta_s - 0) = -1$ otherwise;

(iv) all points of discontinuity $0 < t_1 < \dots < t_{q-1} < \theta_s$ of the function $u_s(t)$ are the roots of the equation

$$\det \begin{pmatrix} \gamma_2 & \gamma_3 & \dots & \gamma_{p+1} \\ \dots & \dots & \dots & \dots \\ \gamma_p & \gamma_{p+1} & \dots & \gamma_{2p-1} \\ 1 & z & \dots & z^{p-1} \end{pmatrix} \cdot \det \begin{pmatrix} -1 & \bar\gamma_1 & \dots & \bar\gamma_p \\ \bar\gamma_1 & \bar\gamma_2 & \dots & \bar\gamma_{p+1} \\ \dots & \dots & \dots & \dots \\ \bar\gamma_{p-1} & \bar\gamma_p & \dots & \bar\gamma_{2p-1} \\ 1 & z & \dots & z^p \end{pmatrix} = 0 \quad \textit{if} \;\; q = 2p,$$

or

$$\det \begin{pmatrix} \bar\gamma_1 & \bar\gamma_2 & \dots & \bar\gamma_p \\ \dots & \dots & \dots & \dots \\ \bar\gamma_{p-1} & \bar\gamma_p & \dots & \bar\gamma_{2p-2} \\ 1 & z & \dots & z^{p-1} \end{pmatrix} \cdot \det \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_p \\ \dots & \dots & \dots & \dots \\ \gamma_{p-1} & \gamma_p & \dots & \bar\gamma_{2p-2} \\ 1 & z & \dots & z^{p-1} \end{pmatrix} = 0 \quad \textit{if} \;\; q = 2p-1,$$

where $\gamma_k = \gamma_k^\pm(\theta_s)$ and $\bar\gamma_k = \bar\gamma_k^\pm(\theta_s)$, and the sign $\pm$ corresponds to the value of $u_s(\theta_s - 0)$ obtained above.

The proof of (iv) can be found in [16].

This theorem gives also the solution of time-optimal control problem (6); in order to find the optimal time and optimal control one substitutes $(-1)^k(k-1)!x_k^0$ instead of s_k, $k = 1, \dots, n$, in all relations of the theorem.

This method allows to solve the optimal synthesis problem. In fact, consider $\widetilde{c}_k^\pm(\theta, x) = c_k^\pm(\theta, s)$ where $s_k = (-1)^k(k-1)!x_k$ as polynomials of θ and x_k and use them for finding of $\widetilde{\gamma}_k^\pm(\theta, x) = \gamma_k^\pm(\theta, s)$ as polynomials of θ and x_k. Hence, $\widetilde{\Delta}_p^\pm(\theta, x) = \Delta_p^\pm(\theta, s)$ also are polynomials of θ and x_k. Denote $\widetilde{\theta}_x = \theta_s$ where $s_k = (-1)^k(k-1)!x_k$.

Then Theorem 2.1 gives the following explicit formula for the time-optimal positional control:

$$u(x) = (-1)^{p-1} \operatorname{sign}(\widetilde{\Delta}_p^-(\widetilde{\theta}_x, x) - \widetilde{\Delta}_p^+(\widetilde{\theta}_x, x)),$$

if

$$\widetilde{\Delta}_p^-(\widetilde{\theta}_x, x) \neq \widetilde{\Delta}_p^+(\widetilde{\theta}_x, x),$$
$$\widetilde{\Delta}_r^-(\widetilde{\theta}_x, x) = \widetilde{\Delta}_r^+(\widetilde{\theta}_x, x), \;\; r = 1, \dots, p-1.$$

2.2. **Power Markov moment problem with gaps.** Developing the technique discussed in the previous subsection, Korobov and Sklyar [13], [14] considered the power Markov moment min-problem *with gaps*,

$$s_k = \int_0^\theta t^{m_k-1} u(t)\,dt, \quad k = 1,\dots,n, \quad |u(t)| \le 1, \quad \theta \to \min, \tag{9}$$

where $m_1 < \cdots < m_n$ are arbitrary positive integers. In particular, the time-optimal control problem for the autonomous system (1) with the matrix $A(t) = A$ having a rational spectrum can be reduced to the min-problem of type (9).

We recall briefly the sketch of solving of the power Markov moment min-problem with gaps given in [13], [14].

Denote by χ_k, $k = 1,\dots,m_n$ the sequence of the form

$$\chi_k = \begin{cases} 1, & \text{if} \quad k \in \{m_j\}_{j=1}^n, \\ 0, & \text{if} \quad k \notin \{m_j\}_{j=1}^n \end{cases}$$

and suppose χ_k is periodic of period p, that is $\chi_k = \chi_{k+p}$. Introduce the *generating function* $P(w)$ which is analytic in a neighborhood of the origin and satisfies the following property

$$\ln P(w) = \sum_{r=0}^{\infty} \sum_{m_k \le p} a_{m_k+rp} w^{m_k+rp}. \tag{10}$$

As it was shown in [13], [14], such function $P(w)$ is rational if and only if the set $\{e_p^k : k \in \{1,\dots,p\} \backslash \{m_j : m_j \le p\}\}$ coincides with the set of roots of some polynomial $r(w)$ which is a divisor of $w^p - 1$ over $\mathbb{Q}$. In this case, if $r(w) = \sum_{k=0}^{p-1} r_k w^k$, $r_k \in \mathbb{Z}$, then $P(w) = \prod_{k=0}^{p-1} (1 - e_p^k w)^{r_k}$ (here e_p denotes a primitive root of unit of power p). Suppose further that $P(w)$ is rational, that is $P(w) = \frac{P_1(w)}{P_2(w)}$ where $P_1(w)$ and $P_2(w)$ are polynomials of power d_1 and d_2 respectively.

Now assume that Markov moment min-problem with gaps (9) has a solution $(\theta_s, u_s(t))$; assume for the simplicity that $u_s(t)$ has $n-1$ points of discontinuity. Consider the case $n = 2m+1$ and introduce the rational function

$$R(z) = \frac{\prod_{j=1}^{m} P(\frac{t_{2j}}{z})}{\prod_{j=1}^{m} P(\frac{t_{2j-1}}{z})} = 1 - \sum_{k=1}^{\infty} \frac{\gamma_k}{z^k}. \tag{11}$$

Observe that function (7) is a partial case of function (11) since for the power moment problem (5) one has $p = 1$ and $P(w) = 1 - w$.

Then the rationality of $R(z)$ implies the equalities $\Gamma_{r,r+m(d_1+d_2)} = 0$, $r \ge 1$.

On the other hand,

$$\ln R(z) = \sum_{j=1}^{m} \ln P\left(\frac{t_{2j}}{z}\right) - \sum_{j=1}^{m} \ln P\left(\frac{t_{2j-1}}{z}\right) = -\frac{\widehat{c}_1^{\pm}(\theta_s, s)}{z} - \cdots - \frac{\widehat{c}_{m_n}^{\pm}(\theta_s, s)}{m_n z^{m_n}} + O(\frac{1}{z^{m_n+1}}),$$

where $\widehat{c}_k^{\pm}(\theta_s, s) = a_k c_k^{\pm}(\theta_s, s)$ and a_k are coefficients of the Taylor series (10) of $\ln P(w)$. In particular, $\widehat{c}_k^{\pm}(\theta_s, s) = 0$ if $k \ne m_j + rp$. Hence, the equality

$$\Gamma^{\pm}_{1,1+m(d_1+d_2)}(\theta) = 0 \tag{12}$$

can be considered as a condition for determining of $\theta = \theta_s$ where $\gamma_k^{\pm} = \gamma_k^{\pm}(\theta)$ are found by formula (8) (where $\widehat{c}_k^{\pm}$ are substituted instead of c_k).

If $1 + 2m(d_1 + d_2) \le m_n$ then equation (12) contains only those γ_k which can be found by (8). In particular, this condition is satisfied for the "problem with even powers" [17]

$$s_k = \int_0^\theta t^{2k-2} u(t)\,dt, \quad k = 1,\dots,n, \quad |u(t)| \le 1, \quad \theta \to \min\,.$$

In this case $p=2$, $P(w)=\frac{1-w}{1+w}$, $d_1=d_2=1$ and $m_k=2k-1$.

If $1+2m(d_1+d_2)>m_n$ then equation (12) includes unknown parameters $\gamma_{m_n+1},\dots,\gamma_{1+2m(d_1+d_2)}$. In order to find them one applies the conditions $\Gamma^{\pm}_{r,r+m(d_1+d_2)}(\theta)=0$, $r\ge 2$, and formula (8) taking into account the equalities $\widehat{c}^{\pm}_k(\theta,s)=0$ if $k\ne m_j+rp$.

In the case $n=2m$ the function

$$R(z)=\frac{\prod\limits_{j=1}^{m}P(\frac{t_{2j-1}}{z})}{z^{d_1-d_2}\prod\limits_{j=1}^{m-1}P(\frac{t_{2j}}{z})}=1-\sum_{k=1}^{\infty}\frac{\gamma_k}{z^k}$$

is used. Analogous arguments are applied if the function $u_s(t)$ has less than $n-1$ points of discontinuity.

Finally, notice that another functions besides of ln can be used in manipulations with the generating function. In particular, for the "problem with even powers" the function arctg was applied in [17].

2.3. **General linear system.** Finally, let us consider the time-optimal control problem for system (1) with arbitrary coefficients. Suppose $A(t)$ and $b(t)$ are real analytic (at least, in a neighborhood of the point $t=0$). Then the time-optimal control problem for system (1) is reduced to the Markov moment min-problem of the form

$$s_k=\sum_{i=0}^{\infty}\frac{1}{i!}g_k^{(i)}(0)\int_0^{\theta}t^iu(t)\,dt,\quad k=1,\dots,n,\quad |u(t)|\le 1,\quad \theta\to\min. \tag{13}$$

Suppose the functions $g_1(t),\dots,g_n(t)$ are linearly independent (what corresponds to the fact that the system is controllable), then they form a Tchebycheff system on some interval $(0,T_1)$ where T_1 is rather small.

Let $m_1<\cdots<m_n$ be indices of the first n linearly independent vectors from the sequence $\{g^{(i)}(0)\}_{i=0}^{\infty}$, and $G=(\frac{1}{m_1!}g^{(m_1)}(0),\dots,\frac{1}{m_n!}g^{(m_n)}(0))^{-1}$. Then (13) is equivalent to

$$\tilde s_k=\int_0^{\theta}t^{m_k}u(t)\,dt+\sum_{j=m_k+1}^{\infty}r_{kj}\int_0^{\theta}t^ju(t)\,dt,\quad k=1,\dots,n,\quad |u(t)|\le 1,\quad \theta\to\min, \tag{14}$$

where $\tilde s_k=(Gs)_k$. Notice that

$$\int_0^{\theta}t^ju(t)\,dt=\theta^{j+1}\int_0^1\tau^j\hat u(\tau)\,d\tau,\quad \text{where}\quad \hat u(\tau)=u(\tau\theta),\quad \tau\in[0,1], \tag{15}$$

hence, the number $j+1$ can be considered as the order of smallness of the power moment (15) when $\theta\to 0$. This means that the power moment $\int_0^{\theta}t^{m_k}u(t)dt$ is the leading term of the right-hand side of (14). Hence, it is natural to expect that the solution of (14) is close to the solution of the power moment min-problem with gaps

$$\tilde s_k=\int_0^{\theta}t^{m_k}u(t)\,dt,\quad k=1,\dots,n,\quad |u(t)|\le 1,\quad \theta\to\min. \tag{16}$$

Notice also that any min-problem (16) corresponds to a linear time-optimal problem. It is not defined uniquely; for example, it can be chosen in the form

$$\dot x_k=-t^{m_k}u,\quad k=1,\dots,n,\quad x(0)=x^0,\quad x(\theta)=0,\quad |u(t)|\le 1,\quad \theta\to\min. \tag{17}$$

This leads to the complete classification of linear systems with real analytic coefficients in the sense of time optimality. More precisely, we adopt the following definition.

Definition 2.2. [27] Consider two Markov moment min-problems

$$s_k=\int_0^{\theta}g_k(t)u(t)\,dt,\quad k=1,\dots,n,\quad |u(t)|\le 1,\quad \theta\to\min, \tag{18}$$

and

$$s_k = \int_0^\theta \widehat{g}_k(t)u(t)\,dt, \quad k = 1,\dots,n, \quad |u(t)| \le 1, \quad \theta \to \min, \tag{19}$$

where $g_k(t)$, $k = 1,\dots,n$, and $\widehat{g}_k(t)$, $k = 1,\dots,n$, are two sequences of real analytic functions in a neighborhood of the point $t = 0$. Let $(\theta_s, u_s(t))$, $(\widehat{\theta}_s, \widehat{u}_s(t))$ be their solutions. These two min-problems are called locally equivalent in a neighborhood of the origin if there exists a linear nonsingular operator $L : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\frac{\widehat{\theta}_{Ls}}{\theta_s} \to 1, \quad \frac{1}{\theta}\int_0^\theta |\widehat{u}_{Ls}(t) - u_s(t)|\,dt \to 0 \quad \text{as} \quad s \to 0,$$

where $\theta = \min\{\widehat{\theta}_{Ls}, \theta_s\}$.

Then the following result holds.

Theorem 2.3. [27] *Two Markov moment min-problems (18) and (19) are locally equivalent in a neighborhood of the origin if and only if the indices of the first n linearly independent vectors from the sequences $\{g^{(i)}(0)\}_{i=0}^\infty$ and $\{\widehat{g}^{(i)}(0)\}_{i=0}^\infty$ coincide. Moreover, suppose $m_1 < \dots < m_n$ are these indices. Then the both Markov moment min-problems are locally equivalent to the power Markov moment min-problem with gaps of the form (16).*

As a consequence, we get the following definition of local equivalence and the classification theorem for time-optimal control problems.

Definition 2.4. [27] Consider two time-optimal control problems of the form

$$\dot{x} = A(t)x + b(t)u, \quad |u| \le 1,\ x(0) = x^0,\ x(\theta) = 0,\ \theta \to \min, \tag{20}$$

and

$$\dot{x} = \widehat{A}(t)x + \widehat{b}(t)u, \quad |u| \le 1,\ x(0) = x^0,\ x(\theta) = 0,\ \theta \to \min, \tag{21}$$

and let $(\theta_{x^0}, u_{x^0}(t))$, $(\widehat{\theta}_{x^0}, \widehat{u}_{x^0}(t))$ be their solutions. These two problems are called locally equivalent in a neighborhood of the origin if there exists a linear nonsingular operator $L : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\frac{\widehat{\theta}_{Lx^0}}{\theta_{x^0}} \to 1, \quad \frac{1}{\theta}\int_0^\theta |\widehat{u}_{Lx^0}(t) - u_{x^0}(t)|\,dt \to 0 \quad \text{as} \quad x^0 \to 0,$$

where $\theta = \min\{\widehat{\theta}_{Lx^0}, \theta_{x^0}\}$.

Theorem 2.5. [27] *Two time-optimal control problems (20) and (21) are locally equivalent in a neighborhood of the origin if and only if the indices of the first n linearly independent vectors from the sequences $\{(-A(t)+\frac{d}{dt})^i b(t)|_{t=0}\}_{i=0}^\infty$ and $\{(-\widehat{A}(t)+\frac{d}{dt})^i \widehat{b}(t)|_{t=0}\}_{i=0}^\infty$ coincide.*

Moreover, it is shown in [15], [27] that under certain conditions the solution of the min-problem (18) (and, therefore, the solution of the time-optimal problem (20)) can be found by the method of successive approximations by use of the corresponding power moment min-problem with gaps (16).

Theorem 2.6. [27] *Let the vectors $\{g^{(m_k)}(0)\}_{k=1}^n$, $m_1 < \dots < m_n$, be linearly independent and $g^{(j)}(0) = 0$ for all $j < m_n$ such that $j \ne m_k$, $k = 1,\dots,n$. Then the solution $(\theta_s, u_s(t))$ of Markov moment min-problem (18) can be found as the limit of the sequence $(\theta^*_{y^i}, u^*_{y^i}(t))$ of solutions of power Markov moment min-problem with gaps (16) where the sequence y^i is defined recursively as*

$$y^{i+1} = G\left(s - \int_0^{\theta^*_{y^i}} g(t)u^*_{y^i}(t)\,dt\right), \quad y^0 = 0,$$

where $G = (\frac{1}{m_1!}g^{(m_1)}(0), \dots, \frac{1}{m_n!}g^{(m_n)}(0))^{-1}$.

In particular, the conditions of the theorem are satisfied if $m_k = k-1$, $k = 1, \dots, n$, and the vectors $\{g^{(k-1)}(0)\}_{k=1}^n$ are linearly independent; the corresponding result was obtained in [15].

3. Nonlinear time optimality and nonlinear power Markov moment min-problem

The next step is to extend the moment approach to nonlinear control systems. The simplest class of nonlinear control systems which are most close to linear ones is the class of affine control systems of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad a(t,0) \equiv 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \tag{22}$$

where $a(t,x)$, $b(t,x)$ are real analytic vector functions in a neighborhood of the origin in $\mathbb{R}^{n+1}$. The condition $a(t,0) \equiv 0$ means that the origin is a point of rest for the system. Let us consider the time-optimal control problem for system (22) to the origin with the control satisfying the constraint $|u(t)| \le 1$.

3.1. **Series representation of nonlinear systems.** In the nonlinear case one gets the power moment series representation as a generalization of the Cauchy formula (2) for the linear case. Namely, if the control $u(t)$ steers the point $x(0) = x^0$ to the origin, $x(\theta) = 0$, then

$$x^0 = S_{a,b}(\theta, u) = \sum_{m=1}^{\infty} \sum_{\substack{m_1+\cdots+m_k+k=m \\ k \ge 1, m_j \ge 0}} v_{m_1\dots m_k} \xi_{m_1\dots m_k}(\theta, u), \tag{23}$$

where $\xi_{m_1\dots m_k}(\theta,u)$ are *nonlinear power moments* of the form

$$\xi_{m_1\dots m_k}(\theta,u) = \int_0^\theta \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} \tau_1^{m_1}\tau_2^{m_2}\cdots\tau_k^{m_k} \prod_{j=1}^k u(\tau_j)\, d\tau_k \cdots d\tau_2 d\tau_1 \tag{24}$$

and $v_{m_1\dots m_k}$ are constant vector coefficients depending on system parameters. More specifically, let R_a and R_b denote operators acting as $R_a c(t,x) = c_t(t,x) + c_x(t,x)a(t,x)$, $R_b c(t,x) = c_x(t,x)b(t,x)$, and $E(x) \equiv x$. Let $\mathrm{ad}_{R_a}^{m+1} R_b = [R_a, \mathrm{ad}_{R_a}^m R_b]$, $\mathrm{ad}_{R_a}^0 R_b = R_b$, where $[\cdot,\cdot]$ is the operator commutator. Then

$$v_{m_1\dots m_k} = \frac{(-1)^k}{m_1!\cdots m_k!}\ \mathrm{ad}_{R_a}^{m_1} R_b \circ \cdots \circ \mathrm{ad}_{R_a}^{m_k} R_b E(x)_{|\substack{t=0\\x=0}}.$$

The series in the right-hand side of (23) is absolutely convergent if $|u(t)| \le 1$ and θ is rather small. Close series representations of nonlinear control systems were given in [28, 29, 30] and other works.

In essence, equality (23) can be considered as *a nonlinear power Markov moment problem* on the interval $[0,\theta]$. In the case of time-optimal control problem one requires $\theta \to \min$ what leads to *the nonlinear power Markov moment min-problem.*

3.2. **Algebra of nonlinear power moments.** If (θ,u) runs through the set $\{(\theta,u) : 0 \le \theta \le T, |u(t)| \le 1\}$ then the functionals (24) are linearly independent. It is convenient to consider the linear span of the nonlinear power moments as *a free associative algebra* $\mathcal{A}$ (over $\mathbb{R}$) generated by the set of linear power moments $\{\xi_m : m \ge 0\}$ and the algebraic operation of "concatenation":

$$\xi_{m_1\dots m_k}\xi_{n_1\dots n_r} = \xi_{m_1\dots m_k n_1\dots n_r}.$$

Below we use the notation $f = \sum \alpha_{m_1\dots m_k}\xi_{m_1\dots m_k}$ (without arguments) when we mean an element of the algebra $\mathcal{A}$ of functionals (where $\alpha_{m_1\dots m_k} \in \mathbb{R}$) and write $f(\theta,u) =$

$\sum \alpha_{m_1\dots m_k}\xi_{m_1\dots m_k}(\theta,u)$ for the value of the functional when θ and $u=u(t)$ are given. Notice that, analogously to (15),

$$\xi_{m_1\dots m_k}(\theta,u)=\theta^{m_1+\cdots+m_k+k}\xi_{m_1\dots m_k}(1,\hat{u})$$

where $\hat{u}(\tau)=u(\tau\theta)$, $\tau\in[0,1]$. Hence, the number $m_1+\cdots+m_k+k$ is the order of smallness of the nonlinear power moment $\xi_{m_1\dots m_k}$. In the algebraic terms, it corresponds to *the graded structure* in the algebra $\mathcal{A}$. Namely, $\mathcal{A}=\sum_{m=1}^{\infty}\mathcal{A}_m$ where $\mathcal{A}_m=\mathrm{Lin}\{\xi_{m_1\dots m_k}:m_1+\cdots+m_k+k=m\}$ are subspaces of homogeneous elements.

In these constructions, the important role is played by the Lie algebra $\mathcal{L}$ (over $\mathbb{R}$) generated by the set of linear moments $\{\xi_m : m\ge 0\}$ with the Lie bracket operation $[\ell_1,\ell_2]=\ell_1\ell_2-\ell_2\ell_1$, $\ell_1,\ell_2\in\mathcal{L}$. Denote $\mathcal{L}_m=\mathcal{L}\cap\mathcal{A}_m$.

Now let us return to the series (23). It generates a linear map $v:\mathcal{A}\to\mathbb{R}^n$ defined on basis elements of $\mathcal{A}$ by the formula $v(\xi_{m_1\dots m_k})=v_{m_1\dots m_k}$. It turns out that the study of many important properties of system (22) can be reduced to the study of "algebraic properties" of the formal series

$$S_{a,b}=\sum_{m=1}^{\infty}\sum_{\substack{m_1+\cdots+m_k+k=m\\ k\ge 1,m_j\ge 0}} v(\xi_{m_1\dots m_k})\xi_{m_1\dots m_k}.$$

In particular, one easily can see that there exists a natural homomorphism φ from $\mathcal{L}$ to the Lie algebra of vector fields generated by the set $\{\mathrm{ad}^m_{R_a}R_bE(x)|_{t=0} : m\ge 0\}$ such that $v(\ell)=\varphi(\ell)(0)$. Moreover, the following important concept is connected with the map v. For any $m\ge 1$, consider the subspace

$$\mathcal{P}_m=\{\ell\in\mathcal{L}_m : v(\ell)\in v(\mathcal{L}_1+\cdots+\mathcal{L}_{m-1})\}$$

(describing linear dependence analogous to that in the sequence $\{(-A(t)+\frac{d}{dt})^i b(t)|_{t=0}\}_{i=0}^{\infty}$ for linear system (1)). Denote by $J_{a,b}$ the right ideal generated by the subspace $\sum_{m=1}^{\infty}\mathcal{P}_m$, that is

$$J_{a,b}=\sum_{m=1}^{\infty}\mathcal{P}_m(\mathcal{A}+\mathbb{R}).$$

Elements of $J_{a,b}$ possess the following property: $v(J_{a,b}\cap\mathcal{A}_m)\subset v(\mathcal{A}_1+\cdots+\mathcal{A}_{m-1})$. Roughly speaking, elements of $J_{a,b}$ cannot be leading terms in the series $S_{a,b}$.

In the linear case only linear changes of variables are considered; in the nonlinear case one can use nonlinear changes of variables. Observe that the change of variables in the system corresponds to the transformation over the series. Namely, suppose an analytic change of variables $y=\Phi(x)$ reduces system (22) to the system $\dot{y}=\bar{a}(t,y)+\bar{b}(t,y)u$. Since $y^0=\Phi(x^0)$ then $S_{\bar{a},\bar{b}}(\theta,u)=\Phi(S_{a,b}(\theta,u))$. To find $\Phi(S_{a,b}(\theta,u))$ one needs to multiply nonlinear power moments (24). Since $\xi_{m_1\dots m_k}$ are integrals over simplex domains then the product of two such integrals can be represented as a linear combination of integrals of the same type. The product of such integrals corresponds to the *shuffle product* operation $\sqcup\!\sqcup$ in the algebra $\mathcal{A}$ defined recursively as

$$\xi_{m_1}\sqcup\!\sqcup\xi_{n_1}=\xi_{m_1n_1}+\xi_{n_1m_1},$$

$$\xi_{m_1}\sqcup\!\sqcup\xi_{n_1\dots n_r}=\xi_{n_1\dots n_r}\sqcup\!\sqcup\xi_{m_1}=\xi_{m_1n_1\dots n_r}+\xi_{n_1}(\xi_{m_1}\sqcup\!\sqcup\xi_{n_2\dots n_r}),\quad r\ge 2,$$

$$\xi_{m_1\dots m_k}\sqcup\!\sqcup\xi_{n_1\dots n_r}=\xi_{m_1}(\xi_{m_2\dots m_k}\sqcup\!\sqcup\xi_{n_1\dots n_r})+\xi_{n_1}(\xi_{m_1\dots m_k}\sqcup\!\sqcup\xi_{n_2\dots n_r}),\quad k,r\ge 2.$$

In fact,

$$\xi_{m_1\dots m_k}(\theta,u)\cdot\xi_{n_1\dots n_r}(\theta,u)=(\xi_{m_1\dots m_k}\sqcup\!\sqcup\xi_{n_1\dots n_r})(\theta,u).$$

Obviously, the shuffle product operation is associative and commutative. The shuffle product was introduced by R. Ree in [31] and used to study properties of Lie elements in an associative algebra; later it was applied for series representing control systems by M. Fliess [29].

This operation allows to express the relation $S_{\bar{a},\bar{b}}(\theta,u) = \Phi(S_{a,b}(\theta,u))$ discussed above via formal series of nonlinear power moments, namely, as $S_{\bar{a},\bar{b}} = \Phi(S_{a,b}) = \sum_{r=1}^{\infty}\frac{1}{r!}\Phi^{(r)}(0)(S_{a,b})^{\sqcup\!\sqcup r}$ where we use the notation $a^{\sqcup\!\sqcup r} = a \sqcup\!\sqcup \cdots \sqcup\!\sqcup a$ (r times).

Moreover, the following important property of the shuffle product was discovered by R. Ree [31]. Introduce the inner product in the algebra $\mathcal{A}$ assuming that the basis $\{\xi_{m_1\dots m_k} : k \geq 1; m_1,\dots,m_k \geq 0\}$ is orthonormal. Then the subspaces $\mathcal{L}$ and $\mathcal{A} \sqcup\!\sqcup \mathcal{A}$ are orthogonal to each other. We essentially used this remarkable result in order to prove the next theorem which describes the leading terms in the series $S_{a,b}$ (analogous to that in (14)). We suppose that system (22) is *accessible*, i.e. its attainability set has a nonempty interior in $\mathbb{R}^n$; this is the case iff $v(\mathcal{L}) = \mathbb{R}^n$.

Theorem 3.1. [23] *Suppose system (22) is such that $v(\mathcal{L}) = \mathbb{R}^n$. Let $\ell_1,\dots,\ell_n \in \mathcal{L}$ be such homogeneous elements that $\mathcal{L} = \mathrm{Lin}\{\ell_1,\dots,\ell_n\} + \sum_{m=1}^{\infty}\mathcal{P}_m$; assume $\ell_k \in \mathcal{A}_{r_k}$, $k = 1,\dots,n$. Denote by $\widetilde{\ell}_k$ the orthogonal projection of ℓ_k on the subspace $J_{a,b}^{\perp}$. Then there exists a map $y = \Phi(x)$, $\Phi(0) = 0$, such that*

$$(\Phi(S_{a,b}))_k = \widetilde{\ell}_k + \rho_k, \quad k = 1,\dots,n,$$

where ρ_k is a sum of elements of $\sum_{j=r_k+1}^{\infty}\mathcal{A}_j$. Moreover, there exists a system $\dot{x} = \hat{a}(t,x) + \hat{b}(t,x)u$ such that

$$(S_{\hat{a},\hat{b}})_k = \widetilde{\ell}_k, \quad k = 1,\dots,n. \tag{25}$$

In particular, this system is accessible.

In terms of the moment problem, this theorem gives the "algebraic" classification of nonlinear power moment min-problems which correspond to affine systems. One naturally expects that the solution of the time-optimal problem for the system $\dot{x} = \hat{a}(t,x) + \hat{b}(t,x)u$ is close to the solution of the time-optimal problem for system (22) as it is in the linear case. In fact, this holds under some additional conditions. The precise results are given in the two next subsections.

3.3. **Systems which are equivalent to linear ones.** The simplest case of the moment problem (25) is a linear case when $\widetilde{\ell}_k = \xi_{m_k}$, $k = 1,\dots,n$, for some set $m_1 < \cdots < m_n$. Notice that this is the case if and only if

$$\mathrm{rank}\{v_i\}_{i=0}^{\infty} = n \ \text{ and } \ v_{m_1\dots m_k} \in \mathrm{Lin}\{v_i\}_{i=0}^{m-2}, \text{ where } m = m_1+\cdots+m_k+k,\ k \geq 2. \tag{26}$$

In [21] we called such systems "essentially linear". The following definition generalizes the concept of the local equivalence of (linear) power Markov moment min-problems.

Definition 3.2. [21] Consider a nonlinear power Markov moment min-problem

$$s = \sum_{m=1}^{\infty}\sum_{\substack{m_1+\cdots+m_k+k=m\\ k\geq 1, m_j\geq 0}} v_{m_1\dots m_k}\xi_{m_1\dots m_k}(\theta,u), \quad |u(t)| \leq 1, \quad \theta \to \min,$$

and suppose it has a solution for any s from a neighborhood of the origin. Denote by $\{(\theta_s, u_s(t)) : u(t) \in U_s\}$ the set of all its solutions. This problem is called locally equivalent to the power moment min-problem with gaps

$$s_k = \int_0^{\theta} t^{m_k}u(t)\,dt, \quad k = 1,\dots,n, \quad |u(t)| \leq 1, \quad \theta \to \min, \tag{27}$$

where $0 \leq m_1 < \cdots < m_n$, if there exists an analytic map Φ of a neighborhood of the origin, $\Phi(0) = 0$, such that

$$\frac{\theta_{\Phi(s)}}{\theta_s^{Lin}} \to 1, \qquad \sup_{u(t)\in U_{\Phi(s)}}\frac{1}{\theta}\int_0^{\theta}|u(t) - u_s^{Lin}(t)|\,dt \to 0 \quad \text{as} \quad s \to 0,$$

where $\theta = \min\{\theta_{\Phi(s)}, \theta_s^{Lin}\}$ and $(\theta_s^{Lin}, u_s^{Lin}(t))$ is the solution of min-problem (27).

Observe that two min-problems with gaps having sequences of powers $m_1 < \cdots < m_n$ and $m'_1 < \cdots < m'_n$ respectively are locally equivalent to each other if and only if these sequences coincide, i.e. $m_k = m'_k$, $k = 1, \ldots, n$.

Since the time-optimal control problem for affine systems of the form (22) is reduced to the nonlinear power Markov moment min-problem, we identify the solutions of the time-optimal problem and the corresponding Markov moment min-problem. Notice that such time-optimal problem (and, hence, the min-problem) has a solution due to the Filippov theorem [32].

Theorem 3.3. [21] *The time-optimal control problem for system (22) is locally equivalent to a certain power Markov moment min-problem with gaps (27) (and, therefore, to the time-optimal control problem for system (17)) if and only if condition (26) holds.*

3.4. **General nonlinear affine system.** In the general case, elements $\widetilde{\ell}_k$ are nonlinear power moments. However, the local equivalence of the initial time-optimal control problem and the Markov moment min-problem

$$s_k = \widetilde{\ell}_k(\theta, u), \quad k = 1, \ldots, n, \quad |u(t)| \le 1, \quad \theta \to \min, \tag{28}$$

can be proved under some additional conditions. First, note that the solvability set of a nonlinear power Markov moment problem of the form

$$s_k = \widetilde{\ell}_k(T, u), \quad k = 1, \ldots, n, \quad |u(t)| \le 1,$$

can be nonconvex, and even in the case of accessibility the origin can belong to the boundary of the solvability set (these properties are well known for the attainability set of a nonlinear affine control system).

We adopt the following version of the definition of local equivalence.

Definition 3.4. [23] Consider a nonlinear power Markov moment min-problem

$$s = \sum_{m=1}^{\infty} \sum_{\substack{m_1+\cdots+m_k+k=m \\ k\ge 1, m_j \ge 0}} v_{m_1\ldots m_k} \xi_{m_1\ldots m_k}(\theta, u), \quad |u(t)| \le 1, \quad \theta \to \min, \tag{29}$$

and for any θ denote by $U_s(\theta)$ the set of all functions $u(t)$ such that $|u(t)| \le 1$, $t \in [0, \theta]$, which satisfy the moment equalities (29). Denote $\theta_s = \inf\{\theta : U_s(\theta) \ne \emptyset\}$.

Consider a nonlinear power Markov moment min-problem

$$s = \sum_{m=1}^{\infty} \sum_{\substack{m_1+\cdots+m_k+k=m \\ k\ge 1, m_j \ge 0}} v^*_{m_1\ldots m_k} \xi_{m_1\ldots m_k}(\theta, u), \quad |u(t)| \le 1, \quad \theta \to \min, \tag{30}$$

and suppose it has the unique solution $(\theta_s^*, u_s^*(t))$ for any s from a domain $\Omega \subset \mathbb{R}^n \backslash \{0\}$, $0 \in \overline{\Omega}$.

We say that nonlinear power Markov moment min-problem (30) approximates nonlinear power Markov moment min-problem (29) in the domain Ω if there exists an analytic map Φ of a neighborhood of the origin, $\Phi(0) = 0$, and a set of pairs $(\widetilde{\theta}_s, \widetilde{u}_s(t))$, $s \in \Omega$, such that $\widetilde{u}_s(t) \in U_{\Phi(s)}(\widetilde{\theta}_s)$ and

$$\frac{\theta_{\Phi(s)}}{\theta_s^*} \to 1, \quad \frac{\widetilde{\theta}_s}{\theta_s^*} \to 1, \quad \frac{1}{\theta} \int_0^{\theta} |u_s^*(t) - \widetilde{u}_s(t)| dt \to 0 \quad \text{as} \quad s \to 0, \quad s \in \Omega,$$

where $\theta = \min\{\widetilde{\theta}_s, \theta_s^*\}$.

In the other words, $\theta_{\Phi(s)}$ and θ_s^* are asymptotically equivalent as $s \to 0$, $s \in \Omega$ like in Definition 3.2 whereas $u_s^*(t)$ is close to an “almost optimal” function $\widetilde{u}_s(t)$.

We proved the following theorem on approximation for nonlinear time-optimal control problems.

Theorem 3.5. [23] *Suppose system (22) is such that $v(\mathcal{L}) = \mathbb{R}^n$. Let $\ell_1, \ldots, \ell_n \in \mathcal{L}$ be such homogeneous elements that $\mathcal{L} = \mathrm{Lin}\{\ell_1, \ldots, \ell_n\} + \sum_{m=1}^{\infty} \mathcal{P}_m$. Denote by $\widetilde{\ell}_k$ the orthogonal projection of ℓ_k on the subspace $J_{a,b}^{\perp}$.*

Suppose $\Omega \subset \mathbb{R}^n$, $0 \in \overline{\Omega}$, is an open domain such that

(i) the nonlinear power Markov moment min-problem

$$s_k = \widetilde{\ell}_k(\theta, u), \quad k = 1, \ldots, n, \quad |u(t)| \le 1, \quad \theta \to \min, \tag{31}$$

has the unique solution $(\theta_s^*, u_s^*(t))$ for any $s \in \Omega$;

(ii) the function θ_s^* is continuous at any $s \in \Omega$;

(iii) for the set $K = \{u_s^*(t\theta_s^*), t \in [0,1] : s \in \Omega\} \subset L_2[0,1]$, the weak convergence of elements from K implies the strong convergence.

Then there exists a set $\{\Omega(\delta)\}_{\delta>0}$ of embedded domains, $\cup_{\delta>0}\Omega(\delta) = \Omega$, such that the moment min-problem (31) approximates the time-optimal control problem for system (22) in any domain $\Omega(\delta)$.

Recall that, as it was stated in Theorem 3.1, there exists a system $\dot{x} = \hat{a}(t,x) + \hat{b}(t,x)u$ such that $(S_{\hat{a},\hat{b}})_k = \widetilde{\ell}_k$, $k = 1, \ldots, n$. Hence, the time-optimal control problem for system (22) is approximated by some (homogeneous) time-optimal control problem.

Hence, the problem arises to study such "special" nonlinear power Markov moment min-problems of the form (31). Since they are realizable as time-optimal control problems then they have solutions due to the Filippov theorem. However, their solutions do not possess important properties like solutions of linear power Markov moment min-problems. In particular, the function $u_s(t)$ can take not only bound values.

Example. [23] Let us consider the time-optimal control problem for the three-dimensional control system

$$\dot{x}_1 = u,\ \dot{x}_2 = x_1,\ \dot{x}_3 = \frac{1}{2}x_1^2, \quad x(0) = x^0,\ x(\theta) = 0, \quad |u(t)| \le 1, \quad \theta \to \min. \tag{32}$$

It corresponds to the nonlinear power Markov moment min-problem

$$\begin{aligned} s_1 &= -\xi_0(\theta, u) = -\int_0^{\theta} u(t)\, dt, \\ s_2 &= \xi_1(\theta, u) = \int_0^{\theta} t u(t)\, dt, \\ s_3 &= -\xi_{01}(\theta, u) = -\int_0^{\theta}\int_0^{\tau_1} \tau_2 u(\tau_1) u(\tau_2)\, d\tau_2 d\tau_1, \quad |u(t)| \le 1, \quad \theta \to \min. \end{aligned} \tag{33}$$

In [33, 23] the complete solution of this time-optimal control problem (and, therefore, the moment min-problem) was obtained. It turns out that the solvability set for this min-problem equals

$$\Omega = \{s : s_3 < -\tfrac{1}{6}|s_1|^3\} \cup \{s = (s_1, -\tfrac{1}{2}s_1|s_1|, -\tfrac{1}{6}|s_1|^3)\},$$

i.e. system (32) is accessible but not locally controllable.

If $s \in \{s : s_3 \le -\frac{1}{6}\sigma s_1^3 - \frac{1}{3}(\frac{1}{2}s_1^2 + \sigma s_2)^{3/2}\}$, where $\sigma = \mathrm{sign}(s_2 + \frac{1}{2}s_1|s_1|)$, then $u_s(t) = \pm 1$ and has no more than two points of discontinuity. If $s_1 = 0$ then such function $u_s(t)$ is not unique.

However, if $s \in \{s : -\frac{1}{6}\sigma s_1^3 - \frac{1}{3}(\frac{1}{2}s_1^2 + \sigma s_2)^{3/2} < s_3 < -\frac{1}{6}|s_1|^3\}$ then $u_s(t)$ is of a singular type

$$u_s(t) = \begin{cases} \pm 1, & t \in [0, a_s], \\ 0, & t \in (a_s, b_s), \\ \pm 1, & t \in [b_s, \theta_s], \end{cases}$$

for some $0 \le a_s < b_s < \theta_s$. Thus, the time-optimal control for the nonlinear affine system is not of the bang-bang type.

For system (32) the conditions of Theorem 3.5 are satisfied. In particular, nonlinear power Markov moment min-problem (33) approximates the time-optimal problem for the (locally controllable) system

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = \frac{1}{2}x_1^2 + x_1^3$$

in the domains $\Omega_1 = \{x : -\delta_1 x_1^3 < x_3 < -\delta_2 x_1^3,\ x_1 > 0\}$ and $\Omega_2 = \{x : \delta_1 x_1^3 < x_3 < \delta_2 x_1^3,\ x_1 < 0\}$ for any $\delta_1 > \delta_2 > \frac{1}{6}$.

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