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OPERATOR-VALUED INTEGRAL OF A VECTOR-FUNCTION AND BASES

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Abstract. In the present paper we are going to introduce an operator-valued integral of a square modulus weakly integrable mappings the ranges of which are Hilbert spaces, as bounded operators. Then, we shall show that each operator-valued integrable mapping of the index set of an orthonormal basis of a Hilbert space H into H can be written as a multiple of a sum of three orthonormal bases.

1. Introduction

Throughout this paper (X, μ) will be a measure space and H will be a Hilbert space over $\mathbb{C}$, where H, in general, is not assumed to be separable. We shall denote the closed unit ball of H by H_1.

Definition 1.1. Let $L^2(X, H)$ be the class of all measurable mappings $f : X \to H$ such that

$$\|f\|_2^2 = \int_X \|f(x)\|^2 d\mu < \infty.$$

By the polar identity we conclude that for each $f, g \in L^2(X, H)$, the mapping $x \mapsto \langle f(x), g(x)\rangle$ of X to $\mathbb{C}$ is measurable, and it can be proved that $L^2(X, H)$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle_{L^2} = \int_X \langle f(x), g(x)\rangle \, d\mu.$$

We shall write $L^2(X)$ when $H = \mathbb{C}$.

The following lemmas can be found in operator theory textbooks.

Lemma 1.2. *Let $u : K \to H$ be a bounded operator with closed range $\mathcal{R}_u$. Then there exists a bounded operator $u^\dagger : H \to K$ for which*

$$uu^\dagger f = f, \quad f \in \mathcal{R}_u.$$

Also, $u^* : H \to K$ has closed range and $(u^*)^\dagger = (u^\dagger)^*$.

Lemma 1.3. *Let $u : K \to H$ be a bounded surjective operator. Given $y \in H$, the equation $ux = y$ has a unique solution of minimal norm, namely, $x = u^\dagger y$.*

The operator $u^\dagger$ is called the pseudo-inverse of u.

Lemma 1.4. *Let $u : H \to K$ be a bounded operator. Then*
- (i) $\|u\| = \|u^*\|$ *and* $\|uu^*\| = \|u\|^2$.
- (ii) $\mathcal{R}_u$ *is closed, if and only if,* $\mathcal{R}_{u^*}$ *is closed.*

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(iii) *u is surjective, if and only if, there exists $c > 0$ such that for each $h \in H$*

$$c\|h\| \leq \|u^*h\|.$$

Lemma 1.5. *Let H be a Hilbert space. Then*

(i) *Every bounded and invertible operator $u : H \to H$ has a unique representation $u = wp$, where w is unitary and p is positive.*

(ii) *Every positive operator p on H with $\|p\| < 1$ can be written $p = 2^{-1}(w + w^*)$, where w is an unitary operator.*

Lemma 1.6. *Let u be a self-adjoint bounded operator on H. Let*

$$m_u = \inf_{\|h\|=1} \langle uh, h\rangle \quad \textit{and} \quad M_u = \sup_{\|h\|=1} \langle uh, h\rangle.$$

Then, $m_u, M_u \in \sigma(u)$.

2. A survey of the operator-valued integral of vector-function

In this section we shall introduce the concept of operator-valued integrability of vector-functions of X to H. Then, we shall define their operator-valued integrals as bounded operators of the Hilbert space $L^2(X)$ to H.

Definition 2.1. Let $f : X \to H$ be a mapping. We say that f is weakly measurable if for each $h \in H$ the mapping $x \mapsto \langle h, f(x)\rangle$ of X to $\mathbb{C}$ is measurable.

Definition 2.2. Let $f : X \to H$ be weakly measurable. We say that f is operator-valued integrable over X if

$$\sup_{h\in H_1} \int_X |\langle h, f(x)\rangle|^2 d\mu < \infty.$$

The class of all operator-valued integrable mappings of X to H will be denoted by $\mathcal{L}(X, H)$. It is clear that $L^2(X, H) \subseteq \mathcal{L}(X, H)$. Also, $\mathcal{L}(X, H)$ is a normed space with the norm defined by

$$\|f\|^2_{\mathcal{L}} = \sup_{h\in H_1} \int_X |\langle h, f(x)\rangle|^2 d\mu.$$

In the normed space $\mathcal{L}(X, H)$, f is a null function if for each $h \in H$

$$\langle h, f\rangle = 0 \quad a.e.$$

Let $f \in \mathcal{L}(X, H)$ and let the mapping $F_f : L^2(X) \to H$ be defined by

$$\langle F_f(g), h\rangle = \int_X g(x)\langle f(x), h\rangle \, d\mu, \quad h \in H, \quad g \in L^2(X). \tag{2.1}$$

It is evident that F_f is well defined and linear. For each $g \in L^2(X)$ and $h \in H$, we have

$$\begin{aligned}\|F_f(g)\| &= \sup_{h\in H_1} |\langle F_f(g), h\rangle| \\ &\leq \Big(\int_X |g(x)|^2 d\mu\Big)^{1/2} \sup_{h\in H_1} \Big(\int_X |\langle f(x), h\rangle|^2 d\mu\Big)^{1/2} \leq \|g\|_2 \|f\|_{\mathcal{L}}.\end{aligned}$$

Hence, F_f is bounded.
For each $g \in L^2(X)$ and $h \in H$ we have

$$\langle F^*_f(h), g\rangle = \langle h, F_f(g)\rangle = \overline{\langle F_f(g), h\rangle} = \int_X \overline{g}(x)\langle h, f(x)\rangle \, d\mu = \langle\langle h, f\rangle, g\rangle_{L^2}.$$

Thus

$$F^*_f(h) = \langle h, f\rangle. \tag{2.2}$$

Also, for each $h \in H$

$$\|F_f^*(h)\|^2 = \langle F_f^*(h), F_f^*(h)\rangle = \int_X |\langle f(x), h\rangle|^2 d\mu. \tag{2.3}$$

Therefore

$$\|F_f\| = \|F_f^*\| = \Big(\sup_{h\in H_1} \int_X |\langle f(x), h\rangle|^2 d\mu\Big)^{1/2} = \|f\|_{\mathcal{L}}. \tag{2.4}$$

Definition 2.3. Let (X, μ) be a measure space and $f \in \mathcal{L}(X, H)$. The unique bounded linear operator $F_f : L^2(X) \to H$ defined by (2.1), will be denoted by

$$\int_{HL(X)} f\, d\mu,$$

and we shall say the operator-valued integral of f over X. Therefore, for each $g \in L^2(X)$, $\int_{HL(X} f\, d\mu$ is defined by

$$\Big\langle \int_{HL(X)} f\, d\mu(g), h\Big\rangle = \int_X g(x)\langle f(x), h\rangle\, d\mu, \quad h \in H.$$

We shall denote the adjoint of $\int_{HL(X)} f\, d\mu$ by $\int^*_{HL(X)} f\, d\mu$, which by (2.2) for each $h \in H$

$$\int^*_{HL(X)} f\, d\mu(h) = \langle h, f\rangle.$$

Remark 2.4. By $(2,3), (2.4)$, for each $f \in \mathcal{L}(X, H)$ we have

(i) $\| \int_{HL(X)} f\, d\mu\| = \|f\|_{\mathcal{L}}$.

(ii) Since, for each $h \in H$

$$\int_X |\langle f(x), h\rangle|^2 d\mu = \Big\langle \int_{HL(X)} f\, d\mu \int^*_{HL(X)} f\, d\mu(h), h\Big\rangle = \Big\| \int^*_{HL(X)} f\, d\mu(h)\Big\|^2,$$

so

$$\inf_{h\in H_1} \Big\| \int^*_{HL(X)} f\, d\mu(h)\Big\|^2 \le \int_{HL(X)} f\, d\mu \int^*_{HL(X)} f\, d\mu \le \sup_{h\in H_1} \Big\| \int^*_{HL(X)} f\, d\mu(h)\Big\|^2.$$

(iii) Let $H = \mathbb{C}$ and $f \in \mathcal{L}(X, \mathbb{C}) = L^2(X)$. Then, the operator-valued integral of f over X is the bounded linear mapping $\int_{HL(X)} f\, d\mu : L^2(X) \to \mathbb{C}$, defined by

$$\int_{HL(X)} f\, d\mu(g) = \int_X f(x)g(x)\, d\mu = \langle f, \overline{g}\rangle_{L^2}, \quad g \in L^2(X),$$

with $\| \int_{HL(X)} f\, d\mu\| = \|f\|_2$. Also, $\int^*_{HL(X)} f\, d\mu : \mathbb{C} \to L^2(X)$ is defined by

$$\int^*_{HL(X)} f\, d\mu(c) = c\overline{f}, \quad c \in \mathbb{C},$$

where

$$\Big\| \int^*_{HL(X)} f\, d\mu(c)\Big\| = |c|\, \|f\|_2, \quad c \in \mathbb{C}.$$

Thus, for each $f \in L^2(X)$, the mapping $\int_{HL(X)} f\, d\mu : L^2(X) \to \mathbb{C}$ is surjective.

Definition 2.5. Let $f, g \in \mathcal{L}(X, H)$. We say that f, g are weakly equal, if

$$\int_{HL(X)} f\, d\mu = \int_{HL(X)} g\, d\mu,$$

which is equivalent with

$$\langle h, f\rangle = \langle h, g\rangle \quad a.e.$$

for each $h \in H$.

According to the definition of the normed space $\mathcal{L}(X,H)$, two members of $\mathcal{L}(X,H)$ are equal, if and only if, they are weakly equal.

Definition 2.6. Let $f,g \in \mathcal{L}(X,H)$. We say that f,g are strongly equal, if

$$\int_{HL(X)} f\,d\mu \int_{HL(X)}^{*} f\,d\mu = \int_{HL(X)} g\,d\mu \int_{HL(X)}^{*} g\,d\mu.$$

It is clear that each weakly equal mapping is also strongly equal, but its converse may be false.

Definition 2.7. Let H be a closed subspace of $L^2(X)$ and $f \in \mathcal{L}(X,H)$. We say that f is positive, if

$$\int_{HL(X)} f\,d\mu : L^2(X) \to L^2(X)$$

is a positive operator.

Lemma 2.8. *Let H be a Hilbert space. Then*

(i) *If $\dim H < \infty$ then $L^2(X,H) = \mathcal{L}(X,H)$.*
(ii) *If there exists $f \in L^2(X,H)$ with $\inf_{h\in H_1} \| \int_{HL(X)}^{*} f\,d\mu(h)\| > 0$ then*

$$\dim H < \infty.$$

Proof. Let $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal basis for H and $\dim H < \infty$. Let $f \in \mathcal{L}(X,H)$. We have

$$\int_X \|f(x)\|^2 d\mu = \int_X \sum_\alpha |\langle f(x), e_\alpha\rangle|^2 d\mu = \sum_\alpha \int_X |\langle f(x), e_\alpha\rangle|^2 d\mu.$$

Thus, we have

$$\begin{aligned} \inf_{h\in H_1} \Big\| \int_{HL(X)}^{*} f\,d\mu(h)\Big\|^2 \sum_\alpha \|e_\alpha\|^2 &\le \int_X \|f(x)\|^2 d\mu \\ &\le \sup_{h\in H_1} \Big\| \int_{HL(X)}^{*} f\,d\mu(h)\Big\|^2 \sum_\alpha \|e_\alpha\|^2. \end{aligned}$$

So

$$\int_X \|f(x)\|^2 d\mu \le \sup_{h\in H_1} \Big\| \int_{HL(X)}^{*} f\,d\mu(h)\Big\|^2 \dim H, \tag{2.5}$$

and

$$\inf_{h\in H_1} \Big\| \int_{HL(X)}^{*} f\,d\mu(h)\Big\|^2 \dim H \le \int_X \|f(x)\|^2 d\mu. \tag{2.6}$$

Hence, by (2.5), $f \in L^2(X,H)$.
(ii) is clear by (2.6). □

Lemma 2.9. *Let $f \in \mathcal{L}(X,H)$. Then the following assertions are equivalent:*

(i) *The operator $\int_{HL(X)} f\,d\mu \int_{HL(X)}^{*} f\,d\mu$ is invertible.*
(ii)

$$\inf_{h\in H_1} \Big\| \int_{HL(X)}^{*} f\,d\mu(h)\Big\| > 0.$$

(iii) *The operator $\int_{HL(X)} f\,d\mu$ is surjective.*

Proof. (i) ⇒ (ii) Let $\int_{HL(X)} f\,d\mu \int^*_{HL(X)} f\,d\mu$ be invertible. We have

$$\begin{aligned}
&\inf_{h\in H_1}\Big\|\int^*_{HL(X)} f\,d\mu(h)\Big\|^2\\
&\quad=\inf_{h\in H_1}\Big\langle \int_{HL(X)} f\,d\mu\int^*_{HL(X)} f\,d\mu(h),h\Big\rangle\in\sigma\Big(\int_{HL(X)} f\,d\mu\int^*_{HL(X)} f\,d\mu\Big).
\end{aligned}$$

So, $\inf_{h\in H_1}\|\int^*_{HL(X)} f\,d\mu(h)\|>0$.

(ii) ⇒ (iii) Let $\inf_{h\in H_1}\|\int^*_{HL(X)} f\,d\mu(h)\|>0$. We have

$$\inf_{h\in H_1}\Big\|\int^*_{HL(X)} f\,d\mu(h)\Big\|\|h\|\le\Big\|\int^*_{HL(X)} f\,d\mu(h)\Big\|,\quad h\in H.$$

Therefore, $\int_{HL(X)} f\,d\mu$ is surjective.

(iii) ⇒ (i) Let $\int_{HL(X)} f\,d\mu$ be surjective. Then, there exists $A>0$ such that

$$A\|h\|\le\Big\|\int^*_{HL(X)} f\,d\mu(h)\Big\|,\quad h\in H.$$

Hence

$$\inf_{h\in H_1}\Big\|\int^*_{HL(X)} f\,d\mu(h)\Big\|\ge A>0.$$

□

Lemma 2.10. *Let H be a Hilbert space. Then*

(i) *Let $f\in\mathcal{L}(X,H)$. Then $\int_{HL(X)} f\,d\mu=0$, if and only if, $f=0$ (weakly).*

(ii) *Let $f_1,f_2\in L^2(X,H)$ and let $\lambda_1,\lambda_2\in\mathbb{C}$. Then*

$$\int_{HL(X)}(\lambda_1f_1+\lambda_2f_2)\,d\mu=\lambda_1\int_{HL(X)} f_1d\mu+\lambda_2\int_{HL(X)} f_2d\mu.$$

Proof. It is evident. □

Lemma 2.11. *Let K be a Hilbert space, $f\in\mathcal{L}(X,H)$ and $u:H\to K$ be a bounded linear mapping. Then*

(i) *$uf\in\mathcal{L}(X,K)$ and*

$$u\int_{HL(X)} f\,d\mu=\int_{HL(X)} uf\,d\mu.$$

(ii) *Let $\inf_{h\in H_1}\|\int^*_{HL(X)} f\,d\mu(h)\|>0$. Then, $\inf_{h\in K_1}\|\int^*_{HL(X)} uf\,d\mu(h)\|>0$, if and only if, u is surjective.*

Proof. (i) Since

$$\sup_{h\in H_1}\int_X|\langle h,u(f(x))\rangle|^2d\mu\le\|u\|^2\sup_{h\in H_1}\int_X|\langle h,f(x)\rangle|^2d\mu,$$

so $uf\in\mathcal{L}(X,K)$. For each $g\in L^2(X)$, we have

$$\begin{aligned}
\Big\langle\int_{HL(X)} uf\,d\mu(g),k\Big\rangle&=\int_X g(x)\langle u(f(x)),k\rangle\,d\mu\\
&=\int_X g(x)\langle f(x),u^*(k)\rangle\,d\mu=\Big\langle u\int_{HL(X)} f\,d\mu(g),k\Big\rangle.
\end{aligned}$$

So, $\int_{HL(X)} uf\,d\mu=u\int_{HL(X)} f\,d\mu$.

(ii) If u is surjective then by Lemma 2.11 (iii), $u\int_{HL(X)} f\,d\mu$ is surjective. So

$$\inf_{h\in K_1}\Big\|\int^*_{HL(X)} uf\,d\mu(h)\Big\|>0.$$

Now, if $\inf_{h\in K_1}\|\int^*_{HL(X)} uf\,d\mu(h)\| > 0$ then $\int_{HL(X)} uf\,d\mu$ is surjective, so u is surjective. $\square$

Corollary 2.12. *Let for each $\alpha \in I, H_\alpha$ be a Hilbert space and $\oplus_{\alpha\in I} H_\alpha$ be the orthogonal sum of $\{H_\alpha\}_{\alpha\in I}$. Let $f \in \mathcal{L}(X, \oplus_{\alpha\in I} H_\alpha)$ and for each $\alpha \in I, f_\alpha = \pi_\alpha \circ f$. Then*

(i) *For each $\alpha \in I, f_\alpha \in \mathcal{L}(X, H_\alpha)$.*
(ii) $(\int_{HL(X)} f\,d\mu)_\alpha = \int_{HL(X)} f_\alpha d\mu$.

Proof. It is evident $\square$

3. Decomposition

In this section, we shall show more properties of operator-valued integrals of vector-functions.

Definition 3.1. Let $f \in \mathcal{L}(X, H)$ and $\mathcal{R}\int_{HL(X)} f\,d\mu$ be closed. We shall denote the pseudo-inverse of $\int_{HL(X)} f\,d\mu$ by $\int^\dagger_{HL(X)} f\,d\mu$. So for each $h \in \mathcal{R}\int_{HL(X)} f\,d\mu$

$$\int_{HL(X)} f\,d\mu \int^\dagger_{HL(X)} f\,d\mu(h) = h.$$

Theorem 3.2. *Let $f \in \mathcal{L}(X, H)$ and $f \neq 0$ (weakly). We have*

(i) *If $g \in \mathcal{L}(X, H)$ then the mapping $U : X \times X \to \mathbb{C}$ defined by*

$$U(x, y) = \langle f(x), g\rangle(y) = \langle f(x), g(y)\rangle,$$

defines a bounded operator on $L^2(X)$.

(ii) *Let $U : X \times X \to \mathbb{C}$ defines a bounded operator $W : L^2(X) \to L^2(X)$ as (i). Let $g : X \to H$ be defined by*

$$g(x) = \int_{HL(X)} f\,d\mu(U(x, .)).$$

Then g is defined for almost all $x \in X$ and $g \in \mathcal{L}(X, H)$. Let

$$\inf_{h\in H_1} \Big\| \int^*_{HL(X)} f\,d\mu(h)\Big\| > 0,$$

then $\inf_{h\in H_1}\|\int^*_{HL(X)} g\,d\mu(h)\| > 0$, if and only if, there exists $c > 0$ such that

$$\inf_{h\in H_1} \Big\| \int^*_{HL(X)} f\,d\mu(h)\Big\| \le c \inf_{h\in H_1} \Big\| \int^*_{HL(X)} g\,d\mu(h)\Big\|.$$

Proof. (i) Let $l \in L^2(X))$ and $x \in X$. We define

$$W_l(x) = \int_X U(x, y)l(y)\,d\mu_y = \int_X \langle f(x), g\rangle l\,d\mu_y.$$

Since, $f \in \mathcal{L}(X, H)$ and $\overline{W}_l(x) = \langle \int_{HL(X)} g\,d\mu(\bar{l}), f(x)\rangle$, W_l is measurable. Also, we have

$$\begin{aligned}\int_X |W_l(x)|^2 d\mu_x &= \int_X \Big|\Big\langle \int_{HL(X)} g\,d\mu(\bar{l}), f(x)\Big\rangle\Big|^2 d\mu_x \\ &\le \Big\|\int_{HL(X)} f\,d\mu\Big\|^2 \Big\|\int_{HL(X)} g\,d\mu(\bar{l})\Big\|^2 \\ &\le \Big\|\int_{HL(X)} f\,d\mu\Big\|^2 \Big\|\int_{HL(X)} g\,d\mu\Big\|^2 \|l\|^2.\end{aligned}$$

Thus, $W : L^2(X) \to L^2(X)$ defined by $W(l) = W_l$ is a bounded operator.

(ii) Since

$$\|W_l\| = \int_X |W_l(x)|^2 d\mu_x = \int_X \Big| \int_X U(x,y) l(y)\, d\mu_y \Big|^2 d\mu_x \le \|W\| \|l\|,$$

for almost all $x \in X$, $U(x,.)l \in L^1(X)$. So, for almost all $x \in X$, $U(x,.) \in L^2(X)$. Hence, g is defined for almost all $x \in X$. Since

$$\overline{\langle h, g(x)\rangle} = \int_X U(x,y) \langle f(y), h\rangle\, d\mu_y = W_{\overline{\langle h, f\rangle}}(x),$$

g is weakly measurable. But

$$\int_X |\langle h, g(x)\rangle|^2 d\mu_x = \int_X |W_{\overline{\langle h, f\rangle}}(x)|^2 d\mu_x \le \|W\| \|\langle h, f\rangle\|.$$

So, $g \in \mathcal{L}(X,H)$. If $\inf_{h\in H_1} \| \int^*_{HL(X)} g\, d\mu(h)\| > 0$ then

$$\begin{aligned} &\left(\inf_{h\in H_1} \Big\| \int^*_{HL(X)} g\, d\mu(h) \Big\|^2 \Big/ \sup_{h\in H_1} \Big\| \int^*_{HL(X)} f\, d\mu(h)\Big\|^2\right) \inf_{h\in H_1} \Big\| \int^*_{HL(X)} f\, d\mu(h)\Big\|^2 \|h\|^4 \\ &\le \left(\inf_{h\in H_1} \Big\| \int^*_{HL(X)} g\, d\mu(h) \Big\|^2 \Big/ \sup_{h\in H_1} \Big\| \int^*_{HL(X)} f\, d\mu(h)\Big\|^2\right) \Big\| \int^*_{HL(X)} f\, d\mu\Big\|^2 \|h\|^2 \\ &= \inf_{h\in H_1} \Big\| \int^*_{HL(X)} g\, d\mu(h)\Big\|^2 \|h\|^2 \le \Big\| \int^*_{HL(X)} g\, d\mu(h)\Big\|^2. \end{aligned}$$

Thus

$$\inf_{h\in H_1} \Big\| \int^*_{HL(X)} f\, d\mu(h)\Big\| \le c \inf_{h\in H_1} \Big\| \int^*_{HL(X)} g\, d\mu(h)\Big\|,$$

where

$$c = \left(\inf_{h\in H_1} \Big\| \int^*_{HL(X)} g\, d\mu(h) \Big\|^2 \Big/ \sup_{h\in H_1} \Big\| \int^*_{HL(X)} f\, d\mu(h)\Big\|^2\right)^{-1/2} > 0.$$

The converse is clear. □

Lemma 3.3. *Let $f \in \mathcal{L}(X,H)$ and $\inf_{h\in H_1} \| \int^*_{HL(X)} f\, d\mu(h)\| > 0$. Let*

$$u = \int_{HL(X)} f\, d\mu \int^*_{HL(X)} f\, d\mu.$$

Then

(i) *Let $l \in L^2(X)$. If $h = \int_{HL(X)} f\, d\mu(l)$ then*

$$\|l\|^2 = \int_X |\langle h, u^{-1} f(x)\rangle|^2 d\mu + \int_X |l(x) - \langle h, u^{-1} f(x)\rangle|^2 d\mu.$$

(ii) *For each $h \in H$, $\int^\dagger_{HL(X)} f\, d\mu(h) = \langle h, u^{-1} f\rangle$.*

(iii) $\| \int^\dagger_{HL(X)} f\, d\mu\|^{-2} = \inf_{h\in H_1} \| \int^*_{HL(X)} f\, d\mu(h)\|^2$.

Proof. (i) By the Lemma 2.11, $\int_{HL(X)} f\, d\mu(l - \langle h, u^{-1} f\rangle) = 0$. So

$$l - \langle h, u^{-1} f\rangle \in \ker \int_{HL(X)} f\, d\mu = \Big(\mathcal{R} \int^*_{HL(X)} f\, d\mu\Big)^\perp.$$

Since $\langle h, u^{-1} f\rangle \in \mathcal{R} \int^*_{HL(X)} f\, d\mu$,

$$\|l\|^2 = \|l - \langle h, u^{-1} f\rangle\|_2^2 + \|\langle h, u^{-1} f\rangle\|_2^2.$$

(ii) Since, $\int^{\dagger}_{HL(X)} f\, d\mu(h)$ is the unique solution of minimal norm of

$$\int_{HL(X)} f\, d\mu(l) = h,$$

so

$$\int_X |\langle l(x) - \langle h, u^{-1} f(x)\rangle|^2 d\mu = 0.$$

Hence $l = \langle h, u^{-1} f\rangle = \int^{\dagger}_{HL(X)} f\, d\mu(h)$.

(iii) Since, $\inf_{h\in H_1} \| \int^{*}_{HL(X)} f\, d\mu(h)\| > 0$, by the Lemma 2.11

$$\inf_{h\in H_1} \Big\| \int^{*}_{HL(X)} u^{-1} f\, d\mu(h)\Big\| > 0.$$

Therefore

$$\Big\| \int^{\dagger}_{HL(X)} f\, d\mu\Big\|^2 = \sup_{h\in H_1} \int_X |\langle h, u^{-1} f(x)\rangle^2 d\mu = \Big\| \int_{HL(X)} u^{-1} f\, d\mu \int^{*}_{HL(X)} u^{-1} f\, d\mu\Big\|$$
$$= \Big\| \Big(\int_{HL(X)} f\, d\mu \int^{*}_{HL(X)} f\, d\mu\Big)^{-1} \Big\| = \Big(\inf_{h\in H_1} \Big\| \int^{*}_{HL(X)} f\, d\mu(h)\Big\|^2 \Big)^{-1}.$$

□

Definition 3.4. Let $f, g \in \mathcal{L}(X, H)$. We define $\langle f, g\rangle_{\mathcal{L}} : X \to L^2(X)$ by

$$\langle f, g\rangle_{\mathcal{L}}(x) = \langle f(x), g\rangle.$$

Theorem 3.5. *Let $f, g \in \mathcal{L}(X, H)$. Then*

(i) $\int^{*}_{HL(X)} g\, d\mu f = \langle f, g\rangle_{\mathcal{L}}$.

(ii) $\langle f, g\rangle_{\mathcal{L}} \in \mathcal{L}(X, L^2(X))$.

(iii) *Let* $\inf_{h\in H_1} \| \int^{*}_{HL(X)} f\, d\mu(h)\| > 0$ *and* $K = \mathcal{R} \int^{*}_{HL(X)} g\, d\mu$ *be closed. Then*

$$\inf_{h\in K_1} \Big\| \int^{*}_{HL(X)} \langle f, g\rangle_{\mathcal{L}} d\mu(h)\Big\| > 0$$

and there exists a surjective bounded operator $u : L^2(X) \to H$ *such that* $g = u\langle g, f\rangle_{\mathcal{L}}$.

Proof. (i) Let $l \in L^2(X)$. For each $x \in X$, we have

$$\Big\langle l, \int^{*}_{HL(X)} g\, d\mu f(x)\Big\rangle = \Big\langle \int_{HL(X)} g\, d\mu(l), f(x)\Big\rangle = \int_X l(y)\langle g(y), f(x)\rangle\, d\mu_y$$
$$= \int_X l(y)\overline{\langle f(x), g(y)\rangle}\, d\mu_y = \langle l, \langle f(x), g\rangle\rangle_{L^2} = \langle l, \langle f, g\rangle_{\mathcal{L}}(x)\rangle_{L^2}.$$

Thus $\int^{*}_{HL(X)} g\, d\mu f = \langle f, g\rangle_{\mathcal{L}}$.

(ii) Let $l \in L^2(X)$. Since, the mapping

$$X \to \mathbb{C}, \quad x \mapsto \langle l, \langle f, g\rangle_{\mathcal{L}}(x)\rangle = \Big\langle l, \int^{*}_{HL(X)} g\, d\mu f(x)\Big\rangle = \Big\langle \int_{HL(X)} g\, d\mu(l), f(x)\Big\rangle$$

is measurable, $\langle f, g\rangle_{\mathcal{L}}$ is weakly measurable. Since

$$\int_X |\langle l, \langle f, g\rangle_{\mathcal{L}}(x)|^2 d\mu = \int_X \Big| \Big\langle l, \int^{*}_{HL(X)} g\, d\mu(f(x))\Big\rangle \Big|^2 d\mu$$
$$= \int_X \Big| \Big\langle \int_{HL(X)} g\, d\mu(l), f(x)\Big\rangle \Big|^2 d\mu \le \sup_{h\in H_1} \Big\| \int^{*}_{HL(X)} f\, d\mu(h)\Big\|^2 \Big\| \int_{HL(X)} g\, d\mu(l)\Big\|^2$$
$$\le \Big\| \int_{HL(X)} f\, d\mu\Big\|^2 \Big\| \int_{HL(X)} g\, d\mu\Big\|^2 \|l\|^2.$$

So, $\langle f,g\rangle_{\mathcal{L}} \in \mathcal{L}(X, L^2(X))$.

(iii) For each $l \in \mathcal{R}\int^*_{HL(X)} g\,d\mu$, we have

$$\begin{aligned}\|l\| &= \Big\|\int^*_{HL(X)} g\,d\mu\Big(\int^*_{HL(X)} g\,d\mu\Big)^{\dagger}(l)\Big\| \\ &= \Big\|\Big(\Big(\int^*_{HL(X)} g\,d\mu\Big)^{\dagger}\Big)^*\Big(\int_{HL(X)} g\,d\mu(l)\Big)\Big\| \\ &\leq \Big\|\int^{\dagger}_{HL(X)} g\,d\mu\Big\|\Big\|\int_{HL(X)} g\,d\mu(l)\Big\|.\end{aligned}$$

Hence

$$\Big\|\int^{\dagger}_{HL(X)} g\,d\mu\Big\|^{-1}\|l\| \leq \Big\|\int_{HL(X)} g\,d\mu(l)\Big\|.$$

Thus

$$\begin{aligned}&\inf_{h\in H_1}\Big\|\int^*_{HL(X)} f\,d\mu(h)\Big\|^2\Big\|\int^{\dagger}_{HL(X)} g\,d\mu\Big\|^{-2}\|l\|^2 \\ &\quad\leq \inf_{h\in H_1}\Big\|\int^*_{HL(X)} f\,d\mu(h)\Big\|^2\Big\|\int_{HL(X)} g\,d\mu(l)\Big\|^2 \\ &\quad\leq \int_X\Big|\Big\langle\int_{HL(X)} g\,d\mu(l), f(x)\Big\rangle\Big|^2 d\mu = \int_X |\langle l, \langle f,g\rangle_{\mathcal{L}}(x)\rangle|^2 d\mu.\end{aligned}$$

Hence

$$\inf_{l\in K_1}\Big\|\int^*_{HL(X)} \langle f,g\rangle_{\mathcal{L}} d\mu(l)\Big\| > 0.$$

We have the following retrieval formula

$$\begin{aligned}g &= \Big(\int_{HL(X)} f\,d\mu\int^*_{HL(X)} f\,d\mu\Big)^{-1}\int_{HL(X)} f\,d\mu\int^*_{HL(X)} f\,d\mu g \\ &= \Big(\int_{HL(X)} f\,d\mu\int^*_{HL(X)} f\,d\mu\Big)^{-1}\int_{HL(X)} f\,d\mu\langle g,f\rangle_{\mathcal{L}}.\end{aligned}$$

So, $g = u\langle g,f\rangle_{\mathcal{L}}$, where, $u = (\int_{HL(X)} f\,d\mu\int^*_{HL(X)} f\,d\mu)^{-1}\int_{HL(X)} f\,d\mu$ is a bounded surjective operator of $L^2(X)$ to H. □

Since, $\langle f,f\rangle_{\mathcal{L}} \in \mathcal{L}(X, L^2(X))$ is positive, we have the following corollary.

Corollary 3.6. *Let $f \in \mathcal{L}(X,H)$ with $\inf_{h\in H_1}\|\int^*_{HL(X)} f\,d\mu(h)\| > 0$, and*

$$K = \mathcal{R}\int^*_{HL(X)} f\,d\mu.$$

Then f can be written as $f = ug$, where $u : K \to H$ is a bounded operator, $g \in \mathcal{L}(X,K)$ is positive with $\inf_{h\in K_1}\|\int^*_{HL(X)} g\,d\mu(h)\| > 0$.

Theorem 3.7. *Let $f \in \mathcal{L}(X,H)$ with $\inf_{h\in H_1}\|\int^*_{HL(X)} f\,d\mu(h)\| > 0$, and $g \in L^2(X)$. Let $u = \int_{HL(X)} f\,d\mu\int^*_{HL(X)} f\,d\mu$. Then, $h = \int_{HL(X)} u^{-1} f\,d\mu(g)$ is the unique vector in H which minimizes the mapping*

$$H \to \mathbb{C}, \quad h \mapsto \int_X |g - \langle h,f\rangle|^2 d\mu.$$

Proof. Since, $\mathcal{R}\int^*_{HL(X)} f d\mu$ is closed and

$$\int_X |g - \langle h, f\rangle|^2 d\mu = \|g - \langle h, f\rangle\|_2^2,$$

it is enough to prove that the mapping

$$L^2(X) \to L^2(X), \quad g \mapsto \Big\langle \int_{HL(X)} u^{-1} f\, d\mu(g), f \Big\rangle$$

is the orthonormal projection of $L^2(X)$ onto $\mathcal{R}\int^*_{HL(X)} f\, d\mu$.

Let $g \in \mathcal{R}\int^*_{HL(X)} f\, d\mu^{\perp}$. Then

$$\Big\langle \int_{HL(X)} u^{-1} f\, d\mu(g), f \Big\rangle = \Big\langle u^{-1} \int_{HL(X)} f\, d\mu(g), f \Big\rangle = \Big\langle \int_{HL(X)} f\, d\mu(g), u^{-1} f \Big\rangle = 0.$$

Because, for each $x \in X$

$$\begin{aligned}\Big\langle \int_{HL(X)} f\, d\mu(g), u^{-1} f \Big\rangle(x) &= \int_X g(y) \langle f(y), u^{-1} f(x)\rangle\, d\mu \\ &= \langle g, \langle u^{-1} f(x), f\rangle\rangle_{L^2} = 0.\end{aligned}$$

Now, let $g \in \mathcal{R}\int^*_{HL(X)} f\, d\mu$. So, there exists $h \in H$ with $g = \langle h, f\rangle$. We have,

$$\Big\langle \int_{HL(X)} u^{-1} f\, d\mu(g), f \Big\rangle = \Big\langle u^{-1} \int_{HL(X)} f\, d\mu \int^*_{HL(X)} f\, d\mu(h), f \Big\rangle = \langle h, f\rangle = g,$$

and the theorem is proved. □

Theorem 3.8. *Let $e = \{e_\alpha\}_{\alpha \in X}$ be an orthonormal basis for H . Let $\{\delta_\alpha\}_{\alpha \in X}$ be the canonical orthonormal basis for $l^2(X)$. Let $u : H \to l^2(X)$ be the isomorphism which maps e_α to δ_α. Then*

(i) *Let $f \in \mathcal{L}(X, H)$ and $0 < \epsilon < 1$. Then, there exist orthonormal bases $e^i = \{e^i_\alpha\}_{\alpha \in X}, i = 1, 2, 3$ for H such that*

$$f = \frac{\| \int_{HL(X)} f\, d\mu\|}{1 - \epsilon}(e^1 + e^2 + e^3). \tag{3.1}$$

(ii) *Let $f \in \mathcal{L}(X, H)$ be positive (i.e. $uf \in \mathcal{L}(X, l^2(X))$ is positive) and $0 < \epsilon < 1$. Then there exist orthonormal bases $e^i = \{e^i_\alpha\}_{\alpha \in X}, i = 1, 2$ for H such that*

$$f = \frac{\| \int_{HL(X)} f\, d\mu\|}{2\epsilon}(e^1 + e^2). \tag{3.2}$$

Proof. (i) If $\| \int_{HL(X)} f\, d\mu\| = 0$ then $f = 0$ and (3.2) is satisfied. Now, let

$$\Big\| \int_{HL(X)} f\, d\mu \Big\| > 0.$$

Let $w : H \to H$ be defined by

$$w = \frac{1}{2} I + \frac{1 - \epsilon}{2} \frac{\int_{HL(X)} f\, d\mu u}{\| \int_{HL(X)} f\, d\mu\|}.$$

Since $\|I - w\| < 1$, w is invertible. So, by using the polar decomposition we can write $w = vp$, where v is a unitary and p is a positive operator. But, $\|p\| < 1$, so we can write $p = \frac{1}{2}(z + z^*)$, where z, z^* are unitary operators. Thus

$$\int_{HL(X)} f\, d\mu u = \frac{\| \int_{HL(X)} f\, d\mu\|}{1 - \epsilon}(vz + vz^* - I).$$

For each $h \in H$ we have

$$\left\langle \int_{HL(X)} f\, d\mu u(e_\alpha), h \right\rangle = \int_X \delta_\alpha(\beta)\langle f(\beta), h\rangle\, d\mu_\beta = \langle f(\alpha), h\rangle, \quad \alpha \in X.$$

Therefore

$$f = \int_{HL(X)} f\, d\mu u e = \frac{\| \int_{HL(X)} f\, d\mu\|}{1-\epsilon}(vze + vz^*e - e).$$

Since, vz and vz^* are unitary operators, vze and vz^*e are orthonormal bases for H. Thus

$$f = \frac{\| \int_{HL(X)} f\, d\mu\|}{1-\epsilon}(e^1 + e^2 + e^3),$$

where $e^i, i = 1, 2, 3$ are orthonormal bases for H.

(ii) Since $u \int_{HL(X)} f\, d\mu : l^2(X) \to l^2(X)$ is positive and u is a unitary,

$$\begin{aligned} u\int_{HL(X)} f\, d\mu &= \frac{\| \int_{HL(X)} uf\, d\mu\|}{2\epsilon}(w + w^*) \\ &= \frac{\|u \int_{HL(X)} f\, d\mu\|}{2\epsilon}(w + w^*) = \frac{\| \int_{HL(X)} f\, d\mu\|}{2\epsilon}(w + w^*), \end{aligned}$$

where w is an unitary operator. We have

$$f(\alpha) = \int_{HL(X)} f\, d\mu(\delta_\alpha) = \frac{\| \int_{HL(X)} f\, d\mu\|}{2\epsilon}(u^{-1}w(\delta_\alpha) + u^{-1}w^*(\delta_\alpha)), \quad \alpha \in X.$$

Thus

$$f = \frac{\| \int_{HL(X)} f\, d\mu\|}{2\epsilon}(e^1 + e^2).$$

where $e^i, i = 1, 2$ are orthonormal bases for H. □

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References

1. Sterling K. Berberian, *Lectures in Functional Analysis and Operator Theory*, Graduate Texts in Mathematics, 15, Springer-Verlag, New York—Heidelberg, 1974.
2. Ole Christensen, *An Introduction to Frames and Riesz Bases*, Applied and Numerical Harmonic Analysis, Birkhauser Boston, Inc., Boston, 2003.
3. Harro G. Heuser, *Functional Analysis*, A Wiley-Interscience Publication, John Wiley & Sons, Ltd., Chichester, 1982.
4. Gert K. Pedersen, *Analysis Now*, Graduate Texts in Mathematics, 118, Springer-Verlag, New York, 1989.
5. W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Co., New York, 1987.
6. W. Rudin, *Functional Analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York—Dusseldorf—Johannesburg, 1973.
7. S. Sakai, *C^*-algebras and W^*-algebras*, Reprint of the 1971 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1998.

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