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A DESCRIPTION OF CHARACTERS ON THE INFINITE WREATH PRODUCT

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Abstract. Let $\mathfrak{S}_\infty$ be the infinity permutation group and Γ an arbitrary group. Then $\mathfrak{S}_\infty$ admits a natural action on Γ^∞ by automorphisms, so one can form a semidirect product $\Gamma^\infty \rtimes \mathfrak{S}_\infty$, known as the *wreath* product $\Gamma \wr \mathfrak{S}_\infty$ of Γ by $\mathfrak{S}_\infty$. We obtain a full description of unitary II_1-factor-representations of $\Gamma \wr \mathfrak{S}_\infty$ in terms of finite characters of Γ. Our approach is based on extending Okounkov's classification method for admissible representations of $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$. Also, we discuss certain examples of representations of type *III*, where the *modular operator* of Tomita-Takesaki expresses naturally by the asymptotic operators, which are important in the theory of characters of $\mathfrak{S}_\infty$.

1. Introduction

1.1. **A definition of the wreath product.** Let $\mathbb{N}$ stand for the natural numbers. A bijection $s : \mathbb{N} \to \mathbb{N}$ is called *finite* if the set $\{i \in \mathbb{N} | s(i) \neq i\}$ is finite. Define $\mathfrak{S}_\infty$ as the group of all finite bijections $\mathbb{N} \to \mathbb{N}$ and set $\mathfrak{S}_n = \{s \in \mathfrak{S}_\infty | \ s(i) = i \ \forall \ i > n\}$. For every group Γ, an element of Γ^n can always be written as an ordered collection $[\gamma_k]_{k=1}^n = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, where $\gamma_k \in \Gamma$. Let e be the unit of Γ. For any $n > 1$ we identify the element $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}) \in \Gamma^{n-1}$ with $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, e) \in \Gamma^n$ and set $\Gamma_e^\infty = \varinjlim \Gamma^n$. One can view Γ_e^∞ as a group of infinite ordered collections $[\gamma_k]_{k=1}^\infty$ such that there are finitely many elements γ_k not equal to e. The *wreath* product $\Gamma \wr \mathfrak{S}_n$ is the semidirect product $\Gamma^n \rtimes \mathfrak{S}_n$ for the natural permutation action of $\mathfrak{S}_n$ on Γ^n (see [4]). In the same way, we define the group $\Gamma \wr \mathfrak{S}_\infty = \Gamma_e^\infty \rtimes \mathfrak{S}_\infty$. $\Gamma \wr \mathfrak{S}_\infty$ can be also viewed as the inductive limit $\varinjlim \Gamma \wr \mathfrak{S}_n$. Using the embedding $\gamma \in \Gamma^n \to (\gamma, \mathrm{id}) \in \Gamma \wr \mathfrak{S}_n$ and $s \in \mathfrak{S}_n \to \left(e^{(n)}, s\right) \in \Gamma \wr \mathfrak{S}_n$, where $e^{(n)} = (e, e, \ldots, e)$ and id is the identical bijection, we may identify Γ^n and $\mathfrak{S}_n$ with the corresponding subgroups in $\Gamma \wr \mathfrak{S}_n$. If Γ is a topological group, then we equip Γ^n with the natural product topology. Furthermore, we will always consider Γ_e^∞ as a topological group with the inductive limit topology. As a set, $\Gamma \wr \mathfrak{S}_\infty$ is just $\Gamma_e^\infty \times \mathfrak{S}_\infty$. Therefore, we equip $\Gamma \wr \mathfrak{S}_\infty$ with the product topology, considering $\mathfrak{S}_\infty$ as a discrete topological space.

1.2. **The results.** In this paper we give a full classification of *indecomposable* characters (see Definitions (3)–(4)) on $\Gamma \wr \mathfrak{S}_\infty$ (Theorem 9). Our approach is based on the semigroup method of Olshanski [7] and the ideas of Okounkov used in the study of *admissible* representations of the groups related to $\mathfrak{S}_\infty$ (see [2],[3]). We have noticed that two double cosets containing the transposition or $\gamma \in \Gamma$ commute, as the elements of Olshanski semigroup. (see Lemma 17). This observation enables one to develop Okounkov's method for the group $\Gamma \wr \mathfrak{S}_\infty$ (see Section 5). In Section 3 we discuss certain examples of representations of type *III*. The corresponding positive definite functions (p.d.f.) φ are not

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characters, but the following holds:

$$\varphi(sg) = \varphi(gs) \quad \text{for all} \quad g \in \Gamma \wr \mathfrak{S}_\infty \quad \text{and} \quad s \in \mathfrak{S}_\infty. \tag{1.1}$$

Hence the restriction $\varphi\big|_{\mathfrak{S}_\infty}$ is a character. At that, the Okounkov's asymptotic operators (see (4.4)) are naturally connected to the Tomita-Takesaki modular operator (see subsection 3.3). In fact, this observation is common for p.d.f. with the property (1.1). For those, we are going to produce a complete classification in a subsequent paper.

1.3. **The basic definition and the conjugacy classes.** Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ an algebra of all bounded operators in $\mathcal{H}$, and $\mathcal{I}_\mathcal{H}$ the identity operator in $\mathcal{H}$. We denote by $\mathcal{U}(\mathcal{H})$ the unitary subgroup in $\mathcal{B}(\mathcal{H})$. By a unitary representation of the topological group G we always mean a *continuous* homomorphism of G into $\mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ is equipped with the strong operator topology.

Definition 1. A unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of G is called a factor-representation if the W^*-algebra $\pi(G)''$ generated by the operators $\pi(g)$ $(g \in G)$, is a factor.

Definition 2. A unitary representation π is called a factor-representation of finite type if $\pi(G)''$ is a factor of type II_1.

Let $\mathcal{M}$ be a factor of type II_1 and $\mathcal{M}$ a subalgebra of $\mathcal{B}(\mathcal{H})$. If $\pi(G) \subset \mathcal{U}(\mathcal{M}) = \mathcal{M} \bigcap \mathcal{U}(\mathcal{H})$ and $\mathrm{tr}_\mathcal{M}$ is the unique normal, normalized $(\mathrm{tr}(I) = 1)$ trace on $\mathcal{M}$, then it determines a *character* $\phi_\pi^\mathcal{M}$ on G by $\phi_\pi^\mathcal{M}(g) = \mathrm{tr}_\mathcal{M}(\pi(g))$.

Definition 3. A continuous function ϕ on G is called a character if it satisfies the following properties:

(a) ϕ is central, that is, $\phi(g_1g_2) = \phi(g_2g_1)$ $\forall$ $g_1, g_2 \in G$;
(b) ϕ is positive definite, that is, for all $g_1, g_2, \ldots, g_n$ the matrix $\left[\phi\left(g_j g_k^{-1}\right)\right]_{j,k=1}^n$ is non-negatively definite;
(c) ϕ is normalized, that is, $\phi(e_G) = 1$, where e_G is the unit of G.

Definition 4. A character ϕ is called *indecomposable* if the group representation corresponding to ϕ (according to the GNS construction) is a factor-representation.

In this paper we obtain a complete description of indecomposable characters on $\Gamma \wr \mathfrak{S}_\infty$ in the case when Γ is a separable topological group.

First, let us describe the conjugacy classes in $\Gamma \wr \mathfrak{S}_\infty$. Recall that the conjugacy classes in $\mathfrak{S}_\infty$ are parametrized by non-increasing sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ of natural numbers such that there are finitely many elements λ_k not equal to 1. Namely, $\lambda_1, \lambda_2, \ldots$ are the orders of cycles of a permutation $s \in \mathfrak{S}_\infty$. Furthermore, an element $\Gamma \wr \mathfrak{S}_\infty$ can be written as a product of an element of $\mathfrak{S}_\infty$ and an element of Γ_e^∞, and the commutation rule between these two kinds of elements is as follows:

$$s \cdot \gamma = s \cdot (\gamma_1, \gamma_2, \ldots) = \left(\gamma_{s^{-1}(1)}, \gamma_{s^{-1}(2)}, \ldots\right) \cdot s, \tag{1.2}$$

where $s \in \mathfrak{S}_\infty$, $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$.

By the analogy with the definition of a cycle in $\mathfrak{S}_\infty$ define the generalized cycle in $\Gamma \wr \mathfrak{S}_\infty$.

Definition 5. Say that element $g = s \cdot \gamma \in \Gamma \wr \mathfrak{S}_\infty$, where $\gamma = (\gamma_1, \gamma_2, \ldots)$ is generalized cycle if s is cycle and $\{i \,|\, \gamma_i \neq e\} \subset \{i \,|\, s(i) \neq i\}$.

Let s be any permutation. Denote $\mathbb{N}/s$ the set of orbits of s on $\mathbb{N}$. Note that for $p \in \mathbb{N}/s$ the permutation s_p given by

$$s_p(k) = \begin{cases} s(k), & \text{if } k \in p, \\ k, & \text{otherwise,} \end{cases}$$

is a cycle of order $|p|$, where $|p|$ stand for the cardinality of p. For $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$ define the element $\gamma(p) = (\gamma_1(p), \gamma_2(p), \ldots) \in \Gamma_e^\infty$ as follows:

$$\gamma_k(p) = \begin{cases} \gamma_k, & \text{if } k \in p, \\ e, & \text{otherwise.} \end{cases} \tag{1.3}$$

Thus, using (1.2), we have the decomposition of $g = s \cdot \gamma$ onto generalized cycles

$$s \cdot \gamma = \prod_{p \in \mathbb{N} \diagup s} s_p \cdot \gamma(p). \tag{1.4}$$

For an arbitrary group G denote by $\mathfrak{c}_G(g)$ the conjugacy class of $g \in G$. Let $g = s \cdot \gamma \in \Gamma \wr \mathfrak{S}_\infty$. Note that for any orbit $p \in \mathbb{N}/s$ and any $k_p \in p$ the conjugacy class $\mathfrak{c}_\Gamma\left(\gamma_{k_p} \cdot \gamma_{s(k_p)} \cdots \gamma_{s^{(l)}(k_p)} \cdots \gamma_{s^{(|p|-1)}(k_p)}\right)$ does not depend on choice of k_p. Define the invariant $\mathfrak{i}(g)$ given by unordered ∞-tuples of pairs $\left\{\left(|p|, \mathfrak{c}_\Gamma\left(\gamma_{k_p} \cdot \gamma_{s(k_p)} \cdots \gamma_{s^{(l)}(k_p)} \cdots \gamma_{s^{(|p|-1)}(k_p)}\right)\right)\right\}_{p \in \mathbb{N} \diagup s}$, where $s^{(l)}$ is l-th iteration of s and k_p – any number from the orbit p. The following statement can be easily proved.

Proposition 6. *Let g_1 and g_2 be elements of $\Gamma \wr \mathfrak{S}_\infty$. Then $\mathfrak{c}(g_1) = \mathfrak{c}(g_2)$ if and only if $\mathfrak{i}(g_1) = \mathfrak{i}(g_2)$.*

For any $g = s \cdot \gamma \in \Gamma \wr \mathfrak{S}_\infty$ denote $\operatorname{supp}(g) = \{i \in \mathbb{N} | \, s(i) \neq i \text{ or } \gamma_i \neq e\}$ and call this set the *support* of g. Define for any $\iota \in \Gamma$ and $k \in \mathbb{N}$ the element $\iota(\{k\}) = (\iota_1(\{k\}), \iota_2(\{k\}), \ldots, \iota_l(\{k\}), \ldots) \in \Gamma_e^\infty$ as follows:

$$\iota_l(\{k\}) = \begin{cases} \iota, & \text{if } l = k, \\ e, & \text{otherwise.} \end{cases} \tag{1.5}$$

1.4. **The multiplicativity.** The following claim gives a useful characterization of the class of indecomposable characters:

Proposition 7. *The following assumptions on a character ϕ of $\Gamma \wr \mathfrak{S}_\infty$ are equivalent:*

(a) ϕ is indecomposable;
(b) $\phi(g) = \prod_{p \in \mathbb{N} \diagup s} \phi(s_p \cdot \gamma(p))$ for any $g = s \cdot \gamma = \prod_{p \in \mathbb{N} \diagup s} s_p \cdot \gamma(p)$ (see 1.4).

Proof. To prove the proposition, we consider the elements $g = s \cdot \gamma$ and $g' = s' \cdot \gamma'$ of $\Gamma \wr \mathfrak{S}_\infty$ such that $\operatorname{supp}(g) \cap \operatorname{supp}(g') = \emptyset$. Then by the properties of the group $\Gamma \wr \mathfrak{S}_\infty$ there exists a sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathfrak{S}_\infty$ such that

$$s_n \cdot g = g \cdot s_n \quad \text{and} \quad s_n \, g' s_n^{-1} \cdot h = h \cdot s_n \, g' s_n^{-1} \quad \text{for all} \quad h \in \Gamma \wr \mathfrak{S}_n. \tag{1.6}$$

Suppose now that (a) holds. Using the GNS-construction, we produce the representation π_ϕ of $\Gamma \wr \mathfrak{S}_\infty$ which acts in a Hilbert space $\mathcal{H}_\phi$ with a cyclic vector ξ_ϕ such that

$$\phi(g) = (\pi_\phi(g) \xi_\phi, \xi_\phi).$$

Let $A = w - \lim_{n \to \infty} \pi_\phi\left(s_n \cdot g' s_n^{-1}\right)$ be a limit of the sequence $\pi_\phi\left(s_n \cdot g' s_n^{-1}\right)$ in the *weak operator* topology. Using (1.6), we deduce by Definition 4 that $A = a\mathcal{I}$, where $\mathcal{I}$ is the identity operator in $\mathcal{H}_\phi$ and a a complex number. Therefore,

$$\phi(g \cdot g') = \lim_{n \to \infty} \phi\left(g \cdot s_n \cdot g' \cdot s_n^{-1}\right) = \phi(g) \cdot \lim_{n \to \infty} \phi\left(s_n \cdot g' \cdot s_n^{-1}\right) = \phi(g) \cdot \phi(g').$$

Thus *(b)* follows from *(a)*.

Conversely, suppose that *(b)* holds. For any subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, define its commutant as follows:

$$\mathcal{S}' = \left\{T \in \mathcal{B}(\mathcal{H}) \,\middle|\, ST = TS \text{ for all } S \in \mathcal{S}\right\}.$$

If $\pi_\phi\left(\Gamma \wr \mathfrak{S}_\infty\right)' \bigcap \pi_\phi\left(\Gamma \wr \mathfrak{S}_\infty\right)'' = \mathcal{Z}$ is larger than the scalars, then it contains a pair of orthogonal projections E and F with the properties

$$\phi\left(E\right) \neq 0, \quad \phi\left(F\right) \neq 0 \quad \text{and} \quad E \cdot F = 0. \tag{1.7}$$

By the von Neumann Double Commutant Theorem, for any $\varepsilon > 0$ there exist $g_k^E, g_k^F \in \Gamma \wr \mathfrak{S}_n \subset \Gamma \wr \mathfrak{S}_\infty$ ($n < \infty$) and complex numbers c_k^E, c_k^F ($k = 1, 2, \ldots, N < \infty$) such that

$$\begin{aligned} \left\|\sum_{k=1}^{N} c_k^E \pi_\phi\left(g_k^E\right)\xi_\phi - E\xi_\phi\right\| &< \varepsilon\phi(E), \\ \left\|\sum_{k=1}^{N} c_k^F \pi_\phi\left(g_k^F\right)\xi_\phi - F\xi_\phi\right\| &< \varepsilon\phi(F). \end{aligned} \tag{1.8}$$

Consider the bijection

$$\tau(j) = \begin{cases} j+n, & \text{if } j \leq n, \\ j-n, & \text{if } n < j \leq 2n, \\ j, & \text{otherwise.} \end{cases}$$

By Definition (3), use (1.8) to obtain

$$\begin{aligned} \left\|\sum_{k=1}^{N} c_k^E \pi_\phi\left(\tau g_k^E \tau\right)\xi_\phi - E\xi_\phi\right\| &< \varepsilon\phi(E), \\ \left\|\sum_{k=1}^{N} c_k^F \pi_\phi\left(\tau g_k^F \tau\right)\xi_\phi - F\xi_\phi\right\| &< \varepsilon\phi(F). \end{aligned} \tag{1.9}$$

Now, using *(b)*, (1.7), (1.8) and (1.9), we have

$$\begin{aligned} &\varepsilon\sqrt{\phi(E)\phi(F)}\left(\varepsilon\sqrt{\phi(E)\phi(F)} + \sqrt{\phi(E)} + \sqrt{\phi(F)}\right) \\ &> \left|\left(\sum_{k=1}^{N} c_k^E \pi_\phi\left(\tau g_k^E \tau\right) \cdot \sum_{k=1}^{N} c_k^F \pi_\phi\left(g_k^F\right)\xi_\phi, \xi_\phi\right)\right| \\ &> \left|\left(\sum_{k=1}^{N} c_k^E \pi_\phi\left(\tau g_k^E \tau\right)\xi_\phi, \xi_\phi\right) \cdot \left(\sum_{k=1}^{N} c_k^F \pi_\phi\left(\tau g_k^F \tau\right)\xi_\phi, \xi_\phi\right)\right| \\ &> \phi(E)\phi(F)(\varepsilon+1)^2. \end{aligned}$$

Hence

$$\varepsilon > \left[\frac{1-\sqrt{\phi(F)}}{\sqrt{\phi(F)}} + \frac{1-\sqrt{\phi(E)}}{\sqrt{\phi(E)}}\right]^{-1}.$$

Then, comparing this to (1.7), we get a contradiction. □

1.5. **The main result.** In [5], E. Thoma obtained the following remarkable description of all *indecomposable* characters of $\mathfrak{S}_\infty$. The characters of $\mathfrak{S}_\infty$ are labeled by pairs of non-increasing positive sequences of numbers $\{\alpha_k\}$, $\{\beta_k\}$ ($k \in \mathbb{N}$) (which are called the Thoma parameters) such that

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k \leq 1. \tag{1.10}$$

The value of the corresponding character on a permutation with a single cycle of length l is

$$\sum_{k=1}^{\infty} \alpha_k^l + (-1)^{l-1} \sum_{k=1}^{\infty} \beta_k^l.$$

Its value on a permutation with several disjoint cycles equals the product of values on each cycle.

Let $g = s \cdot \gamma$, $p \in \mathbb{N}/s$ be one of the orbits of s. Then put

$$\tilde{\gamma}(p) = \gamma_k \cdot \gamma_{s^{-1}(k)} \cdots \gamma_{s^{(-l)}(k)} \cdots \gamma_{s^{(-|p|+1)}(k)}, \quad \text{where} \quad (k \in p). \tag{1.11}$$

Now we define an analog of Thoma parameters for characters of the group $\Gamma \wr \mathfrak{S}_\infty$. Namely, let us call *Thoma parameters* the collection ϱ^0, $\{\varrho^{\alpha_k}\}$, $\left\{\varrho^{\beta_k}\right\}$, $\{\alpha_k\}$, $\{\beta_k\}$, where ϱ^0 is the representation of Γ of *finite* type, $\alpha = \{\alpha_k\}$, $\beta = \{\beta_k\}$ are non-increasing finite or infinite sequences of positive numbers, $\varrho^\alpha = \{\varrho^{\alpha_k}\}$ and $\varrho^\beta = \left\{\varrho^{\beta_k}\right\}$ are sequences of finite-dimensional irreducible representations of Γ such that $\sum\limits_k \left(\alpha_k \cdot \dim \varrho^{\alpha_k} + \beta_k \cdot \dim \varrho^{\beta_k}\right) \leq 1$.

For Thoma parameters ϱ^0, $\{\varrho^{\alpha_k}\}$, $\left\{\varrho^{\beta_k}\right\}$, $\{\alpha_k\}$, $\{\beta_k\}$ we define a function $\phi = \phi_{\varrho^0, \varrho^\alpha, \varrho^\beta, \alpha, \beta}$ by the next three properties:

(1) for $g = s \cdot \gamma \in \Gamma \wr \mathfrak{S}_\infty$ one has

$$\phi(g) = \prod_{p \in \mathbb{N}/s} \phi\left(s(p) \cdot \gamma(p)\right) \quad \text{(see (1.2) – (1.3))};$$

(2) for the generalized cycle $g = s \cdot \gamma$ (see definition 5) with $p = \operatorname{supp}(g)$ and $s \neq \mathrm{id}$ one has

$$\phi(g) = \sum_{k=1}^{\infty} \left(\alpha_k^{|p|} \cdot \operatorname{Tr}\left(\varrho^{\alpha_k}\left(\tilde{\gamma}(p)\right)\right) + (-1)^{|p|-1} \beta_k^{|p|} \cdot \operatorname{Tr}\left(\varrho^{\beta_k}\left(\tilde{\gamma}(p)\right)\right)\right);$$

(3) for each $\iota \in \Gamma$ and $n \in \mathbb{N}$ one has

$$\begin{aligned}\phi(\iota(\{n\})) = &\sum_{k=1}^{\infty} \left(\alpha_k \cdot \operatorname{Tr}\left(\varrho^{\alpha_k}(\iota)\right) + \beta_k \cdot \operatorname{Tr}\left(\varrho^{\beta_k}(\iota)\right)\right) \\ &+ \left(1 - \sum_{k \in \mathbb{N}} \left(\alpha_k \cdot \dim \varrho^{\alpha_k} + \beta_k \cdot \dim \varrho^{\beta_k}\right)\right) \operatorname{tr}_0(\iota) \quad \text{(see (1.5))},\end{aligned}$$

where Tr is the ordinary trace and tr_0 is the normalized character of the representation ϱ^0.

Proposition 8. *The function $\phi_{\varrho^0, \varrho^\alpha, \varrho^\beta, \alpha, \beta}$ is an indecomposable character (see definition 3).*

Proof. The realizations of the corresponding factor-representations we give in the section 2. □

Here is our main result.

Theorem 9. *If ϕ is an indecomposable character on $\Gamma \wr \mathfrak{S}_\infty$, then there exist Thoma parameters ϱ^0, $\{\varrho^{\alpha_k}\}$, $\left\{\varrho^{\beta_k}\right\}$, $\{\alpha_k\}$, $\{\beta_k\}$, such that $\phi = \phi_{\varrho^0, \varrho^\alpha, \varrho^\beta, \alpha, \beta}$.*

2. Realizations of II_1–factor-representations

A complete family of II_1–factor-representations of $\Gamma \wr \mathfrak{S}_\infty$ can be constructed using the Vershik-Kerov [8], Olshanski [7] realizations or Okunkov methods (so called mixtures of representations) [3], found for the II_1–factor-representations of the infinite symmetric group $\mathfrak{S}_\infty$. We follow the approach developed by Olshanski as it leads to less spadework.

2.1. **A construction of representations.** Let $\{\alpha_k\}_{k\in\mathbb{N}}$, $\{\beta_k\}_{k\in\mathbb{N}}$ be two finite or infinite sets of the positive numbers from $(0,1)$ and let ϱ^{α_k} and ϱ^{β_k} be unitary irreducible finite-dimensional representations of Γ that act in the Hilbert spaces $\mathcal{H}^{\alpha_k}$ and $\mathcal{H}^{\beta_k}$ correspondingly. We assume, that

$$\sum_k \alpha_k \cdot \dim \varrho^{\alpha_k} + \sum_k \beta_k \cdot \varrho^{\beta_k} \leq 1.$$

We set

$$\delta = 1 - \sum_k \alpha_k \cdot \dim \varrho^{\alpha_k} - \sum_k \beta_k \cdot \varrho^{\beta_k}.$$

Let $\mathcal{H}^0$ stand for the Hilbert space, where acts the unitary representation of a finite type ϱ^0 of Γ. Then the formula $\operatorname{tr}^0(\gamma) = \left(\varrho^0(\gamma)\xi^{(0)}, \xi^{(0)}\right)_{\mathcal{H}^0}$ defines the character on Γ. We denote by $\left(\varrho^{0k}, \mathcal{H}^{0k}, \xi^{(0k)}\right)$ the k-th copy of the triplet $\left(\varrho^0, \mathcal{H}^0, \xi^{(0)}\right)$.

Let $\left\{\mathrm{e}_j^{(\alpha_k)}\right\}_{1\leq j\leq \dim \mathcal{H}^{\alpha_k}}$ be an orthonormal basis in $\mathcal{H}^{\alpha_k}$. Let

$$\mathbf{H} = \left(\left(\bigoplus_k \mathcal{H}^{\alpha_k}\right) \oplus \left(\bigoplus_k \mathcal{H}^{\beta_k}\right) \oplus \left(\bigoplus_k \mathcal{H}^{0k}\right)\right) \otimes \left(\left(\bigoplus_k \mathcal{H}^{\alpha_k}\right) \oplus \left(\bigoplus_k \mathcal{H}^{\beta_k}\right) \oplus \left(\bigoplus_k \mathcal{H}^{0k}\right)\right)$$

and let

$$\eta^{(m)} = \sum_k \sqrt{\alpha_k}\left(\sum_j \mathrm{e}_j^{(\alpha_k)} \otimes \mathrm{e}_j^{(\alpha_k)}\right) + \sum_k \sqrt{\beta_k}\left(\sum_j \mathrm{e}_j^{(\beta_k)} \otimes \mathrm{e}_j^{(\beta_k)}\right) + \sqrt{\delta}\xi^{(0m)} \otimes \xi^{(0m)}.$$

Define the unitary representation ϱ of Γ in $\mathbf{H}$ as follows

$$\varrho = \left(\left(\bigoplus_k \varrho^{\alpha_k}\right) \oplus \left(\bigoplus_k \varrho^{\beta_k}\right) \oplus \left(\bigoplus_k \varrho^{0k}\right)\right) \otimes I, \tag{2.1}$$

.

We will identify $\mathcal{H}^{\alpha_k} \otimes \mathcal{H}^{\alpha_k}$, $\mathcal{H}^{\beta_k} \otimes \mathcal{H}^{\beta_k}$ and $\left(\bigoplus_k \mathcal{H}^{0k}\right) \otimes \left(\bigoplus_k \mathcal{H}^{0k}\right)$ with their images with respect in the natural embedding to $\mathbf{H}$. Denote by $\mathbf{H}^m$ the $m-$th copy of the Hilbert space $\mathbf{H}$ and consider the infinite tensor product

$$\breve{\mathbf{H}} = \bigotimes_m \left(\mathbf{H}^m, \eta^{(m)}\right).$$

It is convenient to represent $\breve{\mathbf{H}}$ as the closure of the linear span of the vectors of the form

$$\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{m-1} \otimes \eta^{(m)} \otimes \eta^{(m+1)} \otimes \cdots, \text{ where } \zeta_j \text{ is an any vector from } \mathbf{H}^j.$$

We extend the set $\left\{\mathrm{e}_j^{(\alpha_k)}\right\}_{j=1}^{\dim \mathcal{H}^{\alpha_k}} \bigcup \left\{\mathrm{e}_j^{(\beta_k)}\right\}_{j=1}^{\dim \mathcal{H}^{\beta_k}}$ to an orthonormal basis $\mathfrak{A}$ in $\left(\bigoplus_k \mathcal{H}^{\alpha_k}\right) \oplus \left(\bigoplus_k \mathcal{H}^{\beta_k}\right) \oplus \left(\bigoplus_k \mathcal{H}^{0k}\right)$. Now we fix the orthonormal basis

$$\mathfrak{B} = \{\mathrm{e}_j \otimes \mathrm{e}_l : \ \mathrm{e}_j, \mathrm{e}_l \in \mathfrak{A}\}$$

in $\mathbf{H}$ and we assume below $\zeta_j \in \mathfrak{B}$. Let components of the vector $\breve{\zeta} = \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{m-1} \otimes \cdots$ be of the form $\zeta_j = \upsilon_j \otimes \tau_j$. Define for $\mathfrak{s} \in \mathfrak{S}_\infty$ the vector $\mathfrak{s}\left(\breve{\zeta}\right) = \vartheta_1 \otimes \vartheta_2 \otimes \cdots \otimes \vartheta_{m-1} \otimes \cdots$ as follows:

$$\vartheta_j = \upsilon_{s^{-1}(j)} \otimes \tau_j.$$

Now build the sequence $\mathfrak{j}\left(\breve{\zeta}\right) = \{j_1 < j_2 < \cdots\}$ such, that

$$\zeta_{j_l} = \mathrm{e}_m^{(\beta_k)} \otimes \mathrm{f} \quad \text{for some} \quad \beta_k \quad \text{and} \quad m.$$

Let $\mathfrak{t}$ be a permutation for which $\mathfrak{s}\left(\left(j_{\mathfrak{t}(1)}\right)\right) < \mathfrak{s}\left(\left(j_{\mathfrak{t}(2)}\right)\right) < \cdots < \mathfrak{s}\left(\left(j_{\mathfrak{t}(l)}\right)\right) < \cdots$. Finally, we set $\psi\left(\mathfrak{s}, \breve{\zeta}\right) = \operatorname{sgn}(\mathfrak{t})$. The corresponding representation π of $\Gamma \wr \mathfrak{S}_\infty$ can be realized in Hilbert space $\breve{\mathbf{H}}$ as follows:

$$\begin{aligned} &\pi(\gamma)\left(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{m-1} \otimes \eta^{(m)} \otimes \cdots\right) \\ &\quad = \varrho(\gamma_1)\zeta_1 \otimes \varrho(\gamma_2)\zeta_2 \otimes \cdots \otimes \varrho(\gamma_{m-1})\zeta_{m-1} \otimes \varrho(\gamma_m)\eta^{(m)} \otimes \cdots \\ &\text{and for} \quad \mathfrak{s} \in \mathfrak{S}_\infty \quad \pi(\mathfrak{s})\left(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{m-1} \otimes \cdots\right) = \psi\left(\mathfrak{s}, \breve{\zeta}\right)\mathfrak{s}\left(\breve{\zeta}\right). \end{aligned} \tag{2.2}$$

2.2. **The character's formula.** Set $\breve{\eta} = \bigotimes\limits_m \eta^{(m)}$. Assume that $\mathfrak{s}$ is the cycle $(1 \to 2 \to 3 \to \cdots \to k-1 \to k)$, where $k > 1$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k, e_\Gamma, e_\Gamma, \ldots)$. Routine calculations provide that

$$\left(\pi(\mathfrak{s}\gamma)\breve{\eta}, \breve{\eta}\right) = \sum_j \alpha_j^k \operatorname{Tr}\left(\varrho^{\alpha_j}(\gamma_1\gamma_2\cdots\gamma_k)\right) + \sum_j \beta_j^k \operatorname{Tr}\left(\varrho^{\beta_j}(\gamma_1\gamma_2\cdots\gamma_k)\right), \tag{2.3}$$

where $\operatorname{Tr}(\varrho^r(\gamma)) = \sum\limits_{j=1}^{\dim \varrho^r} \varrho^r_{j\,j}(\gamma)$.

It is obvious, that

$$\left(\pi(\gamma)\breve{\eta}, \breve{\eta}\right) = \prod_{j=1}^k \left(\sum_i \alpha_i \operatorname{Tr}\left(\varrho^{\alpha_i}(\gamma_j)\right) + \sum_i \beta_i \operatorname{Tr}\left(\varrho^{\beta_i}(\gamma_j)\right) + \left(\varrho^0(\gamma_j)\xi^{(0)}, \xi^{(0)}\right)\right).$$

Since tr^0 is a character on Γ, one can use (2.3) and the multiplicativity property (see Proposition 7) to obtain the following

Corollary 10. *Let $\chi(g) = \left(\pi(g)\breve{\eta}, \breve{\eta}\right)$. Then χ is an indecomposable character on $\Gamma \wr \mathfrak{S}_\infty$.*

3. Other examples

In this section we construct examples of *infinite* type representations of $\mathbb{Z}_2 \wr \mathfrak{S}_\infty$. The corresponding positive definite functions are not characters. On the other hand they satisfy the following condition:

$$\varphi(sg) = \varphi(gs) \quad \text{for all} \quad g \in G = \Gamma \wr \mathfrak{S}_\infty \quad \text{and} \quad s \in \mathfrak{S}_\infty.$$

In the generic case the representation π_φ built by GNS-construction from φ is of type *III*. Furthermore, the state φ on the W^*-algebra $\pi_\varphi(G)''$ is faithful. These properties allow one to construct the Tomita-Takesaki modular operator Δ_φ. Surprisingly, Δ_φ is naturally related to the Okounkov operator $\mathcal{O}_k$ (see (4.4)), which is an important object in the representation theory of symmetric group (see [2], [3]).

3.1. **A construction.** Let $X_i = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{0,1\} \times \{0,1\}$. Define a probability measure ν_i on X_i by $\nu_i((k,l)) = p_{kl}$. Let $(X, \mu) = \prod\limits_i (X_i, \nu_i)$ and $x = (x_i) \in X$, where $x_i = \left(x_i^{(0)}, x_i^{(1)}\right) \in X_i$, $x_i^{(k)} \in \{0,1\}$. Define an action $\mathfrak{a}$ of $g = (s_0, s_1) \in \mathfrak{S}_\infty \times \mathfrak{S}_\infty$ on (X, μ) as follows:

$$\left(\mathfrak{a}_g(x)\right)_i^{(k)} = x_{s_k(i)}^{(k)} \quad (k = 0, 1).$$

Remark 1. The measure μ is $\mathfrak{S}_\infty \times \mathfrak{S}_\infty-$quasiinvariant if and only if $p_{ij} \neq 0$ for all $i, j = 0, 1$.

We are about to construct a unitary representation π_μ of $G\times G$ in $L^2(X,\mu)$. With $\varsigma\in L^2(X,\mu)$ set

$$(3.1)\qquad \begin{aligned}(\pi_\mu((s_0,s_1))\varsigma)(x) &= \left(\frac{d\mu(\mathfrak{a}_g(x))}{d\mu(x)}\right)^{\frac{1}{2}}\varsigma(\mathfrak{a}_g(x)),\\ \left(\pi_\mu\left(\left(\gamma^{(0)},\gamma^{(1)}\right)\right)\varsigma\right)(x) &= (-1)^{\left(\sum\limits_{i,k}\gamma_i^{(k)}x_i^{(k)}\right)}\varsigma(x),\end{aligned}$$

where $\gamma^{(0)}=\left(\gamma_i^{(0)}\right)\in\mathbb{Z}_2^\infty$, $\gamma^{(1)}=\left(\gamma_i^{(1)}\right)\in\mathbb{Z}_2^\infty$, and $\left(\gamma^{(0)},\gamma^{(1)}\right)\in\mathbb{Z}_2^\infty\times\mathbb{Z}_2^\infty$. Let $\pi_\mu^{(0)}(g)=\pi_\mu((g,e_G))$ and $\pi_\mu^{(1)}(g)=\pi_\mu((e_G,g))$.

Proposition 11. *π_μ is irreducible. Hence, $\pi_\mu^{(0)}$ and $\pi_\mu^{(1)}$ are factor-representations of $\Gamma\wr\mathfrak{S}_\infty$.*

Proof. Obvious. □

3.2. **A cyclic separating vector.** Let $\mathbb{I}$ be an element of $L^2(X,\mu)$ given by the function identically equal to 1.

Theorem 12. *If $\det[p_{ij}]\neq 0$, then $\mathbb{I}$ is a cyclic separating vector for $\pi_\mu^{(0)}(G)''$ and $\pi_\mu^{(1)}(G)''$. That is,*

$$\left[\pi_\mu^{(0)}(G)''\mathbb{I}\right]=\left[\pi_\mu^{(1)}(G)''\mathbb{I}\right]=L^2(X,\mu).$$

Proof. Let (k,l) be a transposition from $\mathfrak{S}_\infty$. First notice that the operator

$$\mathcal{O}_k^{(j)}=\lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^n\pi_\mu^{(j)}((k,l))\qquad\text{(see (4.4))}$$

belongs to $\pi_\mu^{(j)}(G)''$ $(j=0,1)$. Since

$$\left(L^2(X,\mu),\mathbb{I}\right)=\bigotimes_{i=1}^\infty\left(L^2(X_i,\nu_i),\mathbb{I}\right)$$

one can apply the law of large numbers to deduce that

$$\mathcal{O}_i^{(j)}=I\otimes I\otimes\cdots\otimes\underset{i-th}{\mathcal{O}_i^{(j,i)}}\otimes I\otimes\cdots$$

Furthermore, if $\chi_{kl}^{(i)}$ is the indicator of the point $(k,l)\in X_i=\mathbb{Z}_2\times\mathbb{Z}_2$, the matrices of $\mathcal{O}_i^{(0,i)}$ and $\mathcal{O}_i^{(1,i)}$ in the orthonormal basis $\left\{\mathrm{e}_{kl}^{(i)}=\frac{\chi_{kl}^{(i)}}{\sqrt{p_{kl}}}\right\}_{k,l=0,1}$ are as follows:

$$(3.2)\qquad \begin{aligned}\mathcal{O}_i^{(0,i)}&\leftrightarrow\begin{bmatrix}p_{00}+p_{01}&0&\sqrt{p_{00}p_{10}}+\sqrt{p_{01}p_{11}}&0\\0&p_{00}+p_{01}&0&\sqrt{p_{00}p_{10}}+\sqrt{p_{01}p_{11}}\\\sqrt{p_{00}p_{10}}+\sqrt{p_{01}p_{11}}&0&p_{10}+p_{11}&0\\0&\sqrt{p_{00}p_{10}}+\sqrt{p_{01}p_{11}}&0&p_{10}+p_{11}\end{bmatrix},\\ \mathcal{O}_i^{(1,i)}&\leftrightarrow\begin{bmatrix}p_{00}+p_{10}&\sqrt{p_{00}p_{01}}+\sqrt{p_{10}p_{11}}&0&0\\\sqrt{p_{00}p_{01}}+\sqrt{p_{10}p_{11}}&p_{01}+p_{11}&0&0\\0&0&p_{00}+p_{10}&\sqrt{p_{00}p_{01}}+\sqrt{p_{10}p_{11}}\\0&0&\sqrt{p_{00}p_{01}}+\sqrt{p_{10}p_{11}}&p_{01}+p_{11}\end{bmatrix}.\end{aligned}$$

By the construction,

$$\pi_\mu^{(k)}\left(\gamma^{(k)}\right)=\bigotimes_{i=1}^\infty\pi_\mu^{(k,i)}\left(\gamma_i^{(k)}\right),$$

where $\pi_\mu^{(0,i)}\left(\gamma_i^{(0)}\right)$ and $\pi_\mu^{(1,i)}\left(\gamma_i^{(1)}\right)$ are determined by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)^{\gamma_i^{(0)}} & 0 \\ 0 & 0 & 0 & (-1)^{\gamma_i^{(0)}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1)^{\gamma_i^{(1)}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (-1)^{\gamma_i^{(1)}} \end{bmatrix}. \tag{3.3}$$

Use the map

$$\mathfrak{I}_i : \sum_{m,n=0,1} a_{mn}\mathrm{e}_{mn}^{(i)} \to \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}, \tag{3.4}$$

to identify $L^2(X_i, \nu_i)$ to the full matrix algebra $M_2(\mathbb{C})$, so that

$$\mathfrak{I}_i(\mathbb{I}) = \begin{bmatrix} \sqrt{p_{00}} & \sqrt{p_{01}} \\ \sqrt{p_{10}} & \sqrt{p_{11}} \end{bmatrix}.$$

Equip $M_2(\mathbb{C})$ with the Hermitian form

$$\langle a, b \rangle_i = \mathrm{Tr}\,(b^* a),$$

then $\mathfrak{I}_i$ is a unitary and $\mathfrak{I}_i L^2(X_i, \nu_i) = M_2(\mathbb{C})$. Now as an elementary consequence of (3.2) and (3.3) one has:

$$\begin{gathered}
\mathfrak{I}_i\mathcal{O}_i^{(0,i)}\mathfrak{I}_i^{-1}a = \begin{bmatrix} p_{00}+p_{01} & \sqrt{p_{00}p_{10}}+\sqrt{p_{01}p_{11}} \\ \sqrt{p_{00}p_{10}}+\sqrt{p_{01}p_{11}} & p_{10}+p_{11} \end{bmatrix} a, \\
\mathfrak{I}_i\mathcal{O}_i^{(1,i)}\mathfrak{I}_i^{-1}a = a\begin{bmatrix} p_{00}+p_{10} & \sqrt{p_{00}p_{01}}+\sqrt{p_{10}p_{11}} \\ \sqrt{p_{00}p_{01}}+\sqrt{p_{10}p_{11}} & p_{01}+p_{11} \end{bmatrix}, \\
\mathfrak{I}_i\pi_\mu^{(0,i)}\left(\gamma_i^{(0)}\right)\mathfrak{I}_i^{-1}a = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{\gamma_i^{(0)}} \end{bmatrix} a, \\
\mathfrak{I}_i\pi_\mu^{(1,i)}\left(\gamma_i^{(1)}\right)\mathfrak{I}_i^{-1}a = a\begin{bmatrix} 1 & 0 \\ 0 & (-1)^{\gamma_i^{(1)}} \end{bmatrix}, \quad \text{where} \quad a \in M_2(\mathbb{C}).
\end{gathered} \tag{3.5}$$

Thus, in view of Remark 1 (see p. 307), the algebra $\mathfrak{M}_i^k$ generated by the operators $\mathfrak{I}_i\mathcal{O}_i^{(k,i)}\mathfrak{I}_i^{-1}$ and $\mathfrak{I}_i\pi_\mu^{(0,i)}\left(\gamma_i^{(k)}\right)\mathfrak{I}_i^{-1}$ is just $M_2(\mathbb{C})$. Since $\det(\mathfrak{I}_i(\mathbb{I})) \neq 0$, one has finally $\mathfrak{M}_i^0\mathfrak{I}_i(\mathbb{I}) = \mathfrak{M}_i^1\mathfrak{I}_i(\mathbb{I}) = M_2(\mathbb{C})$. □

3.3. **The modular operator.** Consider the Hilbert space $\mathfrak{H} = \bigotimes_{i=1}^{\infty}(M_2(\mathbb{C}), \langle\ \rangle_i, \mathfrak{I}_i(\mathbb{I}))$. It is convenient to represent $\mathfrak{H}$ as the closure of the linear span of the vectors $a_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes \mathfrak{I}_{i+1}(\mathbb{I}) \otimes \mathfrak{I}_{i+2}(\mathbb{I}) \cdots$, where $a_i \in M_2(\mathbb{C})$. If $\mathfrak{I} = \bigotimes_{i=1}^{\infty}\mathfrak{I}_i$, one has by Theorem 12

$$\mathfrak{I}L^2(X, \mu) = \mathfrak{H}.$$

Let $\mathcal{L}(\mathfrak{H})$ and $\mathcal{R}(\mathfrak{H})$ be the W^*-algebras generated in $\mathfrak{H}$ by the operators of left and right multiplication by elements of the form

$$a_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes I_2 \otimes I_2 \otimes \cdots, \quad \text{where} \quad a_i \in M_2(\mathbb{C}), \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proposition 13. *$\pi_\mu^{(0)}(G)'' = \mathfrak{I}^{-1}\mathcal{L}(\mathfrak{H})\mathfrak{I}$ and $\pi_\mu^{(1)}(G)'' = \mathfrak{I}^{-1}\mathcal{R}(\mathfrak{H})\mathfrak{I}$.*

Proof. Let $\mathfrak{A}_n^{(j)}$ stand for the W^*−algebra generated by the operators $\left\{\mathcal{O}_i^{(j)}\right\}_{i=1}^n$ and $\left\{\pi_\mu^{(j)}\left(\Gamma^n\right)\right\}$ $(j = 0,1)$. In view of (3.5), $\mathfrak{A}_n^{(j)}$ is isomorphic $\bigotimes\limits_{i=1}^n M_2(\mathbb{C})$. Therefore, $\pi_\mu^{(j)}\left(\mathfrak{S}_n\right) \subset \mathfrak{A}_n^{(j)}$. Finally, use (3.5) deduce $\mathfrak{A}_n^{(0)} \subset \mathcal{L}(\mathfrak{H})$ and $\mathfrak{A}_n^{(1)} \subset \mathcal{R}(\mathfrak{H})$. □

Let $\xi = \mathfrak{I}_1(\mathbb{I}) \otimes \mathfrak{I}_2(\mathbb{I}) \otimes \cdots \otimes \mathfrak{I}_{i+2}(\mathbb{I}) \otimes \cdots$. Since the vector ξ is cyclic and separating for $\mathcal{L}(\mathfrak{H})$ (Theorem 12), one can construct the modular operator Δ_ξ (see [9]). Namely, if S and F are closures of antilinear operators given by

$$S(a\xi) = a^*\xi \quad \text{for all} \quad a \in \mathcal{L}(\mathfrak{H}) \quad \text{and} \quad F(\xi a') = \xi\left(a'\right)^* \quad \text{for all} \quad a' \in \mathcal{R}(\mathfrak{H}),$$

then

$$F = S^* \quad \text{and} \quad \Delta_\xi = FS.$$

Hence, with $a = a_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes I_2 \otimes I_2 \otimes \cdots$ one has

$$a^*\xi = \xi \cdot \Big(\bigotimes_{j=1}^i \mathfrak{I}_j(\mathbb{I})\Big)^{-1} \otimes I_2 \otimes I_2 \otimes \cdots \cdot a^* \cdot \Big(\bigotimes_{j=1}^i \mathfrak{I}_j(\mathbb{I})\Big) \otimes I_2 \otimes I_2 \otimes \cdots .$$

Therefore,

$$\Delta_\xi\left(a\xi\right) = F\left(a^*\xi\right) = \xi \cdot \Big(\bigotimes_{j=1}^i \mathfrak{I}_j(\mathbb{I})\Big)^* \otimes I_2 \otimes \cdots \cdot a \cdot \Big(\bigotimes_{j=1}^i \left(\mathfrak{I}_j(\mathbb{I})\right)^*\Big)^{-1} \otimes I_2 \otimes \cdots .$$

Finally, use the relation $\mathfrak{I}_j(\mathbb{I})\left(\mathfrak{I}_j(\mathbb{I})\right)^* = \mathfrak{I}_j \mathcal{O}_j^{(0,j)} \mathfrak{I}_j^{-1}$ (see (3.5)) to obtain

$$\begin{aligned}\Delta_\xi\left(a\xi\right) = \bigotimes_{j=1}^i \left(\mathfrak{I}_j \mathcal{O}_j^{(0,j)} \mathfrak{I}_j^{-1}\right) a \Big(\bigotimes_{j=1}^i \mathfrak{I}_j \mathcal{O}_j^{(0,j)} \mathfrak{I}_j^{-1}\Big)^{-1} \\ \otimes \mathfrak{I}_{i+1}(\mathbb{I}) \otimes \mathfrak{I}_{i+2}(\mathbb{I}) \otimes \cdots .\end{aligned} \tag{3.6}$$

Thus the modular operator Δ_ξ is defined in a natural way by the Okounkov operator $\mathcal{O}_j$ (see (4.4), [2], [3]).

4. The characters of G and spherical functions of the pair $(G \times G, \operatorname{diag} G)$

In what follows, $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ is the unitary representation of $G = \Gamma \wr \mathfrak{S}_\infty$ that corresponds by GNS-construction to the character ϕ. In particular, the operators $\pi\left(G\right)$ act in $\mathcal{H}_\phi$ with *cyclic separating* vector ξ_ϕ. That is,

$$\left[\pi_\phi\left(G\right)\xi_\phi\right] = \left[\pi_\phi\left(G\right)'\xi_\phi\right] = \mathcal{H}_\phi, \tag{4.1}$$

where $[\mathcal{S}]$ stands for the closed subspace in $\mathcal{H}_\phi$ generated by $\mathcal{S}$. Moreover $\phi(g) = \left(\pi_\phi\left(g\right)\xi_\phi, \xi_\phi\right)$ for all $g \in G$.

The property (4.1) allows one to produce a unitary spherical representation $\pi_\phi^{(2)}$ of the Olshanski pair $(G \times G, K)$, where $K = \operatorname{diag} G = \{(g,g)\}_{g\in G}$. Namely,

$$\pi_\phi^{(2)}\left(g_1, g_2\right) x\xi_\phi = \pi_\phi\left(g_1\right) x \pi_\phi\left(g_2\right)^* \xi_\phi \quad \text{for all} \quad x \in \pi_\phi\left(G\right)'' . \tag{4.2}$$

Let

$$\begin{gathered}G_n(\infty) = \left\{g = s \cdot \gamma \in G \big| \; s(l) = l \text{ and } \gamma_l = e \text{ for all } l = 1, 2, \ldots, n\right\}, \\ K_n(\infty) = K \cap \left(G_n(\infty) \times G_n(\infty)\right), \quad G_n = \Gamma \wr \mathfrak{S}_n, \quad K_n = \left(G_n \times G_n\right) \cap K.\end{gathered}$$

It follows from the definition that $G_0(\infty) = G_\infty = G$, $K_0(\infty) = K_\infty = K$.

Set

$$\mathcal{H}_\phi^{K_n(\infty)} = \left\{\eta \in \mathcal{H}_\phi \big| \; \pi_\phi^{(2)}\left(g\right)\eta = \eta \text{ for all } g \in K_n(\infty)\right\},$$

and let P_n be the orthogonal projection onto $\mathcal{H}_\phi^{K_n(\infty)}$.

Lemma 14. *$\bigcup\limits_{n=0}^{\infty} \mathcal{H}_\phi^{K_n(\infty)}$ is a dense subspace in $\mathcal{H}_\phi$. In different terms, $\lim\limits_{n\to\infty} P_n = \mathcal{I}_{\mathcal{H}_\phi}$ in the strong operator topology.*

Proof. It follows from the definition of $\pi_\phi^{(2)}$ (see (4.2)) that

$$[\pi_\phi(G_n)\,\xi_\phi] \subset \mathcal{H}_\phi^{K_n(\infty)}. \tag{4.3}$$

On the other hand, ξ_ϕ is a cyclic vector. That is, $\left[\bigcup\limits_{n=1}^{\infty} \pi_\phi(G_n)\,\xi_\phi\right] = \mathcal{H}_\phi$. Now our statement follows from (4.3). □

Remind a construction of asymptotic operators which appears in [2], [3]. Consider the transposition $(i,n) \in \mathfrak{S}_\infty$ and the operator

$$\mathcal{O}_k = \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \pi_\phi\left((k,l)\right). \tag{4.4}$$

The limit exists in the strong operator topology.

Lemma 15. *Let $i(p)$ be an element of $p \in \mathbb{N}\diagup s$. Given any $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots) \in \Gamma_e^\infty$, there exists $\tilde{\gamma} \in \Gamma_e^\infty$ with the property $\tilde{\gamma}\cdot s\cdot\gamma\cdot\tilde{\gamma}^{-1} = s\cdot\gamma'$, where*

$$\gamma'_{s^{(l-1)}(i(p))} = e_\Gamma \quad \textit{for all} \quad l = 1, 2, \ldots, \quad |p|-1 \quad \textit{and} \quad p \in \mathbb{N}\diagup s,$$
$$\gamma'_{s^{(|p|-1)}(i(p))} = \gamma_{s^{(|p|-1)}(i(p))}\cdot\gamma_{s^{(|p|-2)}(i(p))}\cdots\gamma_{i(p)}.$$

Proof. Let the $\tilde{\gamma}$ be defined as follows:

$$\tilde{\gamma}_{i(p)} = e_\Gamma, \tilde{\gamma}_{s(i(p))} = \gamma_{i(p)}^{-1}, \tilde{\gamma}_{s^{(2)}(i(p))} = \gamma_{i(p)}^{-1}\cdot\gamma_{s(i(p))}^{-1}, \cdots,$$
$$\tilde{\gamma}_{s^{(|p|-1)}(i(p))} = \gamma_{i(p)}^{-1}\cdot\gamma_{s(i(p))}^{-1}\cdots\gamma_{s^{(|p|-2)}(i(p))} \quad \text{for all} \quad p \in \mathbb{N}\diagup s.$$

Now our statement can be readily verified. □

Lemma 16. *Let s be a cycle from $\mathfrak{S}_\infty$. Suppose that for $\beta, \gamma \in \Gamma_e^\infty$ the following relations hold:*

$$\beta_k = \gamma_k = e_\Gamma \quad \textit{for all} \quad k \in \left\{ j \in \mathbb{N} \middle| \, s(j) = j \right\}.$$

If $s\beta$ and $s\gamma$ are in the same conjugate class, then there exists $\tilde{\gamma} \in \Gamma_e^\infty$ such that $s\gamma = \tilde{\gamma}\cdot s\beta\cdot\tilde{\gamma}^{-1}$.

Proof. One may assume without loss of generality that

$$s(k) = k+1 \quad \text{for} \quad k = 1, 2, \ldots, m-1, \quad s(m) = 1 \quad \text{and} \quad s(l) = l \quad \text{for all} \quad l > m.$$

By Lemma 15 there exist $\tilde{\gamma}, \tilde{\beta} \in \Gamma_e^\infty$ with the properties

$$\begin{aligned} &\tilde{\gamma}\cdot s\cdot\gamma\cdot\tilde{\gamma}^{-1} = s\cdot\gamma',\ \tilde{\beta}\cdot s\cdot\beta\cdot\tilde{\beta}^{-1} = s\cdot\beta', \quad \text{where} \\ &\gamma'_k = \beta'_k = e_\Gamma \quad \text{for} \quad k = 1, 2, \ldots, m-1, m+1, \ldots. \end{aligned} \tag{4.5}$$

Let $s \in \mathfrak{S}_\infty$ and $\delta \in \Gamma_e^\infty$ be such that

$$(t\delta)\, s\gamma'\, (t\delta)^{-1} = s\beta'.$$

One has the following relations:

$$
\begin{array}{ccc}
\delta_2\gamma'_1 & = & \beta'_{t(1)}\delta_1 \\
\delta_3\gamma'_2 & = & \beta'_{t(2)}\delta_2 \\
\vdots & \vdots & \vdots \\
\delta_m\gamma'_{m-1} & = & \beta'_{t(m-1)}\delta_{m-1} \\
\delta_1\gamma'_m & = & \beta'_{t(m)}\delta_m .
\end{array}
\tag{4.6}
$$

By assumptions of the Lemma, $t(\{1,2,\ldots,m\}) = \{1,2,\ldots,m\}$, and we may assume that $t(k) = k$ for all $k > m$. Hence, there exists a map f from $\mathbb{N}$ to $\mathbb{N}$ such that

$$t(k) = s^{f(k)}(k) \quad \text{for} \quad k \in \mathbb{N}.$$

Now use the relation $ts = st$ to obtain

$$f(k) = l \quad \text{for} \quad k = 1,2,\ldots,m. \tag{4.7}$$

Since s^m is the identity, it suffices to consider the case $l \in \{1,2,\ldots,m-1\}$.

Use (4.6) to obtain

$$\delta_1 = \cdots = \delta_{m-l}, \quad \delta_{m-l+1} = \cdots = \delta_m,$$
$$\beta'_m = \delta_m\delta_1^{-1}, \quad \gamma'_m = \delta_1^{-1}\delta_m.$$

These relations together with (4.5) yield the following relation:

$$\delta'\ s\gamma'\ (\delta')^{-1} = s\beta', \quad \text{where} \quad \delta' = \left(\delta_m^{-1}\delta_1, \delta_m^{-1}\delta_1, \ldots, \delta_m^{-1}\delta_1, \ldots\right).$$

□

5. A proof of the main result

The proof of Theorem 9 splits into a few lemmas.

For each *indecomposable* character ϕ let $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ denote the cyclic representation of the group $\Gamma \wr \mathfrak{S}_\infty$ associated to ϕ via the GNS-construction.

Lemma 17. *If a W^*-algebra $\mathfrak{A}$ is generated by the operators $\pi_\phi(\Gamma_e^\infty)$, $\{\mathcal{O}_j\}_{j\in\mathbb{N}}$, and $\mathcal{C}(\mathfrak{A})$ is a center of $\mathfrak{A}$, then $\{\mathcal{O}_j\}_{j\in\mathbb{N}} \subset \mathcal{C}(\mathfrak{A})$.*

Proof. The relation $\mathcal{O}_k \cdot \mathcal{O}_l = \mathcal{O}_l \cdot \mathcal{O}_k$ allows an easy verification by definition (4.4) (see [2] or [3]).

Now prove the relation

$$\mathcal{O}_l \cdot \pi_\phi(\gamma) = \pi_\phi(\gamma) \cdot \mathcal{O}_l \quad \text{for all} \quad \gamma \in \Gamma_e^\infty \quad \text{and} \quad l \in \mathbb{N}. \tag{5.1}$$

Let $K_n^{\mathfrak{S}}(\infty) = K_n(\infty) \cap (\mathfrak{S}_\infty \times \mathfrak{S}_\infty)$ and $K_n^{\mathfrak{S}}(m) = K_n^{\mathfrak{S}}(\infty) \cap (G_m \times G_m)$, where $m > n$. If $P_n^{\mathfrak{S}}$ stands for the orthogonal projection onto $\mathcal{H}_\phi^{K_n^{\mathfrak{S}}(\infty)}$, then

$$P_n^{\mathfrak{S}} = \lim_{m\to\infty} \frac{1}{(m-n)!} \sum_{g \in K_n^{\mathfrak{S}}(m)} \pi_\phi^{(2)}(g) \tag{5.2}$$

in the strong operator topology and $P_n^{\mathfrak{S}} \geq P_n$[1]. Hence, using (4.4) and (5.2), we obtain for $i \leq n < k$

$$P_n^{\mathfrak{S}}\mathcal{O}_i P_n^{\mathfrak{S}} = P_n^{\mathfrak{S}}\pi_\phi((i,k))\,P_n^{\mathfrak{S}} \quad \text{and} \quad P_n\mathcal{O}_i P_n = P_n\pi_\phi((i,k))\,P_n. \tag{5.3}$$

In the case when $\gamma_l = e$ the equality (5.1) easily follows from (4.4). Therefore, it suffices to prove (5.1) for the elements $\gamma = \gamma(\{l\})$ (see (1.3)).

[1]See the page 311 (4.3) for definition of P_n

If $i \leq n < k$, then, using (4.4), we have

$$\begin{aligned}
P_n\pi_\phi(\gamma(\{i\}))\mathcal{O}_iP_n &\overset{\{P_n^{\mathfrak{S}}\geq P_n\}}{=} P_nP_n^{\mathfrak{S}}\pi_\phi(\gamma(\{i\}))\mathcal{O}_iP_n^{\mathfrak{S}}P_n \\
&\overset{\{(4.4),(5.2)\}}{=} P_n\pi_\phi(\gamma(\{i\}))P_n^{\mathfrak{S}}\pi_\phi((i,k))P_n^{\mathfrak{S}}P_n \\
&= P_nP_n^{\mathfrak{S}}\pi_\phi((i,k))\pi_\phi(\gamma(\{k\}))P_n^{\mathfrak{S}}P_n \\
&= P_nP_n^{\mathfrak{S}}\pi_\phi((i,k))\pi_\phi(\gamma(\{k\}))\pi_\phi^{(2)}\left(\left(\gamma(\{k\})^{-1},\gamma(\{k\})^{-1}\right)\right)P_n \\
&\overset{(4.2)}{=} P_nP_n^{\mathfrak{S}}\pi_\phi^{(2)}\left(\left(e,\gamma(\{k\})^{-1}\right)\right)\pi_\phi((i,k))P_n \\
&= P_n\pi_\phi^{(2)}((\gamma(\{k\}),\gamma(\{k\})))\pi_\phi^{(2)}\left(\left(e,\gamma(\{k\})^{-1}\right)\right)\pi_\phi((i,k))P_n \\
&= P_n\pi_\phi(\gamma(\{k\}))\pi_\phi((i,k))P_n = P_n\pi_\phi((i,k))\pi_\phi(\gamma(\{i\}))P_n \\
&\overset{(4.4)}{=} P_n\mathcal{O}_i\pi_\phi(\gamma(\{i\}))P_n.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} P_n = \mathcal{I}_{\mathcal{H}_\phi}$ (see Lemma 14), the relation

$$\pi_\phi(\gamma(\{i\}))\mathcal{O}_i = \mathcal{O}_i\pi_\phi(\gamma(\{i\}))$$

follows. □

We use the notation $(i_0, i_1, \ldots, i_{q-1})$ for the cyclic permutation s which acts as follows

$$s(i) = \begin{cases} i_{k+1(\operatorname{mod} q)}, & \text{if } i = i_k \in \{i_0, i_1, \ldots, i_{q-1}\}, \\ i, & \text{otherwise.}\end{cases}$$

Lemma 18. *If $\mathcal{O}_i$ is defined as in (4.4) and*

$$\mathbb{D}(m,n,q) = \left\{\overrightarrow{k} = (k_1, k_2, \ldots, k_q) \in \mathbb{N} \middle| k_i \neq k_j \text{ and } m < k_i \leq n\ \forall i, j = 1, \ldots, q\right\},$$

then for every positive integer m

$$\mathcal{O}_i^q = \lim_{n\to\infty}\frac{1}{n^q}\sum_{\overrightarrow{k}\in\mathbb{D}(m,n,q)}\pi_\phi((k_q, k_{q-1}, \ldots, k_1, i)).$$

Proof. If we notice that

$$(i,k_1)\cdot(i,k_2)\cdots(i,k_q) = (k_q, k_{q-1}, \ldots, k_1, i)$$

for pairwise different $i, k_1, k_2, \ldots, k_q$ and $\operatorname{Card}(\mathbb{D}(m,n)) = \prod\limits_{j=0}^{q-1}(n-m-j)$, the proof becomes obvious. □

Lemma 19. *Let $g = \prod\limits_{p\in\mathbb{N}/s} s_p\cdot\gamma(p)$ be a decomposition of $g = s\cdot\gamma \in \Gamma\wr\mathfrak{S}_\infty$ (see (1.4)) and $i(p)$ any element from $p \in \mathbb{N}/s$. Define $\gamma^{(i(p))} \in \Gamma_e^\infty$ as follows:*

$$(5.4)\quad \gamma_k^{(i(p))} = \begin{cases}\gamma_{i(p)}\cdot\gamma_{s^{-1}(i(p))}\cdots\gamma_{s^{(-|p|+2)}(i(p))}\cdot\gamma_{s^{(-|p|+1)}(i(p))}, & \text{if } k = i(p), \\ e, & \text{otherwise.}\end{cases}$$

If ϕ is an indecomposable character on $\Gamma\wr\mathfrak{S}_\infty$, then

$$(5.5)\qquad \left(\pi_\phi(s\cdot\gamma)\prod_j\mathcal{O}_j^{r_j}\xi_\phi,\xi_\phi\right) = \prod_{p\in\mathbb{N}/s}\left(\pi_\phi\left(\gamma^{(i(p))}\right)\mathcal{O}_{i(p)}^{|p|-1+\sum_{j\in p}r_j}\xi_\phi,\xi_\phi\right).$$

Proof. By Proposition 7 we have

$$(5.6)\qquad \left(\pi_\phi(s\cdot\gamma)\prod_j\mathcal{O}_j^{r_j}\xi_\phi,\xi_\phi\right) = \prod_{p\in\mathbb{N}/s}\left(\pi_\phi(s_p\cdot\gamma(p))\prod_{j\in p}\mathcal{O}_j^{r_j}\xi_\phi,\xi_\phi\right).$$

Therefore it suffices to prove (5.5) in the case when s is a single cycle and $\gamma = \gamma(p)$, where $p \in \mathbb{N}/s$ and $|p| > 1$. Let $s = \left(i_1, i_2, \ldots, i_{|p|}\right)$. By a virtue of Lemma 16, we find $\tilde{\gamma} \in \Gamma_e^\infty$ such that

$$\tilde{\gamma} \cdot s \cdot \gamma \cdot \tilde{\gamma}^{-1} = s \cdot \gamma^{(i_1)}. \tag{5.7}$$

Thus, by Lemma 17,

$$\left(\pi_\phi\left(s \cdot \gamma\right) \prod_{j \in p} \mathcal{O}_j^{r_j} \xi_\phi, \xi_\phi\right) = \left(\pi_\phi\left(\gamma^{(i_1)}\right) \pi_\phi\left(s\right) \prod_{j \in p} \mathcal{O}_j^{r_j} \xi_\phi, \xi_\phi\right). \tag{5.8}$$

Let

$$\mathfrak{S}_\infty^j = \left\{\tau \in \mathfrak{S}_\infty \middle| \tau(j) = j\right\}.$$

Now use Lemma 18 to obtain

$$\begin{aligned}
&\left(\pi_\phi\left(\gamma^{(i_1)}\right) \pi_\phi\left(s\right) \prod_{j \in p} \mathcal{O}_j^{r_j} \xi_\phi, \xi_\phi\right) \\
&\quad = \lim_{n \to \infty} \frac{1}{n^q} \sum_{\overrightarrow{k} \in \mathbb{D}(m,n,q)} \Bigg(\pi_\phi\left(\gamma^{(i_1)}\right) \pi_\phi\bigg(\Big(k_{r_{i_1}}^{(i_1)}, k_{r_{i_1}-1}^{(i_1)}, \ldots, k_1^{(i_1)}, i_2, \\
&\quad k_{r_{i_2}}^{(i_2)}, \ldots, k_1^{(i_2)}, i_3, \ldots, i_{|p|}, k_{r_{i_{|p|}}}^{\left(i_{|p|}\right)}, \ldots, k_1^{\left(i_{|p|}\right)}, i_1\Big)\bigg) \xi_\phi, \xi_\phi\Bigg),
\end{aligned}$$

where

$$\overrightarrow{k} = \left(k_{r_{i_1}}^{(i_1)}, k_{r_{i_1}-1}^{(i_1)}, \ldots, k_1^{(i_1)}, k_{r_{i_2}}^{(i_2)}, \ldots, k_1^{(i_2)}, \ldots, k_{r_{i_{|p|}}}^{\left(i_{|p|}\right)}, \ldots, k_1^{\left(i_{|p|}\right)}\right), \quad q = \sum_{j \in p} r_j.$$

Hence, by the relation $\tau \cdot \gamma^{(i_1)} \tau^{-1} = \gamma^{(i_1)}$ $\left(\tau \in \mathfrak{S}_\infty^{i_1}\right)$, we have

$$\begin{aligned}
&\left(\pi_\phi\left(\gamma^{(i_1)}\right) \pi_\phi\left(s\right) \prod_{j \in p} \mathcal{O}_j^{r_j} \xi_\phi, \xi_\phi\right) \\
&\quad = \lim_{n \to \infty} \frac{1}{n^{q'}} \sum_{\overrightarrow{k} \in \mathbb{D}(m,n,q')} \Bigg(\pi_\phi\left(\gamma^{(i_1)}\right) \pi_\phi\bigg(\Big(k_{r_{i_1}}^{(i_1)}, k_{r_{i_1}-1}^{(i_1)}, \ldots, k_1^{(i_1)}, i_2, \\
&\quad k_{r_{i_2}}^{(i_2)}, \ldots, k_1^{(i_2)}, i_3, \ldots, i_{|p|}, k_{r_{i_{|p|}}}^{\left(i_{|p|}\right)}, \ldots, k_1^{\left(i_{|p|}\right)}, i_1\Big)\bigg) \xi_\phi, \xi_\phi\Bigg),
\end{aligned}$$

where

$$\overrightarrow{k} = \left(k_{r_{i_1}}^{(i_1)}, k_{r_{i_1}-1}^{(i_1)}, \ldots, k_1^{(i_1)}, i_2, k_{r_{i_2}}^{(i_2)}, \ldots, k_1^{(i_2)}, i_3, \ldots, i_{|p|}, k_{r_{i_{|p|}}}^{\left(i_{|p|}\right)}, \ldots, k_1^{\left(i_{|p|}\right)}\right),$$
$$q' = |p| - 1 + \sum_{j \in p} r_j.$$

This relation, in view of Lemma 18, implies the statement of Lemma 19. □

We use the notation $\mathfrak{A}_j$ for the W^*-algebra generated by $\pi_\phi(\gamma)$, $\gamma = (e, \cdots, e, \gamma_j, e, \cdots)$, and $\mathcal{O}_j$. Given an operator A from $\mathfrak{A}_j$, denote by $A^{(k)}$ its copy in $\mathfrak{A}_k$:

$$A^{(k)} = \pi_\phi\left((j,k)\right) A \pi_\phi\left((j,k)\right) \quad \left(A^{(j)} = A\right).$$

The next assertion follows from Lemma 19.

Lemma 20. *Let s, $i(p)$ be the same as in Lemma 19. If $A_j, B_j \in \mathfrak{A}_j$, then*

$$\begin{aligned}&\left(\pi_\phi\left(s\right)\prod_j A_j\xi_\phi, \prod_j B_j\xi_\phi\right)\\ &= \prod_{p\in\mathbb{N}/s}\left(A^{(i(p))}_{i(p)}\left(B^{(i(p))}_{i(p)}\right)^* A^{(i(p))}_{s^{-1}(i(p))}\left(B^{(i(p))}_{s^{-1}(i(p))}\right)^*\right.\\ &\left.\cdots A^{(i(p))}_{s^{1-|p|}(i(p))}\left(B^{(i(p))}_{s^{1-|p|}(i(p))}\right)^* \mathcal{O}^{|p|-1}_{i(p)}\xi_\phi,\ \xi_\phi\right).\end{aligned} \tag{5.9}$$

The following lemma is an analogue of Theorem 1 from [3].

Lemma 21. *Let $\Delta = [a, b]$ be an interval in $[-1, 0]$ or in $[0, 1]$ with the property $\min\{|a|, |b|\} > \varepsilon > 0$. If $E^{(i)}_\Delta$ is a spectral projection of $\mathcal{O}_i$ corresponding to Δ, then for any orthogonal projection E from $\mathfrak{A}_i$ one has $\left(EE^{(i)}_\Delta\xi_\phi, \xi_\phi\right)^2 \geq \varepsilon\left(EE^{(i)}_\Delta\xi_\phi, \xi_\phi\right)$.*

Proof. Using Lemmas 17 and 20, we have

$$\begin{aligned}&\left|\left(\pi_\phi\left((i, i+1)\right)EE^{(i)}_\Delta\xi_\phi, EE^{(i)}_\Delta\xi_\phi\right)\right|\\ &= \left|\left(\mathcal{O}_iEE^{(i)}_\Delta\xi_\phi, EE^{(i)}_\Delta\xi_\phi\right)\right| > \varepsilon\left|\left(EE^{(i)}_\Delta\xi_\phi, \xi_\phi\right)\right|.\end{aligned} \tag{5.10}$$

On the other hand, under the assumption $E^{(i+1)} = \pi_\phi\left((i, i+1)\right)E\pi_\phi\left((i, i+1)\right)$, one has

$$EE^{(i)}_\Delta \cdot E^{(i+1)}E^{(i+1)}_\Delta \cdot \pi_\phi\left((i, i+1)\right) = \pi_\phi\left((i, i+1)\right) \cdot EE^{(i)}_\Delta \cdot E^{(i+1)}E^{(i+1)}_\Delta.$$

Therefore,

$$\begin{aligned}&\left|\left(\pi_\phi\left((i, i+1)\right)EE^{(i)}_\Delta\xi_\phi, EE^{(i)}_\Delta\xi_\phi\right)\right|\\ &= \left|\left(\pi_\phi\left((i, i+1)\right)E^{(i+1)}E^{(i+1)}_\Delta EE^{(i)}_\Delta\xi_\phi, \xi_\phi\right)\right|\\ &\leq \left|\left(E^{(i+1)}E^{(i+1)}_\Delta EE^{(i)}_\Delta\xi_\phi, \xi_\phi\right)\right| \overset{(\text{Prop. } 7)}{=} \left(EE^{(i)}_\Delta\xi_\phi, \xi_\phi\right)^2.\end{aligned}$$

Hence, using (5.10), we obtain our statement. □

The following statement is well known (see [3]) and also follows from Lemma 21.

Corollary 22. *There exists at most countable set of numbers α_i, β_i from $(0, 1)$ and a set of pairwise orthogonal projections $\left\{E^{(k)}\left(\alpha_i\right), E^{(k)}\left(\beta_i\right)\right\} \subset \mathfrak{A}_k$ such that*

$$\mathcal{O}_k = \sum \alpha_iE^{(k)}\left(\alpha_i\right) - \sum \beta_iE^{(k)}\left(\beta_i\right). \tag{5.11}$$

The following assertion is an analogue of Theorem 2 from [3].

Lemma 23. *Let r be a number from $\{\alpha_i, \beta_i\}$ and let E be any projection from $\mathfrak{A}_k$. If $\left(E \cdot E^{(k)}(r)\xi_\phi, \xi_\phi\right) = r\nu(r) \neq 0$, then $\nu(r) \in \mathbb{Z}$.*

Proof. For completeness of the proof, we use the arguments of Kerov, Olshanski, Vershik and Okounkov from [1] and [3].

For any $m \in \mathbb{N}$, define the projection $e_m(r)$ as follows:

$$e_m(r) = \prod_{j=1}^m E^{(j)} \cdot E^{(j)}(r), \quad \text{where}$$

$$E^{(j)} = \pi_\phi\left((j, k)\right)E\pi_\phi\left((j, k)\right), \quad E^{(j)}(r) = \pi_\phi\left((j, k)\right)E^{(k)}(r)\pi_\phi\left((j, k)\right).$$

Let $\mathbb{P}_m(s)$ be the set of orbits s on $\{1, 2, \ldots, m\}$. If $s \in \mathfrak{S}_m$, then by Lemma 20 we obtain

$$\left(\pi_\phi(s)e_m(r)\xi_\phi,\ e_m(r)\xi_\phi\right) = \nu(r)^{|\mathbb{P}_m(s)|}\prod_{p\in\mathbb{P}_m(s)} r^{|p|}. \tag{5.12}$$

Set $\phi_r(s) = \dfrac{(\pi_\phi(s)e_m(r)\xi_\phi,\ e_m(r)\xi_\phi)}{(e_m(r)\xi_\phi,\ e_m(r)\xi_\phi)}$. Using (5.12), we have

$$\phi_r(s) = \frac{\nu(r)^{|\mathbb{P}_m(s)|}}{\nu(r)^m}. \tag{5.13}$$

Therefore, ϕ_r is an indecomposable character on $\mathfrak{S}_\infty$ in view of Proposition 7.

We following G. Olshanski (see [6]) in expounding the proof of the following formula:

$$\sum_{s\in\mathfrak{S}_m} \operatorname{sgn}(s)\, t^{|\mathbb{P}_m(s)|} = t(t-1)\cdots(t-m+1). \tag{5.14}$$

For that, we consider the canonical projection $p_{m,m-1}$ from $\mathfrak{S}_m$ onto $\mathfrak{S}_{m-1}$

$$\left(p_{m,m-1}(s)\right)(i) = \begin{cases} s(i), & \text{if } s(i) < m, \\ s(m), & \text{if } s(i) = m. \end{cases}$$

Since $|\mathbb{P}_{m-1}\left(p_{m,m-1}(s)\right)| = |\mathbb{P}_m(s)|$ when $s \notin \mathfrak{S}_{m-1}$, and $|\mathbb{P}_{m-1}\left(p_{m,m-1}(s)\right)| = |\mathbb{P}_m(s)| - 1$ when $s \in \mathfrak{S}_{m-1}$, then

$$\begin{aligned} \sum_{s\in\mathfrak{S}_m} \operatorname{sgn}(s)\, t^{|\mathbb{P}_m(s)|} &= \sum_{s\in\mathfrak{S}_{m-1}} \ \sum_{\tilde{s}\in\mathfrak{S}_m:\, p_{m,m-1}(\tilde{s})=s} \operatorname{sgn}(s)\, t^{|\mathbb{P}_m(s)|} \\ &= t\cdot \sum_{s\in\mathfrak{S}_{m-1}} t^{|\mathbb{P}_m(s)|} - (m-1)\cdot \sum_{s\in\mathfrak{S}_{m-1}} t^{|\mathbb{P}_m(s)|} = (t-m+1) \sum_{s\in\mathfrak{S}_{m-1}} t^{|\mathbb{P}_m(s)|}. \end{aligned}$$

Hence (5.14) is now accessible by an elementary induction argument.

We follow the idea of A. Okounkov in considering the orthogonal projection

$$\operatorname{Alt}_r(m) = \frac{1}{m!} \sum_{s\in\mathfrak{S}_m} \operatorname{sgn}(s)\, \pi_{\phi_r}(s).$$

Since $\sum_{s\in\mathfrak{S}_m} \operatorname{sgn}(s)\, \phi_r(s) \geq 0$, then, using (5.13) and (5.14), we obtain for $r > 0$

$$\nu(r)\cdot(\nu(r)-1)\cdots(\nu(r)-m+1) \geq 0 \quad \text{for all} \quad m \in \mathbb{N}.$$

Thus, we get a contradiction in the case $\nu(r) > 0$. The opposite case $\nu(r) < 0$ can be considered in a similar way. For that, one should use the formula

$$\sum_{s\in\mathfrak{S}_m} t^{|\mathbb{P}_m(s)|} = t(t+1)\cdots(t+m-1) \quad \text{(see [6])}$$

and consider the projection

$$\operatorname{Sym}_r(m) = \frac{1}{m!} \sum_{s\in\mathfrak{S}_m} \pi_{\phi_r}(s).$$

□

Proof of Theorem 9. Let $E_k(r)$ be the spectral projection of $\mathcal{O}_k$ (see (4.4), (5.11)). By Lemma 23, for $r \neq 0$ the W^*-algebra $E_k(r)\mathfrak{A}_k$ (see p. 314) is finite-dimensional. On the other hand, use Lemma 17 to obtain the unitary representation $\left(E_k(r)\pi_\phi\Big|_\Gamma, E_k(r)\mathcal{H}_\phi\right)$ of the group Γ in the space $E_k(r)\mathcal{H}_\phi$. Thus, the representations ϱ^r for $r \neq 0$ as in Theorem 9 are the irreducible components of $\left(E_k(r)\pi_\phi\Big|_\Gamma, E_k(r)\mathcal{H}_\phi\right)$. The formula for characters follows from Lemmas 17 and 20. Finally, for each character as in Theorem 9 we construct the realization as in Section 2. □

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