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p-ADIC FRACTIONAL DIFFERENTIATION OPERATOR WITH POINT INTERACTIONS

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Dedicated to 100th birthday anniversary of Mark Krein.

Abstract. Finite rank point perturbations of the p-adic fractional differentiation operator D^α are studied. The main attention is paid to the description of operator realizations (in $L_2(\mathbb{Q}_p)$) of the heuristic expression $D^\alpha + \sum_{i,j=1}^n b_{ij} < \delta_{x_j}, \cdot > \delta_{x_i}$ in a form that is maximally adapted for the preservation of physically meaningful relations to the parameters b_{ij} of the singular potential.

1. Introduction

The conventional description of the physical space-time uses the field $\mathbb{R}$ of real numbers. In most cases, mathematical models based on $\mathbb{R}$ provide quite satisfactory descriptions of the physical reality. However, the result of a physical measurement is always a rational number, so the use of the completion $\mathbb{R}$ of the field of rational numbers $\mathbb{Q}$ is not more than a mathematical idealization. On the other hand, by Ostrovski's theorem, the only reasonable alternative to $\mathbb{R}$ among completions of $\mathbb{Q}$ is the fields $\mathbb{Q}_p$ of p-adic numbers (definition of $\mathbb{Q}_p$ see below in Section 2). For this reason, it is natural to use p-adic analysis in physical situations, where the conventional space-time geometry is known to fail, for examples in the attempts to understand the matter at sub-Planck distances or time intervals. In order to do this, at first, it is necessary to develop p-adic counterparts of the standard quantum mechanics and quantum field theory.

There are many works devoted to such an activity (see the surveys in [13], [17]). However, in spite of a considerable success obtained in recent years, many interesting problems of p-adic quantum mechanics are still unsolved and wait for a comprehensive study.

In the present paper, we are going to continue the investigation of the p-adic fractional differentiation operator with point interactions started by A. Kochubei [12], [13].

In 'usual' mathematical physics, point interactions Hamiltonians are the operator realizations in $L_2(\mathbb{R}^r)$ of differential expressions $-\Delta+V_Y$ or, more generally, $(-\Delta)^k+V_Y$, where a zero-range potential $V_Y = \sum_{i,j=1}^n b_{ij} < \delta_{x_j}, \cdot > \delta_{x_i}$ $(b_{ij} \in \mathbb{C})$ contains the Dirac delta functions δ_x concentrated at points x_i of the subset $Y = \{x_1, \dots, x_n\} \subset \mathbb{R}^r$ [1].

Since there exists a p-adic analysis based on the mappings from $\mathbb{Q}_p$ into $\mathbb{Q}_p$ and an analysis connected with the mapping $\mathbb{Q}_p$ into the field of complex numbers $\mathbb{C}$, there exist two types of p-adic physical models. The present paper deals with the mapping $\mathbb{Q}_p \to \mathbb{C}$, i.e., complex-valued functions defined on $\mathbb{Q}_p$ will be considered. In this case, the operation of differentiation is *not defined* and the operator of fractional differentiation D^α of order

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α ($\alpha > 0$) plays a corresponding role [13], [17]. In particular, p-adic Schrödinger-type operators with potentials $V(x) : \mathbb{Q}_p \to \mathbb{C}$ are defined as $D^\alpha + V(x)$ [13].

The definition of D^α is given in the framework of the p-adic distribution theory with the help of Schwartz-type distributions $\mathcal{D}'(\mathbb{Q}_p)$. One of remarkable features of this theory is that any distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ with point support $\operatorname{supp} f = \{x\}$ ($x \in \mathbb{Q}_p$) coincides with the Dirac delta function at the point x multiplied by a constant $c \in \mathbb{C}$, i.e., $f = c\delta_x$. For this reason, it is natural to consider the expression $D^\alpha + V_Y$ as a p-adic analogue of Hamiltonians with finite rank point interactions.

In the present paper, the main attention is paid to the description of operator realizations of $D^\alpha + V_Y$ in $L_2(\mathbb{Q}_p)$ in a form that is maximally adapted for the preservation of physically meaningful relations to the parameters b_{ij} of the singular potential $V_Y = \sum_{i,j=1}^n b_{ij} < \delta_{x_j}, \cdot > \delta_{x_i}$.

In Section 2, we recall some elements of p-adic analysis [17], [13] needed for reading the paper and establish a connection between α and the property of functions from $\mathcal{D}(D^\alpha)$ to be continuous. The same problem is also analyzed for the solutions of $D^\alpha + I = \delta$.

Section 3 contains a description of the Friedrichs extension of the symmetric operator associated with $D^\alpha + V_Y$ (this description depends on α) and the description of operator realizations of $D^\alpha + V_Y$ in $L_2(\mathbb{Q}_p)$. Taking into account an intensive development of consistent physical theories of quantum mechanics on the base of pseudo-Hermitian Hamiltonians that are not Hermitian in the standard sense but satisfy a less restrictive and more physical condition of symmetry in last few years [6], [16], we do not restrict ourselves to the case of self-adjoint operators and consider the more general case of η-self-adjoint operator realizations of $D^\alpha + V_Y$ (Theorem 3.1).

We use the following notations: $\mathcal{D}(A)$ and $\ker A$ denote the domain and the null-space of the linear operator A, respectively. $A \upharpoonright_X$ means the restriction of A onto a set X.

2. Fractional differential operator D^α

2.1. **Elements of p-adic analysis.** Basically, we shall use notations from [17]. Let us fix a prime number p. The field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to p-adic norm $|\cdot|_p$, which is defined as follows: $|0|_p = 0$; $|x|_p = p^{-\gamma}$ if an arbitrary rational number $x \neq 0$ is represented as $x = p^\gamma \frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and integers m and n are not divisible by p. The p-adic norm $|\cdot|_p$ satisfies the strong triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$. Moreover, $|x+y|_p = \max(|x|_p, |y|_p)$ if $|x|_p \neq |y|_p$.

Any p-adic number $x \neq 0$ can be uniquely represented as a series,

$$x = p^{\gamma(x)} \sum_{i=0}^{+\infty} x_i p^i, \quad x_i = 0, 1, \ldots, p-1, \quad x_0 > 0, \quad \gamma(x) \in \mathbb{Z}, \tag{2.1}$$

convergent in the p-adic norm (the canonical representation of x). In this case, $|x|_p = p^{-\gamma(x)}$.

The canonical representation (2.1) enables one to determine the fractional part $\{x\}_p$ of $x \in \mathbb{Q}_p$ by the rule $\{x\}_p = 0$ if $x = 0$ or $\gamma(x) \geq 0$; $\{x\}_p = p^{\gamma(x)} \sum_{i=0}^{-\gamma(x)-1} x_i p^i$ if $\gamma(x) < 0$.

Denote by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p \mid |x-a|_p \leq p^\gamma\} \quad \text{and} \quad S_\gamma(a) = \{x \in \mathbb{Q}_p \mid |x-a|_p = p^\gamma\},$$

respectively, the ball and the sphere of radius p^γ with the center at a point $a \in \mathbb{Q}_p$ and set $B_\gamma(0) = B_\gamma$, $S_\gamma(0) = S_\gamma$, $\gamma \in \mathbb{Z}$.

The ring $\mathbb{Z}_p$ of p-adic integers coincides with the disc B_0 ($\mathbb{Z}_p = B_0$), which is the completion of integers with respect to the p-adic norm $|\cdot|_p$.

As usual, in order to define some classes of distributions on $\mathbb{Q}_p$, one has first to introduce an appropriate class of test functions.

A complex-valued function f defined on $\mathbb{Q}_p$ is called *locally-constant* if for any $x \in \mathbb{Q}_p$ there exists an integer $l(x)$ such that $f(x+x') = f(x), \ \forall x' \in B_{l(x)}$.

Denote by $\mathcal{D}(\mathbb{Q}_p)$ the linear space of locally constant functions on $\mathbb{Q}_p$ with compact supports. For any test function $\phi \in \mathcal{D}(\mathbb{Q}_p)$ there exists $l \in \mathbb{Z}$ such that $\phi(x+x') = \phi(x)$, $x' \in B_l$, $x \in \mathbb{Q}_p$. The largest of such numbers $l = l(\phi)$ is called the parameter of constancy of ϕ. Typical examples of test functions are indicator functions of spheres and balls,

$$\delta(|x|_p - p^\gamma) := \begin{cases} 1, & x \in S_\gamma, \\ 0, & x \notin S_\gamma, \end{cases} \qquad \Omega(|x|_p) := \begin{cases} 1, & |x|_p \le 1, \\ 0, & |x|_p > 1. \end{cases} \tag{2.2}$$

In order to furnish $\mathcal{D}(\mathbb{Q}_p)$ with a topology, let us consider a subspace $\mathcal{D}^l_\gamma \subset \mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in the ball B_γ and the parameter of constancy $\ge l$. The convergence $\phi_n \to 0$ in $\mathcal{D}(\mathbb{Q}_p)$ has the following meaning: $\phi_k \in \mathcal{D}^l_\gamma$, where the indices l and γ do not depend on k and ϕ_k tends uniformly to zero. This convergence determines the Schwartz topology in $\mathcal{D}(\mathbb{Q}_p)$.

Denote by $\mathcal{D}'(\mathbb{Q}_p)$ the set of all linear functionals (Schwartz-type distributions) on $\mathcal{D}(\mathbb{Q}_p)$. In contrast to distributions on $\mathbb{R}^n$, any linear functional $\mathcal{D}(\mathbb{Q}_p) \to \mathbb{C}$ is automatically continuous. The action of a functional f upon a test function ϕ will be denoted by $< f, \phi >$.

It follows from the definition of $\mathcal{D}(\mathbb{Q}_p)$ that any test function $\phi \in \mathcal{D}(\mathbb{Q}_p)$ is continuous on $\mathbb{Q}_p$. This means the Dirac delta function $< \delta_x, \phi >= \phi(x)$ is well defined for any point $x \in \mathbb{Q}_p$.

On $\mathbb{Q}_p$ there exists the Haar measure, i.e., a positive measure $d_p x$ invariant under shifts, $d_p(x+a) = d_p x$, and normalized by the equality $\int_{|x|_p \le 1} d_p x = 1$.

Denote by $L_2(\mathbb{Q}_p)$ the set of measurable functions f on $\mathbb{Q}_p$ satisfying the condition $\int_{\mathbb{Q}_p} |f(x)|^2 d_p x < \infty$. The set $L_2(\mathbb{Q}_p)$ is a Hilbert space with the scalar product $(f, g)_{L_2(\mathbb{Q}_p)} = \int_{\mathbb{Q}_p} f(x)\overline{g(x)} d_p x$.

The Fourier transform of $\phi \in \mathcal{D}(\mathbb{Q}_p)$ is defined by the formula

$$F[\phi](\xi) = \widetilde{\phi}(\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x)\phi(x)\, d_p x, \quad \xi \in \mathbb{Q}_p,$$

where $\chi_p(\xi x) = e^{2\pi i \{\xi x\}_p}$ is an additive character of the field $\mathbb{Q}_p$ for any fixed $\xi \in \mathbb{Q}_p$. The Fourier transform $F[\cdot]$ maps $\mathcal{D}(\mathbb{Q}_p)$ onto $\mathcal{D}(\mathbb{Q}_p)$. Its extension by continuity onto $L_2(\mathbb{Q}_p)$ determines an unitary operator in $L_2(\mathbb{Q}_p)$.

The Fourier transform $F[f]$ of a distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ is defined by the standard relation $< F[f], \phi >=< f, F[\phi] >, \ \forall \phi \in \mathcal{D}(\mathbb{Q}_p)$. The Fourier transform is a linear isomorphism of $\mathcal{D}'(\mathbb{Q}_p)$ onto $\mathcal{D}'(\mathbb{Q}_p)$.

2.2. **The operator D^α.** The operator of differentiation is not defined in $L_2(\mathbb{Q}_p)$. Its role is played by the operator of fractional differentiation D^α (the Vladimirov pseudo-differential operator) which is defined as

$$D^\alpha f = \int_{\mathbb{Q}_p} |\xi|_p^\alpha F[f](\xi)\chi_p(-\xi x)\, d_p \xi, \quad \alpha > 0. \tag{2.3}$$

It is easy to see [13] that $D^\alpha f$ is well defined for $f \in \mathcal{D}(\mathbb{Q}_p)$. Note that $D^\alpha f$ need not to belong necessarily to $\mathcal{D}(\mathbb{Q}_p)$ (since the function $|\xi|_p^\alpha$ is not locally constant), however $D^\alpha f \in L_2(\mathbb{Q}_p)$.

Since $\mathcal{D}(\mathbb{Q}_p)$ is not invariant with respect to D^α we cannot define D^α on the whole space $\mathcal{D}'(\mathbb{Q}_p)$. For a distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$, the operation D^α is well defined only if the right-hand side of (2.3) exists.

In what follows we will consider the operator D^α, $\alpha > 0$, as an unbounded operator in $L_2(\mathbb{Q}_p)$. In this case, its domain of definition $\mathcal{D}(D^\alpha)$ consists of those $u \in L_2(\mathbb{Q}_p)$ for which $|\xi|_p^\alpha F[u](\xi) \in L_2(\mathbb{Q}_p)$. Since D^α is unitarily equivalent to the operator of multiplication by $|\xi|_p^\alpha$, it is a positive self-adjoint operator in $L_2(\mathbb{Q}_p)$, its spectrum consists of the eigenvalues $\lambda_\gamma = p^{\alpha\gamma}$ ($\gamma \in \mathbb{Z}$) of infinite multiplicity, and their accumulation point $\lambda = 0$.

It is easy to see from (2.3) that an arbitrary (normalized) eigenfunction ψ of D^α corresponding to the eigenvalue $\lambda_\gamma = p^{\alpha\gamma}$ admits the description

$$\widetilde{\psi}(\xi) = \delta(|\xi|_p - p^\gamma)\rho(\xi), \quad \int_{S_\gamma} |\rho(\xi)|^2 d_p\xi = 1,$$

where the function $\rho(\xi)$ defined on the sphere S_γ serves as a parameter of the description. Choosing $\rho(\xi)$ in different ways one can obtain various orthonormal bases in $L_2(\mathbb{Q}_p)$ formed by eigenfunctions of D^α [13], [14] [17]. In particular, the choice of $\rho(\xi)$ as a system of locally constant functions on S_γ leads to the well-known Vladimirov functions [13], [17]. The selection of $\rho(\xi)$ as indicators of a special class of subsets of S_γ gives the p-adic wavelet basis $\{\psi_{Nj\epsilon}\}$ recently constructed in [14]. Precisely, it was shown [14] that the set of eigenfunctions of D^α

$$\psi_{Nj\epsilon}(x) = p^{-\frac{N}{2}}\chi(p^{N-1}jx)\Omega(|p^N x - \epsilon|_p), \quad N \in \mathbb{Z}, \quad \epsilon \in \mathbb{Q}_p/\mathbb{Z}_p, \quad j = 1, \dots, p-1, \tag{2.4}$$

forms an orthonormal basis in $L_2(\mathbb{Q}_p)$ such that

$$D^\alpha \psi_{Nj\epsilon} = p^{\alpha(1-N)}\psi_{Nj\epsilon}. \tag{2.5}$$

Here the indexes N, j, ϵ serve as parameters of the basis. In particular, elements $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ can be described as $\epsilon = \sum_{i=1}^m \epsilon_i p^{-i}$ ($m \in \mathbb{N}, \epsilon_i = 0, \dots, p-1$).

Theorem 2.1. *An arbitrary function $u \in \mathcal{D}(D^\alpha)$ is continuous on $\mathbb{Q}_p$ if and only if $\alpha > 1/2$.*

Proof. Let $u \in \mathcal{D}(D^\alpha)$ and let

$$u(x) = \sum_{N=1}^{\infty}\sum_{j=1}^{p-1}\sum_{\epsilon}(u, \psi_{Nj\epsilon})\psi_{Nj\epsilon}(x) + \sum_{N=-\infty}^{0}\sum_{j=1}^{p-1}\sum_{\epsilon}(u, \psi_{Nj\epsilon})\psi_{Nj\epsilon}(x) \tag{2.6}$$

be its expansion into the p-adic wavelet basic (2.4).

It is easy to see that $\psi_{Nj\epsilon}(x) \in \mathcal{D}(\mathbb{Q}_p)$ and, hence, the functions $\psi_{Nj\epsilon}(x)$ are continuous on $\mathbb{Q}_p$. Thus, to prove the continuity of $u(x)$, it suffices to verify that the series in (2.6) converges uniformly.

First of all we remark that for fixed N and x there is at most one ϵ such that $\psi_{Nj\epsilon}(x) \neq 0$. Indeed, if there exist ϵ_1 and ϵ_2 such that $\psi_{Nj\epsilon_i}(x) \neq 0$, then $\Omega(|p^N x - \epsilon_i|_p) = 1$. But then $|p^N x - \epsilon_1|_p \le 1$ and $|p^N x - \epsilon_2|_p \le 1$. By the strong triangle inequality, $|\epsilon_1 - \epsilon_2|_p \le 1$. The latter relation and the condition $\epsilon_i \in \mathbb{Q}_p/\mathbb{Z}_p$ imply the equality $\epsilon_1 = \epsilon_2$.

Thus, for fixed N and x, the sum corresponding to the parameter ϵ consists of at most one non-zero term.

Further, it follows from (2.4) and (2.6) that

$$|\psi_{Nj\epsilon}(x)| \le p^{-N/2} \quad \text{and} \quad |(u, \psi_{Nj\epsilon})| \le \|u\|_{L_2(\mathbb{Q}_p)}. \tag{2.7}$$

For a fixed $N > 0$, relations (2.7) ensure the following estimate:

$$\left|\sum_{j=1}^{p-1}\sum_{\epsilon}(u, \psi_{Nj\epsilon})\psi_{Nj\epsilon}(x)\right| \le p^{-N/2}\|u\|_{L_2(\mathbb{Q}_p)}(p-1), \quad \forall x \in \mathbb{Q}_p, \tag{2.8}$$

which gives the uniform convergence of the first series in (2.6).

The condition $u \in \mathcal{D}(D^\alpha)$ and (2.5) imply $(u, \psi_{Nj\epsilon}) = p^{\alpha(N-1)}(D^\alpha u, \psi_{Nj\epsilon})$. Using this equality and (2.7), we obtain

$$\begin{aligned}
\Big|\sum_{j=1}^{p-1}\sum_{\epsilon}(u,\psi_{Nj\epsilon})\psi_{Nj\epsilon}(x)\Big| &= \Big|\sum_{j=1}^{p-1}\sum_{\epsilon}p^{\alpha(N-1)}(D^\alpha u,\psi_{Nj\epsilon})\psi_{Nj\epsilon}(x)\Big| \\
&\le \Big\{\sum_{j=1}^{p-1}\sum_{\epsilon}|(D^\alpha u,\psi_{Nj\epsilon})|^2\Big\}^{1/2}\Big\{\sum_{j=1}^{p-1}\sum_{\epsilon}p^{2\alpha(N-1)}|\psi_{Nj\epsilon}(x)|^2\Big\}^{1/2} \\
&\le \|D^\alpha u\|_{L_2(\mathbb{Q}_p)}\Big\{\sum_{j=1}^{p-1}p^{-N+2\alpha(N-1)}\Big\}^{1/2}.
\end{aligned}$$

The obtained estimate implies that the second series in (2.6) is uniformly convergent for $\alpha > 1/2$. Therefore, any function $u \in \mathcal{D}(D^\alpha)$ is continuous on $\mathbb{Q}_p$ for $\alpha > 1/2$.

In the case $a \le 1/2$, we show that the function

$$f(x) = \sum_{N=-\infty}^{-1}\frac{1}{|N|}p^{(N-1)/2}\psi_{N10}(x) \tag{2.9}$$

(determined in p-adic wavelet basis) belongs to $\mathcal{D}(D^\alpha)$ but $f(x)$ is not continuous on $\mathbb{Q}_p$.

Obviously, $f \in L_2(\mathbb{Q}_p)$ and its Fourier transform is

$$\tilde{f}(\xi) = \sum_{N=-\infty}^{-1}\frac{1}{|N|}p^{(N-1)/2}\tilde{\psi}_{N10}(\xi).$$

By (2.3) and (2.5), $|\xi|_p^\alpha\tilde{\psi}_{N10}(\xi) = p^{\alpha(1-N)}\tilde{\psi}_{N10}(\xi)$. Hence,

$$|\xi|_p^\alpha\tilde{f}(\xi) = \sum_{N=-\infty}^{-1}\frac{1}{|N|}p^{(N-1)/2}\cdot p^{\alpha(1-N)}\tilde{\psi}_{N10}(\xi)$$

and, since $\{\tilde{\psi}_{N10}(\xi)\}_{N\le -1}$ is orthonormal, $|\xi|_p^\alpha\tilde{f}(\xi) \in L_2(\mathbb{Q}_p)$ for $\alpha \le 1/2$ and $|\xi|_p^\alpha\tilde{f}(\xi) \notin L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$. Hence, $f(x) \in \mathcal{D}(D^\alpha)$ for $\alpha \le 1/2$ only.

Let us show that $f(x)$ is not continuous on $\mathbb{Q}_p$. First of all, using (2.4), we rewrite the definition (2.9) of f as

$$f(x) = \sum_{N=-\infty}^{-1}\frac{1}{|N|}p^{-1/2}\chi(p^{N-1}x)\Omega(|p^N x|_p). \tag{2.10}$$

It is easy to see that the restriction of the left-hand side of (2.10) onto any ball $B_\gamma(a) \subset \mathbb{Q}_p \setminus \{0\}$ contains a finite number of non-zero terms. Therefore, $f(x)$ is continuous on $\mathbb{Q}_p \setminus \{0\}$ and it is represented by point-wise convergent series (2.9).

Let us consider the sequence $x_n = p^n$, $n \in \mathbb{N}$. Obviously, $x_n \to 0, (n \to \infty)$ in the p-adic norm $|\cdot|_p$. Furthermore, $\Omega(|p^N x_n|_p) = \Omega(|p^N p^n|_p) = 0$ when $N + n \le -1$. On the other hand, if $N + n \ge 1$, then $p^{N-1}x_n$ is an integer p-adic number and, hence, $\chi(p^{N-1}x_n) = 1$. Taking these relations into account, we deduce from (2.10) that

$$f(x_n) = f(p^n) = p^{-1/2}\left[\frac{\chi(p^{-1})}{n} + \sum_{N=-n+1}^{-1}\frac{1}{|N|}\right] \to \infty \quad \text{as} \quad n \to \infty.$$

Thus, $f(x)$ cannot be continuous at $x = 0$. Theorem 2.1 is proved. $\square$

2.3. **Properties of solutions of $D^\alpha + I = \delta$.** Let us consider the equation

$$D^\alpha h + h = \delta_{x_k}, \quad h \in L_2(\mathbb{Q}_p), \quad x_k \in \mathbb{Q}_p, \quad \alpha > 0, \tag{2.11}$$

where $D^\alpha : L_2(\mathbb{Q}_p) \to \mathcal{D}'(\mathbb{Q}_p)$ is understood in the distribution sense.

It is known [13] that Eq. (2.11) has a unique solution $h = h_k \in L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$ and has no solutions belonging to $L_2(\mathbb{Q}_p)$ for $\alpha \le 1/2$. The next statement continues the investigation of h_k.

Lemma 2.1. *The solution h_k of (2.11) is a function continuous on $\mathbb{Q}_p$ when $\alpha > 1$ and continuous on $\mathbb{Q}_p \backslash \{x_k\}$ when $1/2 < \alpha \le 1$.*

Proof. Reasoning as in the proof of ([13, Lemma 3.7]), where the basis of Vladimirov eigenfunctions was used, we establish an expansion of δ_{x_k} in terms of the p-adic wavelet basis.

Let $u \in \mathcal{D}(D^\alpha)$. By analogy with the proof of Theorem 2.1 we expand u in an uniformly convergent series with respect to the complex-conjugated p-adic wavelet basis $\{\overline{\psi_{Nj\epsilon}}\}$. Since $\{\overline{\psi_{Nj\epsilon}}\}$ are continuous functions on $\mathbb{Q}_p$ we can write $u(x_k) = \sum_{N=-\infty}^{\infty}\sum_{j=1}^{p-1}\sum_\epsilon (u, \overline{\psi_{Nj\epsilon}})\overline{\psi_{Nj\epsilon}}(x_k)$ for $x = x_k$.

Consider

$$\overline{\psi_{Nj\epsilon}}(x_k) = p^{-N/2}\overline{\chi(p^{N-1}jx_k)}\Omega(|p^N x_k - \epsilon|_p) = p^{-N/2}\chi(-p^{N-1}jx_k)\Omega(|p^N x_k - \epsilon|_p).$$

Obviously, $\overline{\psi_{Nj\epsilon}}(x_k) \neq 0 \iff |p^N x_k - \epsilon|_p \le 1$. Here $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ and, hence, $|\epsilon|_p > 1$ for $\epsilon \neq 0$. It follows from the strong triangle inequality and the condition $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ that $|p^N x_k - \epsilon|_p \le 1 \iff \epsilon = \{p^N x_k\}_p$ (if $\epsilon \neq 0$). Moreover, if $\epsilon = 0$, then condition $|p^N x_k|_p \le 1$ implies $\{p^N x_k\}_p = 0$. Combining these two cases we arrive at the conclusion that

$$\overline{\psi_{Nj\epsilon}}(x_k) = \begin{cases} 0, & \epsilon \neq \{p^N x_k\}_p, \\ p^{-N/2}\chi(-p^{N-1}jx_k), & \epsilon = \{p^N x_k\}_p. \end{cases}$$

But then

$$\begin{aligned} < \delta_{x_k}, u >= u(x_k) &= \sum_{N=-\infty}^{\infty}\sum_{j=1}^{p-1} p^{-N/2}\chi(-p^{N-1}jx_k)\big(u, \overline{\psi_{Nj\{p^N x_k\}_p}}\big) \\ &= \sum_{N=-\infty}^{\infty}\sum_{j=1}^{p-1} p^{-N/2}\chi(-p^{N-1}jx_k) < \psi_{Nj\{p^N x_k\}_p}, u > . \end{aligned} \tag{2.12}$$

Since $\mathcal{D}(\mathbb{Q}_p) \subset \mathcal{D}(D^\alpha)$, the equality (2.12) means that

$$\delta_{x_k} = \sum_{N=-\infty}^{\infty}\sum_{j=1}^{p-1} p^{-N/2}\chi(-p^{N-1}jx_k)\psi_{Nj\{p^N x_k\}_p}, \tag{2.13}$$

where the series converges in $\mathcal{D}'(\mathbb{Q}_p)$.

Suppose that a function $h_k \in L_2(\mathbb{Q}_p)$ is represented as a convergent series in $L_2(\mathbb{Q}_p)$,

$$h_k(x) = \sum_{N=-\infty}^{\infty}\sum_{j=1}^{p-1}\sum_\epsilon c_{Nj\epsilon}\psi_{Nj\epsilon}(x).$$

Applying the operator $D^\alpha + I$ term-wise, we get the series

$$D^\alpha h_k + h_k = \sum_{N=-\infty}^{\infty}\sum_{j=1}^{p-1}\sum_\epsilon c_{Nj\epsilon}\big(1 + p^{\alpha(1-N)}\big)\psi_{Nj\epsilon}, \tag{2.14}$$

converging in $\mathcal{D}'$, since $D^\alpha\mathcal{D}(\mathbb{Q}_p) \subset L_2(\mathbb{Q}_p)$. Comparing the terms of (2.13) and (2.14) gives

$$c_{Nj\epsilon} = \begin{cases} 0, & \epsilon \neq \{p^N x_k\}_p, \\ p^{-N/2}\chi(-p^{N-1}jx_k)\big[p^{\alpha(1-N)} + 1\big]^{-1}, & \epsilon = \{p^N x_k\}_p. \end{cases}$$

Thus,

$$h_k(x) = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) \big[p^{\alpha(1-N)} + 1\big]^{-1} \psi_{Nj\{p^N x_k\}_p}(x). \tag{2.15}$$

Let us show that the series (2.15) is uniformly convergent on $\mathbb{Q}_p$ for $\alpha > 1$ and is uniformly convergent on any ball not containing x_k for $1/2 < \alpha \leq 1$.

Indeed, by virtue of (2.7) the general term of (2.15) does not exceed

$$p^{-N} \big[p^{\alpha(1-N)} + 1\big]^{-1} \leq p^{-N}. \tag{2.16}$$

Hence, the subseries of (2.15) formed by terms with $N \geq 0$ converges uniformly.

For $N < 0$ the general term of (2.15) does not exceed

$$p^{-N} \big[p^{\alpha(1-N)} + 1\big]^{-1} \leq \frac{1}{p^\alpha} p^{-N(1-\alpha)}.$$

The obtained estimate implies that for $\alpha > 1$ the subseries of (2.15) formed by terms with $N < 0$ also converges uniformly. So, the series (2.15) converges uniformly for $\alpha > 1$. This proves the assertion of Lemma 2.1 for $\alpha > 1$, since $\psi_{Nj\epsilon}$ are continuous on $\mathbb{Q}_p$.

Let $B_\gamma(a)$ be a ball such that $x_k \notin B_\gamma(a)$. To prove Lemma 2.1 for $1/2 < \alpha \leq 1$ it suffices to verify that the restriction of (2.15) onto $B_\gamma(a)$ contains a finite number of terms with negative parameter $N < 0$.

Indeed, it follows from the strong triangle inequality and the definitions of $\{\cdot\}_p$ and $\Omega(\cdot)$ (see (2.2)) that

$$\Omega(|p^N x - \{p^N x_k\}_p|_p) = \Omega(|p^N x - p^N x_k|_p). \tag{2.17}$$

Hence, the restriction of $\psi_{Nj\{p^N x_k\}_p}(x)$ onto $B_\gamma(a)$ is equal to 0 if $|x - x_k|_p > p^N$ for all $x \in B_\gamma(a)$. Since $|x - x_k|_p > p^\gamma$, the relation $\psi_{Nj\{p^N x_k\}_p}(x) \equiv 0$ $(\forall x \in B_\gamma(a))$ holds for all $N \leq \gamma$. Using the estimation (2.16) we arrive at the conclusion that the series (2.15) converges uniformly for any ball $B_\gamma(a) \subset \mathbb{Q}_p \setminus \{x_k\}$. Lemma 2.1 is proved. ☐

Remark. The solution $h_k(x)$ of (2.11) constructed in Lemma 2.1 is a real-valued function. This fact can be obtained directly from the expansion (2.15). Another way to establish this is based on the invariance of the space $\mathcal{D}(\mathbb{Q}_p)$ and the operator D^α with respect to the complex conjugation. Combining these properties with the uniqueness of the solution of $D^\alpha + I = \delta_{x_k}$ in $L_2(\mathbb{Q}_p)$, we get $\overline{h}_k(x) = h_k(x)$.

Corollary 2.1. *Let the index $\alpha > 1/2$ and points $x_1, \dots, x_n \in \mathbb{Q}_p$ be fixed and let $\mathrm{Sp}\{h_k\}_1^n$ be the linear span of solutions h_k $(1 \leq k \leq n)$ of (2.11). Then $\mathrm{Sp}\{h_k\}_1^n \cap \mathcal{D}(D^{\alpha/2}) = \{0\}$ for $1/2 < \alpha \leq 1$ and $\mathrm{Sp}\{h_k\}_1^n \subset \mathcal{D}(D^{\alpha/2})$ for $\alpha > 1$.*

Proof. The solution h_k of (2.11) is determined by (2.15). Taking the expansion (2.15) and the "semigroup property"

$$D^{\alpha_1} D^{\alpha_2} = D^{\alpha_1 + \alpha_2}, \quad \alpha_1, \alpha_2 > 0, \tag{2.18}$$

of D^α into account, it is easy to see that $h_k \in \mathcal{D}(D^{\alpha/2})$ if and only if the following series converge in $L_2(\mathbb{Q}_p)$:

$$\sum_{N=1}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) \big[p^{\alpha(1-N)} + 1\big]^{-1} p^{\frac{\alpha}{2}(1-N)} \psi_{Nj\{p^N x_k\}_p}$$
$$+ \sum_{N=-\infty}^{0} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) \big[p^{\alpha(1-N)} + 1\big]^{-1} p^{\frac{\alpha}{2}(1-N)} \psi_{Nj\{p^N x_k\}_p}$$

(if the limit exists then it coincides with $D^{\alpha/2} h_k$). For the general term of the first series we have

$$\big|p^{-N/2} p^{\frac{\alpha}{2}(1-N)} \chi(-p^{N-1} j x_k) \big[p^{\alpha(1-N)} + 1\big]^{-1}\big|^2 \leq p^{-N(\alpha+1)+\alpha}, \quad N \geq 1,$$

which implies its convergence in $L_2(\mathbb{Q}_p)$ for any $\alpha > 1/2$.

Similarly, the general term of the second series can be estimated as follows:

$$\left|p^{-N/2}p^{\frac{\alpha}{2}(1-N)}\chi(-p^{N-1}jx_k)\left[p^{\alpha(1-N)}+1\right]^{-1}\right|^2 \le Cp^{(\alpha-1)N}, \quad N \le 0.$$

Obviously this series converges for $\alpha > 1$. Thus $\mathrm{Sp}\{h_k\}_1^n \subset \mathcal{D}(D^{\alpha/2})$ for $\alpha > 1$.

Since $p^{\alpha(1-N)}+1 \le 2p^{\alpha(1-N)}$ for $N \le 0$, we can estimate from below the general term of the second series,

$$\frac{1}{4p^\alpha}p^{(\alpha-1)N} \le \left|p^{-N/2}p^{\frac{\alpha}{2}(1-N)}\chi(-p^{N-1}jx_k)\left[p^{\alpha(1-N)}+1\right]^{-1}\right|^2 \quad (N \le 0) \tag{2.19}$$

which implies its divergence in $L_2(\mathbb{Q}_p)$ for $\alpha \le 1$.

Thus $h_k \notin \mathcal{D}(D^{\alpha/2})$. From this, taking into account that the estimate (2.19) does not depend on the choice of h_k and the functions $\{\psi_{Nj\{p^N x_k\}_p}(x)\}$ $(N<0)$ of the basis $\{\psi_{Nj\epsilon}(x)\}$ corresponding to h_k $(1 \le k \le n)$ in (2.15) are different for sufficiently small negative indexes N, we conclude that $\mathrm{Sp}\{h_k\}_1^n \cap \mathcal{D}(D^{\alpha/2}) = \{0\}$ for $1/2 < \alpha \le 1$. Corollary 2.1 is proved. □

3. Operator D^α with point interactions

3.1. **The Friedrichs extension.** Let $\mathfrak{H}_2 \subset \mathfrak{H}_1 \subset L_2(\mathbb{Q}_p) \subset \mathfrak{H}_{-1} \subset \mathfrak{H}_{-2}$ be the standard scale of Hilbert spaces (A-scale) associated with the positive self-adjoint operator $A = D^\alpha$. Here $\mathfrak{H}_s = \mathcal{D}(A^{s/2})$, $s = 1, 2$, with the norm $\|u\|_s = \|(D^\alpha + I)^{s/2}u\|$ and $\mathfrak{H}_{-s}$ are the completion of $L_2(\mathbb{Q}_p)$ with respect to the norm $\|u\|_{-s}$ (see [3], [7] for details).

Recalling that $h_k(x)$ is a real-valued function and employing (2.12), (2.14) (with u and $\overline{\psi_{Nj\epsilon}}$ instead of h_k and $\psi_{Nj\epsilon}$, respectively), and (2.15), we get

$$< \delta_{x_k}, u >= u(x_k) = ((D^\alpha + I)u, h_k)_{L_2(\mathbb{Q}_p)}, \quad \forall u \in \mathcal{D}(D^\alpha), \quad x_k \in \mathbb{Q}_p. \tag{3.1}$$

Thus, the Dirac delta function δ_{x_k} is well defined on $\mathfrak{H}_2 = \mathcal{D}(D^\alpha)$ and $\delta_{x_k} \in \mathfrak{H}_{-2}$ for $\alpha > 1/2$.

Let us fix points $x_1, \dots, x_n$ $(n < \infty)$ from $\mathbb{Q}_p$ and consider a positive symmetric operator

$$A_{\mathrm{sym}} = D^\alpha \restriction_{\mathcal{D}}, \quad \mathcal{D} = \{u \in \mathcal{D}(D^\alpha) \mid u(x_1) = \dots = u(x_n) = 0\}. \tag{3.2}$$

By Theorem 2.1 the formula (3.2) is well-defined for $\alpha > 1/2$. In this case, (3.1) implies that A_{sym} is a closed densely defined operator in $L_2(\mathbb{Q}_p)$ and its defect subspace $\mathcal{H} = \ker(A^*_{\mathrm{sym}} + I)$ coincides with the linear span of $\{h_k\}_{k=1}^n$. Hence, the deficiency index of A_{sym} is equal to (n, n).

It is clear that the domain of the adjoint A^*_{sym} has the form $\mathcal{D}(A^*_{\mathrm{sym}}) = \mathcal{D}(D^\alpha)\dot{+}\mathcal{H}$ and

$$A^*_{\mathrm{sym}}f = A^*_{\mathrm{sym}}(u+h) = D^\alpha u - h, \quad \forall f = u + h \in \mathcal{D}(A^*_{\mathrm{sym}}) \tag{3.3}$$

$(u \in \mathcal{D}(D^\alpha),\ h \in \mathcal{H})$.

Proposition 3.1. *Let A_F be the Friedrichs extension of A_{sym}. Then $A_F = D^\alpha$ when $1/2 < \alpha \le 1$ and*

$$A_F = A^*_{\mathrm{sym}} \restriction_{\mathcal{D}(A_F)}, \quad \mathcal{D}(A_F) = \{f(x) \in \mathcal{D}(A^*_{\mathrm{sym}}) \mid f(x_1) = \dots = f(x_n) = 0\}$$

when $\alpha > 1$.

Proof. It follows from (2.18) that $\mathfrak{H}_1 = \mathcal{D}(D^{\alpha/2})$. This relation and Corollary 2.1 mean that $\mathcal{H} \subset \mathfrak{H}_1$ $(\alpha > 1)$ and $\mathcal{H} \cap \mathfrak{H}_1 = \{0\}$ $(1/2 < \alpha \le 1)$.

After such a preparatory work, the proof is a direct consequence of some 'folklore' results of the extension theory. For the convenience of the reader some principal stages are repeated below.

First of all, we recall that the Friedrichs extension A_F of $A_{\rm sym}$ is defined as the restriction of the adjoint $A^*_{\rm sym}$ onto $\mathcal{D}(A_F)=\mathcal{D}\cap\mathcal{D}(A^*_{\rm sym})$, where $\mathcal{D}$ is the completion of $\mathcal{D}(A_{\rm sym})$ in the Hilbert space $\mathfrak{H}_1$.

Using the obvious equality $\mathfrak{H}_1=\mathcal{D}\oplus_1\mathcal{H}'$ (here $\mathcal{H}'=\mathcal{H}\cap\mathfrak{H}_1$ and $\oplus_1$ denotes the orthogonal sum in $\mathfrak{H}_1$), we describe $\mathcal{D}(A_F)$ as follows:

$$\mathcal{D}(A_F)=\{f\in\mathfrak{H}_1\cap\mathcal{D}(A^*_{\rm sym}) \mid ((D^\alpha+I)^{1/2}f,\ (D^\alpha+I)^{1/2}h')_{L_2(\mathbb{Q}_p)}=0,\ \forall h'\in\mathcal{H}'\}.$$

If $\mathcal{H}\cap\mathfrak{H}_1=\{0\}$ (the case $1/2<\alpha\le 1$), then $\mathcal{H}'=\{0\}$ and $\mathcal{D}(A_F)=\mathfrak{H}_1\cap\mathcal{D}(A^*_{\rm sym})=\mathcal{D}(D^\alpha)$. Thus $A_F=D^\alpha$.

If $\mathcal{H}\subset\mathfrak{H}_1$ (the case $\alpha>1$), then $\mathcal{H}'=\mathcal{H}$, $\mathcal{D}(A^*_{\rm sym})\subset\mathfrak{H}_1$ and

$$\mathcal{D}(A_F)=\{f\in\mathcal{D}(A^*_{\rm sym}) \mid ((D^\alpha+I)^{1/2}f,(D^\alpha+I)^{1/2}h_k)_{L_2(\mathbb{Q}_p)}=0,\ 1\le k\le n\}.$$

Repeating the same arguments as in the proof of (3.1) it is easy to see that $((D^\alpha+I)^{1/2}f,(D^\alpha+I)^{1/2}h_k)_{L_2(\mathbb{Q}_p)}=f(x_k)$. Proposition 3.1 is proved. $\square$

3.2. **Operator realizations of $D^\alpha+V_Y$ in $L_2(\mathbb{Q}_p)$.** In the additive singular perturbations theory, the algorithm for determining operator realizations of finite rank point perturbations of D^α is determined by the general expression

$$A_Y=D^\alpha+V_Y,\quad V_Y=\sum_{i,j=1}^n b_{ij}<\delta_{x_j},\cdot>\delta_{x_i},\quad b_{ij}\in\mathbb{C}, \tag{3.4}$$

$Y=\{x_1,\dots,x_n\}$ is well known [3] and it is based on the construction of some extension (regularization) $A_{Y{\rm reg}}:=D^\alpha+V_{Y{\rm reg}}$ of (3.4) onto the domain $\mathcal{D}(A^*_{\rm sym})=\mathcal{D}(D^\alpha)\dot{+}\mathcal{H}$.

The $L_2(\mathbb{Q}_p)$-part

$$\widetilde{A}=A_{Y{\rm reg}}\restriction_{\mathcal{D}(\widetilde{A})},\quad \mathcal{D}(\widetilde{A})=\{f\in\mathcal{D}(A^*_{\rm sym}) \mid A_{Y{\rm reg}}f\in L_2(\mathbb{Q}_p)\} \tag{3.5}$$

of the regularization $A_{Y{\rm reg}}$ is called the *operator realization* of $D^\alpha+V_Y$ in $L_2(\mathbb{Q}_p)$.

Since the action of D^α on elements of $\mathcal{H}$ is defined by (2.11), the regularization $A_{Y{\rm reg}}$ depends on the determination of $V_{Y{\rm reg}}$.

If $\alpha>1$, the singular potential $V_Y=\sum_{i,j=1}^n b_{ij}<\delta_{x_j},\cdot>\delta_{x_i}$ is form bounded (since all $h_k\in\mathfrak{H}_1$ and, hence, all $\delta_{x_k}\in\mathfrak{H}_{-1}$). In this case, $\mathcal{D}(A^*_{\rm sym})\subset\mathfrak{H}_1$ consists of functions continuous on $\mathbb{Q}_p$ (Lemma 2.1) and the delta functions δ_{x_k} are uniquely determined on elements $f\in\mathcal{D}(A^*_{\rm sym})$ by continuity (cf. (3.1))

$$<\delta_{x_k},f>=((D^\alpha+I)^{1/2}f,(D^\alpha+I)^{1/2}h_k)_{L_2(\mathbb{Q}_p)}=f(x_k). \tag{3.6}$$

Thus, for $\alpha>1$, the regularization $A_{Y{\rm reg}}$ is uniquely defined and formula (3.5) provides a unique operator realization of (3.4) in $L_2(\mathbb{Q}_p)$ corresponding to a fixed singular potential V_Y.

The case $1/2<\alpha\le 1$ is more complicated, because δ_{x_k} cannot be extended onto $\mathcal{D}(A^*_{\rm sym})$ by continuity.

Since any function $f\in\mathcal{D}(A^*_{\rm sym})=\mathcal{D}(D^\alpha)\dot{+}\mathcal{H}$ admits a decomposition $f=u+\sum_{j=1}^n c_jh_j$ $(u\in\mathcal{D}(D^\alpha),\ c_i\in\mathbb{C})$, the extension of δ_{x_k} to $\mathcal{D}(A^*_{\rm sym})$ is well determined if the entries $r_{kj}=<\delta_{x_k},h_j>$ of the matrix $\mathcal{R}=(r_{kj})_{k,j=1}^n$ are known. In this case, the extended delta-function δ_{x_k} acts on functions $f\in\mathcal{D}(A^*_{\rm sym})$ by the rule

$$<\delta_{x_k},f>=u(x_k)+c_1r_{k1}+\dots+c_nr_{kn},\quad 1\le k\le n. \tag{3.7}$$

(We preserve the same notation δ_{x_k} for the extension.)

Since $\mathcal{H}\cap\mathfrak{H}_1=\{0\}$, the system $\{\delta_{x_k}\}_{k=1}^n$ is $\mathfrak{H}_{-1}$-independent (i.e., its linear span $\mathrm{Sp}\{\delta_{x_k}\}_1^n\cap\mathfrak{H}_{-1}=\{0\}$). Therefore, the natural restrictions on the choice of r_{kj} in (3.7) induced by the fact that a functional $<\phi,\cdot>$ where $\phi\in\mathrm{Sp}\{\delta_{x_k}\}_1^n\cap\mathfrak{H}_{-1}$ admits a natural extension by continuity onto $\mathfrak{H}_1\cap\mathcal{D}(A^*_{\rm sym})$ do not appear in our case (see [3] for details). This means that, in general, any Hermitian matrix $\mathcal{R}=(r_{kj})_{k,j=1}^n$ can be used for the determination of the extended functionals $<\delta_{x_k},\cdot>$ in (3.7).

One of the possible approaches to the definition of r_{kj} deals with the fact that the functions $h_j(x)$ turn out to be continuous at the point $x = x_k$ if $j \neq k$ (see Lemma 2.1). In view of this, it is natural to assume that

$$r_{kj} = < \delta_{x_k}, h_j > = h_j(x_k), \quad j \neq k. \tag{3.8}$$

However this formula cannot be used for the definition of r_{kk} because the substitution of x_k for x in (2.15) leads to the formal equality

$$h_k(x_k) = (p-1) \sum_{N=-\infty}^{\infty} \frac{p^{-N}}{p^{\alpha(1-N)} + 1} \tag{3.9}$$

with a divergent series in the right-hand side. Note that this series does not depend on k. For this reason, some choice of a real number $r = r_{kk}$ $(1 \leq k \leq n)$ can be interpreted as a certain regularization of $\sum_{-\infty}^{\infty} \frac{p^{-N}}{p^{\alpha(1-N)}+1}$.

It follows from (2.15), (2.17), and (3.8) that $r_{kj} = \overline{r}_{jk}$. Hence, the matrix $\mathcal{R} = (r_{kj})_{k,j=1}^n$ constructed in such a way is Hermitian.

It should be noted that if we will use (3.7) instead of the direct formula (3.6) for the definition of extensions $< \delta_{x_k}, \cdot >$ in the case $\alpha > 1$, then we arrive at just the same form of the matrix $\mathcal{R}$. The difference only is in the convergence of the series in (3.9) for $\alpha > 1$ and, hence, $r_{kk} = h_k(x_k)$.

3.3. **Description of operator realizations.** Let η be an invertible bounded self-adjoint operator in $L_2(\mathbb{Q}_p)$.

An operator A is called *η-self-adjoint* in $L_2(\mathbb{Q}_p)$ if $A^* = \eta A \eta^{-1}$, where A^* denotes the adjoint of A [5]. Obviously, self-adjoint operators are a particular case of η-self-adjoint ones for $\eta = I$. In this case, we will use notation 'self-adjoint' instead of 'I-self-adjoint'.

We are going to describe η-self-adjoint operator realizations $\widetilde{A}$ (see (3.5)) of $D^\alpha + V_Y$ in $L_2(\mathbb{Q}_p)$.

To do this, we determine linear mappings $\Gamma_i : \mathcal{D}(A^*_{\mathrm{sym}}) \to \mathbb{C}^n$ $(i = 0, 1)$

$$\Gamma_0 f = \begin{pmatrix} < \delta_{x_1}, f > \\ \vdots \\ < \delta_{x_n}, f > \end{pmatrix}, \quad \Gamma_1 f = -\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \forall f = u + \sum_{i=1}^n c_i h_i \in \mathcal{D}(A^*_{\mathrm{sym}}). \tag{3.10}$$

In what follows we will assume that

$$D^\alpha \eta = \eta D^\alpha \quad \text{and} \quad \eta : \mathcal{H} \to \mathcal{H}. \tag{3.11}$$

By the second relation in (3.11), the action of η on elements of $\mathcal{H}$ can be described with the help of a matrix $\mathcal{Y} = (y_{ij})_{i,j=1}^n$, i.e.,

$$\eta \sum_{i=1}^n c_i h_i = (h_1, \ldots, h_n)\mathcal{Y}(c_1, \ldots, c_n)^{\mathrm{t}} \quad (c_i \in \mathbb{C}), \tag{3.12}$$

where the upper index $^{\mathrm{t}}$ denotes the operation of transposition. Since, in general, the basis $\{h_i\}_{i=1}^n$ of $\mathcal{H}$ is not orthogonal, the matrix $\mathcal{Y}$ is not Hermitian $(\mathcal{Y} \neq \overline{\mathcal{Y}}^{\mathrm{t}})$.

Lemma 3.1. *If $\alpha > 1$, then*

$$\Gamma_0 \eta f = \overline{\mathcal{Y}}^{\mathrm{t}} \Gamma_0 f \quad \text{and} \quad \Gamma_1 \eta f = \mathcal{Y} \Gamma_1 f \quad (\forall f \in \mathcal{D}(A^*_{\mathrm{sym}})).$$

These equalities also hold for $1/2 < \alpha \leq 1$ if $\mathcal{R}\mathcal{Y} = \overline{\mathcal{Y}}^{\mathrm{t}}\mathcal{R}$, where the matrix $\mathcal{R}$ determines the extended functionals $< \delta_{x_k}, \cdot >$ in (3.7).

Proof. Let $f = u + \sum_{i=1} c_i h_i \in \mathcal{D}(A^*_{\mathrm{sym}})$. By (3.12)

$$\eta f = \eta u + (h_1, \ldots, h_n)\mathcal{Y}(c_1, \ldots, c_n)^{\mathrm{t}}, \tag{3.13}$$

where $\eta u \in \mathcal{D}(D^\alpha)$ (see the first relation in (3.11)). In view of (3.10),

$$\Gamma_1 \eta f = -\mathcal{Y}(c_1, \ldots, c_n)^{\mathrm{t}} = \mathcal{Y}\Gamma_1 f.$$

It follows from the first relation in (3.11) that $\eta(D^\alpha + I)^{1/2} = (D^\alpha + I)^{1/2}\eta$. Taking this equality into account, we deduce from (3.6)

$$\begin{aligned} < \delta_{x_k}, \eta f > &= ((D^\alpha + I)^{1/2} f, (D^\alpha + I)^{1/2}\eta h_k)_{L_2(\mathbb{Q}_p)} \\ &= (\overline{y}_{1k}, \ldots, \overline{y}_{nk})(< \delta_{x_1}, f >, \ldots, < \delta_{x_n}, f >)^{\mathrm{t}} \end{aligned}$$

that implies $\Gamma_0 \eta f = \overline{\mathcal{Y}}^{\mathrm{t}}\Gamma_0 f$ for $\alpha > 1$.

Similar arguments with the employing (3.7), (3.13), and $\mathcal{R}\mathcal{Y} = \overline{\mathcal{Y}}^{\mathrm{t}}\mathcal{R}$ give

$$\begin{aligned} \Gamma_0 \eta f &= \Gamma_0 \eta u + \mathcal{R}\mathcal{Y}(c_1, \ldots, c_n)^{\mathrm{t}} = \overline{\mathcal{Y}}^{\mathrm{t}}\Gamma_0 u + \overline{\mathcal{Y}}^{\mathrm{t}}\mathcal{R}(c_1, \ldots, c_n)^{\mathrm{t}} \\ &= \overline{\mathcal{Y}}^{\mathrm{t}}\Gamma_0[u + (h_1, \ldots, h_n)(c_1, \ldots, c_n)^{\mathrm{t}}] = \overline{\mathcal{Y}}^{\mathrm{t}}\Gamma_0 f \end{aligned}$$

for $1/2 < \alpha \leq 1$. Lemma 3.1 is proved. □

Theorem 3.1. *Let $\widetilde{A}$ be the operator realization of $D^\alpha + V_Y$ defined by (3.5). Then $\widetilde{A}$ coincides with the operator*

$$A_{\mathcal{B}} = A^*_{\mathrm{sym}} \upharpoonright \mathcal{D}(A_{\mathcal{B}}), \quad \mathcal{D}(A_{\mathcal{B}}) = \{f \in \mathcal{D}(A^*_{\mathrm{sym}}) \mid \mathcal{B}\Gamma_0 f = \Gamma_1 f\}, \tag{3.14}$$

where $\mathcal{B} = (b_{ij})_{i,j=1}^n$ is the coefficient matrix of the singular potential V_Y.

The operator $A_{\mathcal{B}}$ is self-adjoint if and only if the matrix $\mathcal{B}$ is Hermitian.

If η satisfy (3.11) and $\alpha > 1$, then $A_{\mathcal{B}}$ is η-self-adjoint if and only if the matrix $\mathcal{Y}\mathcal{B}$ is Hermitian. This statement is also true for the case $1/2 < \alpha \leq 1$ under the additional condition that the matrix $\mathcal{R}\mathcal{Y}$ is Hermitian, where $\mathcal{R}$ determines the extended functionals $< \delta_{x_k}, \cdot >$ in (3.7).

Proof. It follows from (2.11), (3.3), and (3.10) that

$$A_{V\mathrm{reg}} f = A^*_{\mathrm{sym}} f + (\delta_{x_1}, \ldots, \delta_{x_n})(\mathcal{B}\Gamma_0 f - \Gamma_1 f), \quad f \in \mathcal{D}(A^*_{\mathrm{sym}}).$$

This equality and (3.5) mean that the operator realization $\widetilde{A}$ of $D^\alpha + V_Y$ coincides with the operator $A_{\mathcal{B}}$ determined by (3.14).

It is known (see, e.g., [4], [8]) that the triple $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$, where Γ_i are defined by (3.10), is a boundary value space (BVS) of A_{sym}. This means that the abstract Green identity

$$(A^*_{\mathrm{sym}} f, g) - (f, A^*_{\mathrm{sym}} g) = (\Gamma_1 f, \Gamma_0 g)_{\mathbb{C}^n} - (\Gamma_0 f, \Gamma_1 g)_{\mathbb{C}^n}, \quad f, g \in \mathcal{D}(A^*_{\mathrm{sym}}) \tag{3.15}$$

is satisfied and the map $(\Gamma_0, \Gamma_1) : \mathcal{D}(A^*_{\mathrm{sym}}) \to \mathbb{C}^n \oplus \mathbb{C}^n$ is surjective.

It follows from the general results of the BVS-theory [9], [10], [15] that the operator $A_{\mathcal{B}}$ determined by (3.14) is self-adjoint $\iff$ the matrix $\mathcal{B}$ is Hermitian.

Conditions (3.11) imposed on η ensure the commutativity of η with A_{sym} and A^*_{sym}, i.e.,

$$\eta A_{\mathrm{sym}} = A_{\mathrm{sym}}\eta, \quad \eta A^*_{\mathrm{sym}} = A^*_{\mathrm{sym}}\eta. \tag{3.16}$$

Relations (3.16) and the definition of η-self-adjoint operators imply that $A_{\mathcal{B}}$ is η-self-adjoint $\iff$ $\eta A_{\mathcal{B}}$ is a self-adjoint extension of the symmetric operator $F_{\mathrm{sym}} = \eta A_{\mathrm{sym}}$.

Thus, the description of η-self-adjoint operators is reduced to the similar problem for self-adjoint ones.

It immediately follows from Lemma 3.1 and relations (3.15), (3.16) that the triple $(\mathbb{C}^n, \Gamma_0, \mathcal{Y}\Gamma_1)$ is a BVS for the symmetric operator F_{sym}. In this BVS, the operator $\eta A_{\mathcal{B}}$ is described by the formula (cf. (3.14)):

$$\eta A_{\mathcal{B}} = \eta A^*_{\mathrm{sym}} \upharpoonright_{\mathcal{D}(\eta A_{\mathcal{B}})}, \quad \mathcal{D}(\eta A_{\mathcal{B}}) = \{f \in \mathcal{D}(A^*_{\mathrm{sym}}) \mid \mathcal{Y}B\Gamma_0 f = \mathcal{Y}\Gamma_1 f\}$$

that completes the proof of Theorem 3.1. □

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