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INVERSE SPECTRAL PROBLEMS FOR COUPLED OSCILLATING SYSTEMS: RECONSTRUCTION FROM THREE SPECTRA

S. ALBEVERIO, R. HRYNIV, AND YA. MYKYTYUK

Dedicated to M. G. Krein, with admiration.

Abstract. We study an inverse spectral problem for a compound oscillating system consisting of a singular string and N masses joined by springs. The operator $\mathcal{A}$ corresponding to this system acts in $L_2(0,1)\times\mathbb{C}^N$ and is composed of a Sturm–Liouville operator in $L_2(0,1)$ with a distributional potential and a Jacobi matrix in $\mathbb{C}^N$ that are coupled in a special way. We solve the problem of reconstructing the system from three spectra—namely, from the spectrum of $\mathcal{A}$ and the spectra of its decoupled parts. A complete description of possible spectra is given.

1. Introduction

The main aim of the present paper is to solve an inverse spectral problem for a class of oscillating systems composed of a singular string and N masses joined by springs. Mathematically such a system is described by a Sturm–Liouville operator S coupled in a special way to a Jacobi operator J.

Namely, assume that q is a real-valued distribution from $W_2^{-1}(0,1)$ and denote by S a Sturm–Liouville operator in $L_2(0,1)$ that is formally given by the differential expression

$$l := -\frac{d^2}{dx^2} + q$$

and the Robin or the Dirichlet boundary condition at the point $x=0$. The precise definition of S is based on regularisation of l by quasi-derivatives [19, 20] and goes as follows. We fix a real-valued distributional primitive $\sigma \in L_2(0,1)$ of q and rewrite ly as

$$l_\sigma y := -(y' - \sigma y)' - \sigma y'$$

on the natural domain

$$\mathcal{D}(l_\sigma) = \{y \in W_1^1(0,1) \mid y' - \sigma y \in W_1^1(0,1),\ l_\sigma y \in L_2(0,1)\}.$$

In what follows, we shall abbreviate the *quasi-derivative* $y' - \sigma y$ to $y_\sigma^{[1]}$ or simply to $y^{[1]}$ when σ is fixed by the context. We define now the operator S by $Sy = l_\sigma y$ on the domain

$$\mathcal{D}(S) = \{y \in \mathcal{D}(l_\sigma) \mid y^{[1]}(0) = hy(0)\}$$

for some $h \in \mathbb{R}\cup\{\infty\}$, $h=\infty$ corresponding to the Dirichlet boundary condition $y(0)=0$.

Assume that J is a Jacobi matrix in $\mathbb{C}^N$, $N\in\mathbb{N}$, i. e., that J in the standard basis $e_1,\dots,e_N$ of $\mathbb{C}^N$ is a symmetric matrix with real entries $b_1,\dots,b_N$ on the main diagonal and positive entries $a_1,\dots,a_{N-1}$ on the main sub- and super-diagonals.

Denote also by B the intertwining operator between $L_2(0,1)$ and $\mathbb{C}^N$ given on $\mathcal{D}(S)$ by $By = a_0 y^{[1]}(1)e_1$ for some $a_0>0$.

Finally, we consider the operator

$$\mathcal{A} := \begin{pmatrix} S & 0 \\ B & J \end{pmatrix} \tag{1.1}$$

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that acts in the product space $\mathcal{H} := L_2(0,1) \times \mathbb{C}^N$ on the domain

$$\mathcal{D}(\mathcal{A}) := \{(y,d)^{\mathrm{t}} \in \mathcal{H} \mid y \in \mathcal{D}(S),\ d = (d(1),\dots,d(N)),\ y(1) = a_0 d(1)\}.$$

It is known [1] that $\mathcal{A}$ is self-adjoint and bounded below in $\mathcal{H}$ and has a simple discrete spectrum. Adding if necessary a sufficiently large constant to the potential q and to the numbers $b_1,\dots,b_N$, we can make the operator $\mathcal{A}$ positive and shall assume this without loss of generality.

Remark 1.1. We observe that although the Sturm–Liouville differential expression l_σ is independent of the particular choice of the primitive σ, the quasi-derivative $y^{[1]}$ in the interface condition and in the boundary condition for S if h is finite, and thus the whole operator $\mathcal{A}$, do depend on σ. We notice, however, that $\mathcal{A}$ is invariant under the simultaneous change $\sigma \mapsto \sigma + C$, $h \mapsto h + C$, and $b_1 \mapsto b_1 + a_0 C$ for any real C. This invariance will be used in Section 4.

Along with $\mathcal{A}$ we consider two operators, $\mathcal{A}_0 := S_{\mathrm{N}} \oplus J$ and $\mathcal{A}_\infty := S_{\mathrm{D}} \oplus J_{(1)}$, where S_{N} and S_{D} are the restrictions of S by the “Neumann” boundary condition $y^{[1]}(1) = 0$ and the Dirichlet boundary condition $y(1) = 0$ respectively, and $J_{(1)}$ is the Jacobi matrix obtained by removing the top row and the most-left column of J. The operators $\mathcal{A}_0$ and $\mathcal{A}_\infty$ formally correspond to two extreme cases of the coupling not allowed in $\mathcal{A}$: first with no coupling at all, and the second with infinite, i.e., rigid coupling. It is easily seen that $\mathcal{A}$ and $\mathcal{A}_0$ are self-adjoint extensions of the same symmetric operator with deficiency indices $(1,1)$ specifying the interface condition at the point $x = 1$, and the same holds for $\mathcal{A}$ and $\mathcal{A}_\infty$. Therefore, as in the papers [8, 13, 16, 17], it is natural to study the question, to which extent $\mathcal{A}$ is determined by the spectra of $\mathcal{A}$ and $\mathcal{A}_0$, or those of $\mathcal{A}$ and $\mathcal{A}_\infty$. As in the purely continuous case of a Sturm–Liouville operator [8, 13, 17] or of a purely discrete case of a Jacobi matrix [16], one has to know the spectra of S_{N} and J or those of S_{D} and $J_{(1)}$ separately—and not just their union—in order to reconstruct $\mathcal{A}$.

Thus the **inverse spectral problem** we are going to solve is that of the reconstruction of the operator $\mathcal{A}$ from the spectra of $\mathcal{A}$, S_{N}, and J or from those of $\mathcal{A}$, S_{D}, and $J_{(1)}$. It generalizes the inverse spectral problems by three spectra for the standard Sturm–Liouville operators or for Jacobi matrices treated in the above-cited papers and is related to the inverse spectral problem for Sturm–Liouville operators with rationally dependent boundary conditions, see [1, 3, 4, 5, 6].

We shall solve the above inverse problem by reducing it to that of reconstructing $\mathcal{A}$ from its spectrum and the sequence of the corresponding norming constants. The latter problem was studied in detail in [1] (see also [3, 4, 5] for the related inverse problem for a Sturm–Liouville operator with rationally dependent boundary conditions), and this allows a complete description of the spectra for the operators involved. We shall prove that the operator $\mathcal{A}$ is recovered uniquely if and only if the three spectra do not intersect. This establishes in this special case the conjecture raised in [8] for Sturm–Liouville operators, which was later proved in [13]; the case of finite Jacobi matrices was studied in [16].

The treatments of the Dirichlet boundary condition ($h = \infty$) and the Robin boundary condition ($h \in \mathbb{R}$) at the point $x = 0$ are completely analogous, and we shall consider in detail only the Dirichlet case. In the next section we shall derive some useful formulae (e.g., for the resolvent of $\mathcal{A}$ and the norming constants) that will be used in the subsequent analysis. In Sections 3 and 4 we reconstruct the operator $\mathcal{A}$ from the spectra of $\mathcal{A}$, S_{D}, and $J_{(1)}$ and from the spectra of $\mathcal{A}$, S_{N}, and J respectively.

Notations. Throughout the paper, the prime will denote the derivative in $x \in [0,1]$, and the overdot will stand for differentiation in the complex variable λ or z. Given two strictly increasing (finite or infinite) sequences (a_n) and (b_n), we shall denote by $(c_n) := (a_n) \amalg (b_n)$ the non-decreasing sequence obtained by amalgamating the sequences

(a_n) and (b_n) and listing the common elements twice. We shall write $\sigma(T)$ for the spectrum of a linear operator T acting in a Hilbert space.

2. Preliminaries

It is known [1] that the operator $\mathcal{A}$ of (1.1) is self-adjoint, lower semi-bounded, and has discrete spectrum $\lambda_1 < \lambda_2 < \dots$; we recall our standing and nonrestrictive assumption that $\lambda_1 > 0$.

For every nonzero $\lambda \in \mathbb{C}$, we define the "fundamental system of solutions" $Y_-(\cdot,\lambda)$ and $Y_+(\cdot,\lambda)$ corresponding to the eigenvalue problem $\mathcal{A}Y = \lambda Y$. Namely, the element $Y_-(\cdot,\lambda) := (y_-(\cdot,\lambda), d_-(\cdot,\lambda))^{\mathrm{t}}$ belongs to $\mathcal{D}(\mathcal{A})$, is normalised by the initial conditions $y_-(0,\lambda) = 0$, $y_-^{[1]}(0,\lambda) = \sqrt{\lambda}$, and satisfies the relation $\mathcal{A}Y = \lambda Y$ in the $L_2(0,1)$-component and in the first $N-1$ components of $\mathbb{C}^N$. In other words, there is a unique $c = c(\lambda) \in \mathbb{C}$ such that

$$(\mathcal{A} - \lambda)Y_-(\cdot,\lambda) = \begin{pmatrix} 0 \\ ce_N \end{pmatrix};$$

in particular, $c(\lambda) = 0$ if and only if λ is in the spectrum of $\mathcal{A}$, in which case $Y_-(\cdot,\lambda)$ is a corresponding eigenelement. The element $Y_+(\cdot,\lambda) := (y_+(\cdot,\lambda), d_+(\cdot,\lambda))^{\mathrm{t}}$ is normalized by the terminal condition $d_+(N,\lambda) = 1$, satisfies the system

$$\begin{aligned} ly_+ - \lambda y_+ &= 0, \\ a_0 y_+^{[1]}(1)e_1 + (J-\lambda)d_+ &= 0, \end{aligned}$$

and the interface condition $y_+(1,\lambda) = a_0 d_+(1,\lambda)$, but needn't satisfy the initial condition $y_+(0,\lambda) = 0$. Moreover, $y_+(0,\lambda) = 0$ holds if and only if λ is in the spectrum of $\mathcal{A}$, in which case $Y_+(\cdot,\lambda)$ is a corresponding eigenelement.

Using the elements $Y_\pm(\cdot,\lambda)$, it is possible to construct the Green function of the operator $\mathcal{A}$ and to find the explicit form of its resolvent, similarly to such constructions for a Sturm–Liouville equation.

Lemma 2.1. *Assume that $\lambda \in \mathbb{C}$ belongs to the resolvent set of the operators J and $\mathcal{A}$ and that $(g,v)^{\mathrm{t}}$ is an arbitrary element of $\mathcal{H}$. Then the element*

$$\begin{pmatrix} y \\ d \end{pmatrix} := (\mathcal{A} - \lambda)^{-1} \begin{pmatrix} g \\ v \end{pmatrix}$$

is given by

$$\begin{aligned} y(x) &= \frac{y_-(x,\lambda)}{W(\lambda)} \Big[\int_x^1 y_+ g + \big(v, d_+(\cdot,\overline{\lambda})\big)_{\mathbb{C}^N} \Big] + \frac{y_+(x,\lambda)}{W(\lambda)} \int_0^x y_- g, \\ d(k) &= (J-\lambda)^{-1} v(k) + \frac{d_+(k,\lambda)}{W(\lambda)} \Big[\int_0^1 y_- g + \frac{y_-^{[1]}(1,\lambda)}{y_+^{[1]}(1,\lambda)} \big(v, d_+(\cdot,\overline{\lambda})\big)_{\mathbb{C}^N} \Big], \end{aligned}$$

where $W(\lambda) := y_+(x,\lambda) y_-^{[1]}(x,\lambda) - y_+^{[1]}(x,\lambda) y_-(x,\lambda)$ is the Wronskian of the solutions y_+ and y_-.

Proof. The function y solves the equation $Sy = \lambda y + g$ and thus is equal to $y_0 + \alpha y_-$, with

$$y_0(x) := \frac{y_-(x,\lambda)}{W(\lambda)} \int_x^1 y_+ g + \frac{y_+(x,\lambda)}{W(\lambda)} \int_0^x y_- g$$

being a particular solution to the above non-homogeneous problem and α some complex number. Since $d_+(\cdot,\lambda) = -a_0 y_+^{[1]}(1,\lambda)(J-\lambda)^{-1}e_1$, the relation

$$a_0 y^{[1]}(1)e_1 + (J-\lambda)d = v \tag{2.1}$$

implies that $d = d_0 + \beta d_+(\cdot,\lambda)$ with $d_0 := (J-\lambda)^{-1}v$ and some $\beta \in \mathbb{C}$.

The constants α and β must be such that the interface condition $y(1) = a_0 d(1)$ and relation (2.1) hold. By virtue of the relation

$$d_0(1) = \big((J-\lambda)^{-1}v, e_1\big) = -\frac{\big(v, d_+(\cdot,\overline{\lambda})\big)_{\mathbb{C}^N}}{a_0 y_+^{[1]}(1,\lambda)}$$

the interface condition transforms into

$$\alpha y_-(1,\lambda) - \beta y_+(1,\lambda) = -\frac{y_+(1,\lambda)}{W(\lambda)}\int_0^1 y_- g - \frac{\big(v, d_+(\cdot,\overline{\lambda})\big)_{\mathbb{C}^N}}{y_+^{[1]}(1,\lambda)}.$$

Similarly, equation (2.1) can be recast as

$$\alpha y_-^{[1]}(1,\lambda) - \beta y_+^{[1]}(1,\lambda) = -\frac{y_+^{[1]}(1,\lambda)}{W(\lambda)}\int_0^1 y_- g.$$

The above two equations form a linear system for α and β, solving which we find that

$$\alpha = \frac{\big(v, d_+(\cdot,\overline{\lambda})\big)_{\mathbb{C}^N}}{W(\lambda)}, \quad \beta = \frac{\int_0^1 y_- g}{W(\lambda)} + \frac{y_-^{[1]}(1,\lambda)}{y_+^{[1]}(1,\lambda)}\frac{\big(v, d_+(\cdot,\overline{\lambda})\big)_{\mathbb{C}^N}}{W(\lambda)},$$

and the required formula for $(y,d)^{\mathrm{t}}$ follows. □

The sequence $\big(Y_-(\cdot,\lambda_n)\big)_{n\in\mathbb{N}}$ forms an orthogonal basis of the space $\mathcal{H}$. We denote by $\alpha_n := \|Y_-(\cdot,\lambda_n)\|^{-2}$ the *norming constant* corresponding to the eigenvalue λ_n. A useful formula for the norming constants is given by the following lemma.

Lemma 2.2. *Assume that $\lambda_n \in \sigma(\mathcal{A})$ is not in the spectrum of J. Then the corresponding norming constant $\alpha_n := \|Y_-(\cdot,\lambda_n)\|^{-2}$ satisfies the equalities*

$$\alpha_n = -\frac{y_+^{[1]}(1,\lambda_n)}{\sqrt{\lambda_n}\, y_-^{[1]}(1,\lambda_n)\dot{y}_+(0,\lambda_n)}. \tag{2.2}$$

Similarly, if $\lambda_n \in \sigma(\mathcal{A})$ is not in the spectrum of $J_{(1)}$, then

$$\alpha_n = -\frac{y_+(1,\lambda_n)}{\sqrt{\lambda_n}\, y_-(1,\lambda_n)\dot{y}_+(0,\lambda_n)}. \tag{2.3}$$

Proof. We take an arbitrary function $g \in L_2(0,1)$, put $G := (g,0)^{\mathrm{t}}$, and calculate the L_2-component $\hat{g}$ of the element $(\mathcal{A}-\lambda)^{-1}G$ in two ways. On the one hand, the resolution of identity of the operator $\mathcal{A}$ gives

$$\hat{g}(x) = \sum_{k=1}^{\infty}\frac{\alpha_k\big(g, y_-(\cdot,\lambda_k)\big)_{\mathbb{C}^N} y_-(x,\lambda_k)}{\lambda_k - \lambda}.$$

On the other hand, using Lemma 2.1, we find that

$$\hat{g}(x) = \frac{y_-(x,\lambda)}{W(\lambda)}\int_x^1 y_+ g + \frac{y_+(x,\lambda)}{W(\lambda)}\int_0^x y_- g.$$

Equating the residues at the point $\lambda = \lambda_n$ and noting that the functions $y_-(\cdot,\lambda_n)$ and $y_+(\cdot,\lambda_n)$ are collinear and that λ_n is a simple zero of W, we conclude that

$$\alpha_n y_-(x,\lambda_n)\big(g, y_-(\cdot,\lambda_n)\big)_{\mathbb{C}^N} = -\frac{y_+(x,\lambda_n)}{\dot{W}(\lambda_n)}\big(g, y_-(\cdot,\lambda_n)\big)_{\mathbb{C}^N},$$

or, on account of the relation $W(\lambda) \equiv \sqrt{\lambda}\, y_+(0,\lambda)$,

$$\alpha_n = -\frac{y_+(x,\lambda_n)}{\sqrt{\lambda_n}\, y_-(x,\lambda_n)}\frac{1}{\dot{y}_+(0,\lambda_n)}.$$

Finally, the ratio $y_+(x,\lambda_n)/y_-(x,\lambda_n)$ does not depend on x, and, moreover,

$$\frac{y_+(x,\lambda_n)}{y_-(x,\lambda_n)} = \frac{y_+(1,\lambda_n)}{y_-(1,\lambda_n)}$$

if $y_-(1,\lambda_n) \neq 0$ and

$$\frac{y_+(x,\lambda_n)}{y_-(x,\lambda_n)} = \frac{y_+^{[1]}(1,\lambda_n)}{y_-^{[1]}(1,\lambda_n)},$$

if $y_-^{[1]}(1,\lambda_n)\ne 0$, and the required formulae follow. It remains to recall [1] that, for λ_n in the spectrum of $\mathcal{A}$, the equality $y_-(1,\lambda_n)=0$ holds if and only if λ_n is an eigenvalue of J and that $y_-^{[1]}(1,\lambda_n)=0$ if and only if λ_n is an eigenvalue of $J_{(1)}$. □

It is known (see [6] for the case $q\in L_1(0,1)$ and [1] for the case $q\in W_2^{-1}(0,1)$) that the eigenvalues (λ_n) and the corresponding norming constants (α_n) determine the operator $\mathcal{A}$ uniquely. Moreover, the cited papers give the algorithm of reconstruction of $\mathcal{A}$ from these spectral data. The next proposition gives also the complete description of the spectral data, cf. [1, 6].

Proposition 2.3. *The eigenvalues (λ_n) of $\mathcal{A}$ and the corresponding norming constants (α_n) obey the asymptotics*

$$\lambda_n=[\pi(n-N)+\tilde\lambda_n]^2,\qquad \alpha_n=2+\tilde\alpha_n,$$

where the sequences $(\tilde\lambda_n)$ and $(\tilde\alpha_n)$ belong to ℓ_2.

Conversely, any sequences (λ_n) and (α_n) of real numbers such that

(a) *the λ_n strictly increase and have the representation $\lambda_n=[\pi(n-N)+\tilde\lambda_n]^2$ for some $N\in\mathbb{N}$ and an ℓ_2-sequence $(\tilde\lambda_n)$;*
(b) *the α_n are positive and equal $2+\tilde\alpha_n$ for some ℓ_2-sequence $(\tilde\alpha_n)$*

are the sequences of eigenvalues and the norming constants for a unique operator $\mathcal{A}$ of the form (1.1).

In the following, we denote by $\mu_{n,\mathrm{D}}$ (resp. $\mu_{n,\mathrm{N}}$) the eigenvalues of the operator S_{D} (resp. of the operator S_{N}), and by $\nu_{1,\mathrm{J}},\dots,\nu_{N,\mathrm{J}}$ (resp. $\nu_1^1,\dots,\nu_{N-1}^1$) the eigenvalues of J (resp. of $J_{(1)}$), all labeled in increasing order. It is well known that the operators S_{D} and S_{N} and the Jacobi matrices J and $J_{(1)}$ have simple discrete spectra. We recall and derive next some properties of these spectra.

Proposition 2.4. ([10, 19, 20]). *There exist sequences $(\tilde\mu_{n,\mathrm{D}})$ and $(\tilde\mu_{n,\mathrm{N}})$ belonging to $\ell_2(\mathbb{N})$ such that*

(a) $\mu_{n,\mathrm{D}}=[\pi n+\tilde\mu_{n,\mathrm{D}}]^2$*;*
(b) $\mu_{n,\mathrm{N}}=[\pi(n-\frac12)+\tilde\mu_{n,\mathrm{N}}]^2$.

We observe that the numbers $\mu_{n,\mathrm{D}}$ are zeros of the function $y_-(1,\lambda)$ and $\mu_{n,\mathrm{N}}$—those of $y_-^{[1]}(1,\lambda)$. Since both functions are exponential in λ of order $\frac12$, they can be reconstructed from their zeros in the following way.

Proposition 2.5. ([11]). *The following equalities hold:*

$$y_-(1,\lambda)=\sqrt{\lambda}\prod_{k=1}^{\infty}\frac{\mu_{k,\mathrm{D}}-\lambda}{\pi^2k^2},\qquad y_-^{[1]}(1,\lambda)=\sqrt{\lambda}\prod_{k=1}^{\infty}\frac{\mu_{k,\mathrm{N}}-\lambda}{\pi^2(k-\frac12)^2}.$$

Simple considerations show that the functions $y_+(1,\lambda)$ and $y_+^{[1]}(1,\lambda)$ are related to the eigenvalues of J and $J_{(1)}$ as follows.

Lemma 2.6. *The following equalities hold:*

$$y_+(1,\lambda)=\frac{a_0}{a_1\cdots a_{N-1}}\prod_{k=1}^{N-1}(\lambda-\nu_k^1),\qquad y_+^{[1]}(1,\lambda)=\frac{1}{a_0a_1\cdots a_{N-1}}\prod_{k=1}^{N}(\lambda-\nu_{k,\mathrm{J}}).$$

Proof. To find the representation for $y_+^{[1]}(1,\lambda)$, it suffices to establish an analogous formula for $d_+(1,\lambda)$. Using the relation $(J-\lambda)d_+(\cdot,\lambda)=-a_0y_+^{[1]}(1,\lambda)e_1$ and the normalization $d_+(N,\lambda)=1$, we find recursively that $d_+(N-k,\lambda)$ is a polynomial in λ of degree k with leading coefficient $(a_{N-1}\cdots a_{N-k})^{-1}$. Therefore $y_+^{[1]}(1,\lambda)$ is a polynomial in λ of degree N with leading coefficient $(a_0a_1\cdots a_{N-1})^{-1}$, and since it vanishes at the points $\nu_{1,\mathrm{J}},\dots,\nu_{N,\mathrm{J}}$, the above formula follows.

Analogously $y_+(1,\lambda) = a_0 d_+(1,\lambda)$ is a polynomial in λ of degree $N-1$ and with leading coefficient $a_0/(a_1\cdots a_{N-1})$. Since $d_+(1,\lambda)$ vanishes at the points $\nu_1^1,\dots,\nu_{N-1}^1$, the result follows. □

Finally, we find below an explicit expression for $y_+(0,\lambda)$ in terms of the eigenvalues λ_n of the operator $\mathcal{A}$.

Lemma 2.7. *The following holds:*

$$y_+(0,\lambda) = -(a_0a_1\cdots a_{N-1})^{-1}\prod_{k=1}^{N}(\lambda-\lambda_k)\prod_{k=1}^{\infty}\frac{\lambda_{k+N}-\lambda}{\pi^2k^2}.$$

Proof. In what follows, λ is an arbitrary nonzero complex number. We recall that $y_+(x,\lambda)$ is a solution of the equation $ly=\lambda y$ satisfying the terminal conditions $y(1) = y_+(1,\lambda)$ and $y^{[1]}(1) = y_+^{[1]}(1,\lambda)$, whence

$$y_+(x,\lambda) = y_+(1,\lambda)u(x,\lambda) + y_+^{[1]}(1,\lambda)v(x,\lambda),$$

where $u(\cdot,\lambda)$ and $v(\cdot,\lambda)$ are solutions of the problems

$$\begin{cases} l_\sigma u = \lambda u,\\ u(1)=1,\\ u^{[1]}(1)=0,\end{cases}\qquad \begin{cases} l_\sigma v = \lambda v,\\ v(1)=0,\\ v^{[1]}(1)=1.\end{cases}$$

Recalling [12] that $u(x,\lambda)$ and $v(x,\lambda)$ have the integral representations

$$u(x,\lambda) = \cos\sqrt{\lambda}(x-1) + \int_x^1 k_1(x,t)\cos\sqrt{\lambda}(t-1)\,dt,$$
$$v(x,\lambda) = \frac{\sin\sqrt{\lambda}(x-1)}{\sqrt{\lambda}} + \int_x^1 k_2(x,t)\frac{\sin\sqrt{\lambda}(t-1)}{\sqrt{\lambda}}\,dt$$

for some upper-diagonal kernels k_j such that $k(x,\cdot)$ belongs to $L_2(0,1)$ for every $x\in[0,1]$, and that by Lemma 2.6 $y_+^{[1]}(1,\lambda)$ and $y_+(1,\lambda)$ are polynomials in λ of degrees N and $N-1$ respectively, we find that

$$y_+(0,\lambda) = -\frac{\lambda^{N-\frac12}\sin\sqrt{\lambda}}{a_0a_1\cdots a_{N-1}}\,[1+\mathrm{o}(1)]$$

as $\lambda\to-\infty$.

Since $y_+(0,\lambda)$ is an entire function of λ of exponential type $\frac12$ and since its zeros coincide with the eigenvalues of $\mathcal{A}$, we conclude that

$$y_+(0,\lambda) = C_1\prod_{k\in\mathbb{N}}\Bigl(1-\frac{\lambda}{\lambda_n}\Bigr)$$

for some constant $C_1\in\mathbb{C}$. Now we find that

$$\begin{aligned}
-1 &= \lim_{\lambda\to-\infty}\frac{a_0a_1\cdots a_{N-1}\,y_+(0,\lambda)}{\lambda^{N-\frac12}\sin\sqrt{\lambda}}\\
&= \lim_{\lambda\to-\infty}\frac{C_1a_0a_1\cdots a_{N-1}}{\lambda^N}\prod_{k\in\mathbb{N}}\Bigl(1-\frac{\lambda}{\lambda_k}\Bigr)\Big/\prod_{k\in\mathbb{N}}\Bigl(1-\frac{\lambda}{\pi^2k^2}\Bigr)\\
&= \lim_{\lambda\to-\infty}\frac{C_1a_0a_1\cdots a_{N-1}}{\lambda^N}\prod_{k=1}^{N}\Bigl(1-\frac{\lambda}{\lambda_k}\Bigr)\prod_{k\in\mathbb{N}}\frac{\lambda_{k+N}-\lambda}{\lambda_{k+N}}\,\frac{\pi^2k^2}{\pi^2k^2-\lambda}\\
&= (-1)^N\frac{C_1a_0a_1\cdots a_{N-1}}{\lambda_1\cdots\lambda_N}\prod_{k\in\mathbb{N}}\frac{\pi^2k^2}{\lambda_{k+N}},
\end{aligned}$$

since the above products converge uniformly on $\mathbb{C}$, whence

$$\begin{aligned}y_+(0,\lambda) &= -\frac{(-1)^N\lambda_1\cdots\lambda_N}{a_0a_1\cdots a_{N-1}}\prod_{k=1}^{N}\Big(1-\frac{\lambda}{\lambda_k}\Big)\prod_{k=1}^{\infty}\frac{\lambda_{k+N}-\lambda}{\pi^2k^2}\\ &= -(a_0a_1\cdots a_{N-1})^{-1}\prod_{k=1}^{N}(\lambda-\lambda_k)\prod_{k=1}^{\infty}\frac{\lambda_{k+N}-\lambda}{\pi^2k^2}.\end{aligned}$$

The lemma is proved. □

We shall use several times the following statement about integral representations of some entire functions, cf. [15, Lemma 3.4.2].

Proposition 2.8. ([11]). *Assume that the numbers a_n and b_n are such that $a_n = \pi n+\tilde a_n$ and $b_n = \pi(n-\frac12)+\tilde b_n$ with some ℓ_2-sequences $(\tilde a_n)$ and $(\tilde b_n)$. Put*

$$\phi(z) := \sqrt{z}\prod_{n\in\mathbb{N}}\frac{a_n^2-z}{\pi^2n^2},\qquad \psi(z) := \prod_{n\in\mathbb{N}}\frac{b_n^2-z}{\pi^2(n-\frac12)^2};$$

then there exist functions $\tilde\phi$ and $\tilde\psi$ in $L_2(0,1)$ such that

$$\phi(z) = \sin\sqrt{z} + \int_0^1\tilde\phi(t)\sin\sqrt{z}t\,dt,\qquad \psi(z) = \cos\sqrt{z} + \int_0^1\tilde\psi(t)\cos\sqrt{z}t\,dt.$$

3. Reconstruction from $\mathcal{A}$, S_{D}, and $J_{(1)}$

Given an arbitrary operator matrix $\mathcal{A}$ of the form (1.1), we denote by $(\lambda_n)_{n\in\mathbb{N}}$, $(\mu_{n,\mathrm{D}})_{n\in\mathbb{N}}$, and $(\nu_n^1)_{n=1}^{N-1}$ the eigenvalue sequences of $\mathcal{A}$, the operator S_{D}, and the Jacobi matrix $J_{(1)}$ respectively. Put also $(\lambda'_n)_{n\in\mathbb{N}} := (\mu_{n,\mathrm{D}})\amalg(\nu_n^1)_{n=1}^{N-1}$, where the amalgamation operation $\amalg$ was defined in the Introduction. An interesting property of the spectra involved is that every multiple element of (λ'_n) is an eigenvalue of $\mathcal{A}$ and any eigenvalue of $\mathcal{A}$ that belongs also to (λ'_n) occurs therein twice. In other words, the following statement holds true.

Proposition 3.1. ([1]). $\sigma(\mathcal{A})\cap\sigma(S_{\mathrm{D}}) = \sigma(\mathcal{A})\cap\sigma(J_{(1)}) = \sigma(S_{\mathrm{D}})\cap\sigma(J_{(1)})$.

This allows us to establish the weak interlacing property of the sequences (λ_n) and (λ'_n) in the following sense.

Lemma 3.2. *The sequences (λ_n) and (λ'_n) weakly interlace, i.e., $\lambda_1 < \lambda'_1$ and for every $n\in\mathbb{N}$ either $\lambda'_n < \lambda_{n+1} < \lambda'_{n+1}$ or $\lambda'_n = \lambda_{n+1} = \lambda'_{n+1}$.*

Proof. Denote by δ_1 the Dirac delta-function at the point $x = 1$ and put $D_1 := (\delta_1, 0)^{\mathrm{t}}$. It is known that the domain of the Sturm–Liouville operator S is contained in $W_2^1(0,1)$; in particular, the functions $y_\pm(\cdot,\lambda)$ belong to $W_2^1(0,1)$. The explicit formula for the resolvent of the operator $\mathcal{A}$ derived in Lemma 2.1 shows that the expression

$$f(\lambda) := \langle(\mathcal{A}-\lambda)^{-1}D_1, D_1\rangle$$

makes sense for any λ not in the spectrum of $\mathcal{A}$ and, moreover, that

$$f(\lambda) = \frac{y_+(1,\lambda)y_-(1,\lambda)}{W(\lambda)}.$$

Since $f(\lambda)$ is a Nevanlinna function, its zeros and poles interlace. On the other hand, the zeros of f coincide with λ'_k and the poles with those λ_k which do not appear in (λ'_n). In view of Proposition 3.1 and the known asymptotics of λ_n and $\mu_{n,\mathrm{D}}$ this justifies the claim. □

The asymptotics of λ_n and $\mu_{n,\mathrm{D}}$ shows that $\lambda_{n+N}-\mu_{n,\mathrm{D}} = \mathrm{o}(n)$ as $n\to\infty$. In fact, this result can be improved, cf. [5] for the case $q\in L_1(0,1)$.

Lemma 3.3. *There exists an ℓ_2-sequence (b_n) such that*

$$\lambda_{n+N} - \mu_{n,\mathrm{D}} = 2a_0^2(1+b_n).$$

Proof. It suffices to consider only large enough n such that λ_{n+N} is not in the spectrum of $J_{(1)}$ and thus formula (2.3) for the norming constant α_{n+N} holds. Using the representation of the functions $y_+(0,\lambda)$ and $y_+(1,\lambda)$, we find that

$$\alpha_{n+N} = \frac{a_0^2\prod_{k=1}^{N-1}(\lambda_{n+N}-\nu_k^1)}{y_-(1,\lambda_{n+N})\prod_{k=1}^{N}(\lambda_{n+N}-\lambda_k)} \Big/ \frac{d}{d\lambda}\Big(\sqrt{\lambda}\prod_{k\in\mathbb{N}}\frac{\lambda_{k+N}-\lambda}{\pi^2k^2}\Big)\Big|_{\lambda=\lambda_{n+N}}. \tag{3.1}$$

Due to the asymptotics of λ_n and Proposition 2.8 the function

$$\phi(\lambda) := \sqrt{\lambda}\prod_{k\in\mathbb{N}}\frac{\lambda_{k+N}-\lambda}{\pi^2k^2}$$

can be represented in the form

$$\phi(\lambda) = \sin\sqrt{\lambda} + \int_0^1 f(t)\sin\sqrt{\lambda}t\,dt$$

for some $f\in L_2(0,1)$, whence

$$\dot\phi(\lambda_{n+N}) = \frac{1}{2\sqrt{\lambda_{n+N}}}\Big(\cos\sqrt{\lambda_{n+N}} + \int_0^1 tf(t)\cos\sqrt{\lambda_{n+N}}t\,dt\Big).$$

By Propositions 2.5 and 2.8, there is also $g\in L_2(0,1)$ such that

$$\psi(\lambda) := y_-(1,\lambda) = \sin\sqrt{\lambda} + \int_0^1 g(t)\sin\sqrt{\lambda}t\,dt.$$

In view of the mean value theorem there are numbers ξ_n between $\mu_{n,\mathrm{D}}$ and λ_{n+N} such that

$$\begin{aligned}\psi(\lambda_{n+N}) &= (\lambda_{n+N}-\mu_{n,\mathrm{D}})\dot\psi(\xi_n)\\ &= \frac{\lambda_{n+N}-\mu_{n,\mathrm{D}}}{2\sqrt{\xi_n}}\Big(\cos\sqrt{\xi_n} + \int_0^1 tg(t)\cos\sqrt{\xi_n}t\,dt\Big).\end{aligned}$$

Due to the asymptotics of λ_n and ξ_n the sequences $(\cos\sqrt{\lambda_{n+N}}t)_{n\in\mathbb{N}}$ and $(\cos\sqrt{\xi_n}t)_{n\in\mathbb{N}}$ form Riesz bases of $L_2(0,1)$ [9] and hence

$$\cos\sqrt{\lambda_{n+N}} + \int_0^1 tf(t)\cos\sqrt{\lambda_{n+N}}t\,dt = (-1)^{n+N}(1+c_n),$$

$$\cos\sqrt{\xi_n} + \int_0^1 tg(t)\cos\sqrt{\xi_n}t\,dt = (-1)^{n+N}(1+d_n)$$

with square summable sequences $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$. Therefore (3.1) can be recast as

$$\alpha_{n+N}(1+c_n)(1+d_n) = \frac{4a_0^2}{\lambda_{n+N}-\mu_{n,\mathrm{D}}}\frac{\sqrt{\lambda_{n+N}\xi_n}}{\lambda_{n+N}-\lambda_N}\prod_{k=1}^{N-1}\frac{\lambda_{n+N}-\nu_k^1}{\lambda_{n+N}-\lambda_k},$$

which, on account of the asymptotics of α_n of Proposition 2.3, implies that

$$\frac{2a_0^2}{\lambda_{n+N}-\mu_{n,\mathrm{D}}} = 1+\hat\alpha_n, \quad (\hat\alpha_n)\in\ell_2,$$

and the result follows. □

Definition 3.4. We denote by $\mathfrak{L}_N$ the set of all triples $\Lambda := \big((\lambda_n)_{n=1}^\infty, (\mu_n)_{n=1}^\infty, (\nu_n)_{n=1}^{N-1}\big)$ of strictly monotone sequences such that the following holds:

(1) there is an ℓ_2-sequence $(\tilde\lambda_n)$ such that $\lambda_n = [\pi(n-N)+\tilde\lambda_n]^2$;
(2) the sequences (λ_n) and $(\lambda'_n) := (\mu_n)\amalg(\nu_n)$ weakly interlace in the sense of Lemma 3.2;
(3) there exist $\gamma_0>0$ and a sequence $(\gamma_n)\in\ell_2$ such that $\lambda_{k+N}-\mu_k = \gamma_0+\gamma_k$.

For a given $\Lambda\in\mathfrak{L}_N$, we denote by A_Λ the set of $n\in\mathbb{N}$ such that $\lambda_n = \lambda'_n$ and put $B_\Lambda := \mathbb{N}\setminus A_\Lambda$.

The results of this and the previous sections show that, for any operator $\mathcal{A}$ of the form (1.1), the corresponding spectral triple $\big((\lambda_n),(\mu_{n,\mathrm{D}}),(\nu_n^1)\big)$ forms an element of $\mathfrak{L}_N$. In the reverse direction, we shall prove that any element of $\mathfrak{L}_N$ is the spectral triple of the above form.

Theorem 3.5. *For any $\Lambda := \big((\lambda_n),(\mu_n),(\nu_n)\big) \in \mathfrak{L}_N$ there exists an operator $\mathcal{A}$ of the form (1.1) such that (λ_n), (μ_n), and (ν_n) are the eigenvalues of the operators $\mathcal{A}$, S_{D}, and $J_{(1)}$ respectively. Such an operator $\mathcal{A}$ is unique if and only if the set A_Λ is empty.*

Proof. We start with constructing the functions

$$\phi(z) := \sqrt{z}\prod_{n\in\mathbb{N}}\frac{\lambda_{n+N}-z}{\pi^2n^2}, \qquad \psi(z) := \sqrt{z}\prod_{n\in\mathbb{N}}\frac{\mu_n-z}{\pi^2n^2},$$

and for $n\in B_\Lambda$ put (cf. (3.1))

$$\beta_n := \frac{\gamma_0\prod_{k=1}^{N-1}(\lambda_n-\nu_k)}{2\psi(\lambda_n)}\Big/\frac{d}{d\lambda}\Big(\phi(\lambda)\prod_{k=1}^{N}(\lambda-\lambda_k)\Big)\Big|_{\lambda=\lambda_n}.$$

Due to the weak interlacing property of Λ the numbers β_n are positive and the proof of Lemma 3.3 shows that $\beta_n = 2+\tilde\beta_n$ for a sequence $(\tilde\beta_n)$ belonging to $\ell_2(B_\Lambda)$.

Now we define the sequence (α_n) with $\alpha_n=\beta_n$ if $n\in B_\Lambda$ and take α_n to be an arbitrary positive number if $n\in A_\Lambda$. The sequences (λ_n) and (α_n) satisfy all the requirements of Proposition 2.3 and thus there exists an operator $\mathcal{A}$ of the form (1.1) whose eigenvalues and norming constants coincide respectively with (λ_n) and (α_n).

It remains to prove that the sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n=1}^{N-1}$ we have started with coincide with the eigenvalues $(\mu_{n,\mathrm{D}})_{n\in\mathbb{N}}$ and $(\nu_n^1)_{n=1}^{N-1}$ of the related operator S_{D} and Jacobi matrix $J_{(1)}$ respectively. Since for the norming constants α_n with $n\in B_\Lambda$ formula (2.3) holds, we conclude that, for such n,

$$\frac{a_0^2\prod_{k=1}^{N-1}(\lambda_n-\nu_k^1)}{y_-(1,\lambda_n)} = \frac{\gamma_0\prod_{k=1}^{N-1}(\lambda_n-\nu_k)}{2\psi(\lambda_n)},$$

i.e., that

$$\frac{2a_0^2\psi(\lambda_n)}{\sqrt{\lambda_n}}\prod_{k=1}^{N-1}(\lambda_n-\nu_k^1) - \frac{\gamma_0 y_-(1,\lambda_n)}{\sqrt{\lambda_n}}\prod_{k=1}^{N-1}(\lambda_n-\nu_k) = 0. \tag{3.2}$$

Recalling that $\psi(\lambda_n)=0=\prod_{k=1}^{N-1}(\lambda_n-\nu_k)$ for $n\in A_\Lambda$, we conclude that equality (3.2) holds for all $n\in\mathbb{N}$. We observe that (3.2) takes the form $\Phi(\lambda_n)=0$, where the function Φ satisfies the relation

$$\Phi(z) = \mathrm{O}(|z|^{N-3/2}\mathrm{e}^{|\operatorname{Im}\sqrt{z}|}) \tag{3.3}$$

as $|z|\to\infty$. We shall prove that $\Phi\equiv 0$.

Assume not, and observe that Φ has then no zeros other than λ_n, $n\in\mathbb{N}$. Indeed, in view of (3.3) Jensen's formula gives

$$\begin{aligned}\int_1^r\frac{n(t)}{t}dt &\le \frac{1}{2\pi}\int_0^{2\pi}\log|\Phi(re^{i\theta})|\,d\theta + C_1\\ &\le (N-\tfrac32)\log r + \frac{\sqrt{r}}{2\pi}\int_0^{2\pi}|\sin\theta/2|\,d\theta + C_2\\ &= (N-\tfrac32)\log r + \frac{2\sqrt{r}}{\pi} + C_2,\end{aligned} \tag{3.4}$$

where $n(t)$ denotes the number of zeros of Φ in the closed circle of radius t centered at the origin and C_1 and C_2 are some positive constants. On the other hand, if Φ had at least one additional zero, then for any $\varepsilon>0$ and all sufficiently large t we would have

$$n(t) \ge \Big[\frac{\sqrt{t}}{\pi}-\varepsilon\Big] + N + 1 \ge \frac{\sqrt{t}}{\pi} + N - \varepsilon,$$

which contradicts (3.4). Now Φ, being of exponential type $\frac{1}{2}$, equals

$$\Phi(z) = C_3 \prod_{n=1}^{\infty}\Big(1 - \frac{z}{\lambda_n}\Big)$$

for some constant C_3. Using the canonical product for $\sin\sqrt{z}$ and the asymptotics of λ_k, we conclude that

$$\lim_{z\to-\infty} \frac{\Phi(z)}{z^{N-\frac{1}{2}}\sin\sqrt{z}} =: C_4 \neq 0,$$

which contradicts (3.3).

Thus we have proved that $\Phi \equiv 0$, i.e., that

$$2a_0^2 \prod_{k\in\mathbb{N}} \frac{\mu_k - z}{\pi^2 k^2} \prod_{k=1}^{N-1} (z - \nu_k^1) \equiv \gamma_0 \prod_{k=1}^{\infty} \frac{\mu_{k,\mathrm{D}} - z}{\pi^2 k^2} \prod_{k=1}^{N-1} (z - \nu_k).$$

It follows that every ν_n that does not occur in (μ_k) is an eigenvalue of $J_{(1)}$ and, similarly, every μ_n that does not occur in (ν_k) is an eigenvalue of S_{D}. Since the sequences (λ_n) and $(\mu_{n,\mathrm{D}}) \amalg (\nu_n^1)$ weakly interlace in the sense of Lemma 3.2, and since the same is true of (λ_n) and (λ'_n), simple considerations show that every multiple element of (λ'_n) belongs to the spectra of both S_{D} and $J_{(1)}$, cf. [13, Sect. 6]. Thus all μ_n are eigenvalues of S_{D} and all ν_n—those of $J_{(1)}$. Since neither $J_{(1)}$ nor S_{D} can have other eigenvalues due to the size and asymptotics limitations respectively, Λ is the spectral triple for the operator $\mathcal{A}$ found.

If the set A_Λ is empty, then the norming constants α_n are uniquely determined by Λ, so that $\mathcal{A}$ is unique in view of Proposition 2.3. If A_Λ is non-empty, then different choices of α_n for $n \in A_\Lambda$ lead to different operators $\mathcal{A}$. The proof is complete. □

Remark 3.6. It follows from the proof of Theorem 3.5 that the set of Λ-isospectral operators $\mathcal{A}$ of the form (1.1) (i.e., the set of operators $\mathcal{A}$ such that the spectra of $\mathcal{A}$, S_{D}, and $J_{(1)}$ form the prescribed triple $\Lambda \in \mathfrak{L}_N$) is a manifold of dimension equal to the cardinality of the set A_Λ.

4. Reconstruction from the spectra of $\mathcal{A}$, S_{N}, and J

Treatment of the inverse problem of reconstructing the operator $\mathcal{A}$ from the spectra of the operators $\mathcal{A}$, S_{N}, and the Jacobi matrix J parallels in general that of the inverse problem of Section 3. One essential difference is that the invariance of $\mathcal{A}$ with respect to changing the primitive σ to $\sigma + C$ and b_1 to $b_1 + a_0 C$ (mentioned in Remark 1.1) is important here as it changes the spectra of both the operator S_{N} and the Jacobi matrix J. Thus the more correct inverse problem should be not only to reconstruct the operator $\mathcal{A}$ per se, but also to fix the appropriate quasi-derivative σ of the potential q and the corresponding Jacobi matrix J.

Given an arbitrary operator matrix $\mathcal{A}$ of the form (1.1) (with fixed σ), we denote by $(\lambda_n)_{n\in\mathbb{N}}$, $(\mu_{n,\mathrm{N}})_{n\in\mathbb{N}}$, and $(\nu_{n,\mathrm{J}})_{n=1}^N$ the eigenvalue sequences of $\mathcal{A}$, S_{N}, and J respectively. Put also $(\lambda'_n)_{n\in\mathbb{N}} := (\mu_{n,\mathrm{D}}) \amalg (\nu_{n,\mathrm{J}})_{n=1}^N$. The above three spectra have the same intersection property as those of Section 3, namely

Proposition 4.1. ([1]). $\sigma(\mathcal{A}) \cap \sigma(S_{\mathrm{N}}) = \sigma(\mathcal{A}) \cap \sigma(J) = \sigma(S_{\mathrm{N}}) \cap \sigma(J)$.

Lemma 4.2. *The sequences (λ'_n) and (λ_n) weakly interlace, i.e., for every $n \in \mathbb{N}$ either $\lambda'_n < \lambda_n < \lambda'_{n+1}$ or $\lambda'_n = \lambda_n = \lambda'_{n+1}$.*

Proof. We observe that (λ'_n) is the sequence of eigenvalues of the operator $\mathcal{A}_0 = S_{\mathrm{N}} \oplus J$ counting multiplicities and that $\mathcal{A}$ and $\mathcal{A}_0$ are self-adjoint extensions of the symmetric operator $\mathcal{A}'$, which is the restriction of $\mathcal{A}$ onto the domain

$$\mathcal{D}(\mathcal{A}') := \{(y, d)^{\mathrm{t}} \in \mathcal{D}(\mathcal{A}) \mid y^{[1]}(1) = 0\}$$

and has deficiency indices $(1,1)$. We denote by $\mathcal{H}'$ a maximal subspace of $\mathcal{D}(\mathcal{A}')$ that is invariant with respect to $\mathcal{A}'$ and put $\mathcal{H}'' := \mathcal{H} \ominus \mathcal{H}'$. The restrictions of the operators $\mathcal{A}$ and $\mathcal{A}_0$ onto $\mathcal{H}'$ coincide (with $\mathcal{A}'$) and $\dim \mathcal{H}' \le N$ since if $Y = (y, d)^{\mathrm{t}}$ is an eigenvector of $\mathcal{A}$ that belongs to $\mathcal{H}'$, then d is an eigenvector of J. It follows from [7] (see also [2, Ch. 1.2]) that the spectra of the restrictions of $\mathcal{A}$ and $\mathcal{A}_0$ onto the subspace $\mathcal{H}''$ strictly interlace. Combining the two parts together, we see that either $\lambda'_n \le \lambda_n$ for all $n \in \mathbb{N}$ or $\lambda_n \le \lambda'_n$ for all $n \in \mathbb{N}$; however, the inequality $\lambda_n \le \lambda'_n$ is ruled out for all n sufficiently large by the asymptotics of λ_n and $\mu_{n,\mathrm{N}}$, see Propositions 2.3 and 2.4. Taking into account the intersection property of Proposition 4.1, we conclude that the spectra weakly interlace in the specified sense. ☐

Definition 4.3. We denote by $\mathfrak{L}'_N$ the set of all triples of strictly monotone sequences $\Lambda := \big((\lambda_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}, (\nu_n)_{n=1}^N\big)$ satisfying the following properties:

(1) there is an ℓ_2-sequence $(\tilde\lambda_n)$ such that $\lambda_n = [\pi(n-N) + \tilde\lambda_n]^2$;
(2) there is an ℓ_2-sequence $(\tilde\mu_n)$ such that $\mu_n = [\pi(n-\frac12) + \tilde\mu_n]^2$;
(3) the sequences (λ_n) and $(\lambda'_n) := (\mu_n) \amalg (\nu_n)$ weakly interlace in the sense of Lemma 4.2.

We denote by A_Λ the set of $n \in \mathbb{N}$ such that $\lambda_n = \lambda'_n$ and put $B_\Lambda := \mathbb{N} \setminus A_\Lambda$.

The results obtained so far show that, for any operator $\mathcal{A}$ of the form (1.1), the corresponding spectral triple $\big((\lambda_n), (\mu_{n,\mathrm{N}}), (\nu_{n,\mathrm{J}})\big)$ form an element of $\mathfrak{L}'_N$. In the reverse direction, we shall prove that any element of $\mathfrak{L}'_N$ is the spectral triple of the above form. The approach lies in reducing the problem to that of reconstruction of $\mathcal{A}$ from the eigenvalues and the norming constants. Lemmas 2.2, 2.6, and 2.7 imply that the three spectra determine uniquely the norming constants α_n for $n \in B_\Lambda$. Hence, if a given triple $\Lambda \in \mathfrak{L}'_N$ is composed of the spectra of some $\mathcal{A}$ and its two parts, then the corresponding norming constants must be related to Λ via the formulae established in Section 2. As a preliminary, we show that any triple in $\mathfrak{L}'_N$ produces in this way the numbers with correct asymptotics.

Lemma 4.4. *Assume that* $\Lambda = \big((\lambda_n), (\mu_n), (\nu_n)\big) \in \mathfrak{L}'_N$ *and define the functions* ϕ, ψ, *and* χ *by the formulae*

$$\phi(\lambda) = \prod_{k=1}^N (\lambda - \lambda_k) \prod_{k=1}^\infty \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2},$$

$$\psi(\lambda) = \sqrt{\lambda} \prod_{k=1}^\infty \frac{\mu_k - \lambda}{\pi^2 (k - \frac12)^2},$$

$$\chi(\lambda) = \prod_{k=1}^N (\lambda - \nu_k).$$

Then the numbers

$$\beta_n := \frac{\chi(\lambda_n)}{\sqrt{\lambda_n}\,\dot\phi(\lambda_n)\psi(\lambda_n)}, \quad n \in B_\Lambda, \tag{4.1}$$

have the asymptotics

$$\beta_n = 2 + \tilde\beta_n$$

where the sequence $(\tilde\beta_n)$ *belongs to* $\ell_2(B_\Lambda)$.

Proof. It clearly suffices to prove that

$$\frac{1}{\beta_n} = \frac{\sqrt{\lambda_n}\,\dot\phi(\lambda_n)\psi(\lambda_n)}{\chi(\lambda_n)} = \frac12 + \hat\beta_n$$

for some sequence $(\hat\beta_n) \in \ell_2$. In view of the asymptotics of (λ_k) and Proposition 2.8, there exists a function $f \in L_2(0,1)$ such that

$$\phi(\lambda) = \Big(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \int_0^1 f(t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,dt\Big)\prod_{k=1}^N(\lambda-\lambda_k)$$

and thus, for $n > N$,

$$\dot\phi(\lambda_n) = \frac{\cos\sqrt{\lambda_n} + \int_0^1 tf(t)\cos\sqrt{\lambda_n}t\,dt}{2\lambda_n}\prod_{k=1}^N(\lambda-\lambda_k).$$

Similarly, for some $g \in L_2(0,1)$ it holds

$$\psi(\lambda) = \sqrt{\lambda}\cos\sqrt{\lambda} + \sqrt{\lambda}\int_0^1 g(t)\cos\sqrt{\lambda}t\,dt.$$

Since the system $\{\sin\sqrt{\lambda_n}t\}_{n>N}$ forms a Riesz basis of $L_2(0,1)$ [9], for any $h \in L_2(0,1)$ the sequence

$$\int_0^1 h(t)\cos\sqrt{\lambda_n}t\,dt, \quad n > N,$$

is square summable. The asymptotics of λ_n implies that $\cos\sqrt{\lambda_n} = (-1)^{n+N}(1+b_n)$, where the sequence (b_n) is in ℓ_2. Combining these relations, we arrive at the representation

$$\frac{\sqrt{\lambda_n}\dot\phi(\lambda_n)\psi(\lambda_n)}{\chi(\lambda_n)} = \frac{1}{2}\big(1+d_n\big)\prod_{k=1}^N\frac{\lambda_n-\lambda_k}{\lambda_n-\nu_k}$$

with $(d_n) \in \ell_2$, which yields the result. □

Theorem 4.5. *For any $\Lambda := \big((\lambda_n),(\mu_n),(\nu_n)\big) \in \mathfrak{L}'_N$ there exist $a_0 > 0$, a function $\sigma \in L_2(0,1)$ and a Jacobi matrix J of size N such that (λ_n) is the spectrum of the corresponding operator $\mathcal{A}$ in $L_2(0,1)\times\mathbb{C}^N$ of the form (1.1), (μ_n) is the spectrum of the operator S_{N}, and (ν_k) is the spectrum of the Jacobi matrix J. The operator $\mathcal{A}$ is unique if and only if the set A_Λ is empty.*

Proof. We start with constructing the functions ϕ,ψ, and χ of Lemma 4.4 and defining the numbers β_n as in (4.1). Next, we put $\alpha_n = \beta_n$ for $n \in B_\Lambda$, and take α_n arbitrary positive for $n \in A_\Lambda$. According to Lemma 4.4, α_n obey the asymptotics $\alpha_n = 2 + \tilde\alpha_n$ with some $(\tilde\alpha_n) \in \ell_2$.

By Proposition 2.3, there exists an operator $\mathcal{A}$ of the form (1.1), whose eigenvalues are λ_n and the corresponding norming constants are α_n. We claim that one can fix a primitive of the potential q of the operator S and a Jacobi matrix J in the representation of $\mathcal{A}$ in such a way that μ_n are the eigenvalues of the operator S_{N} and ν_n are the eigenvalues of J.

We take k^* such that μ_{k^*} is not an eigenvalue of $\mathcal{A}$ just found, fix the unique primitive σ of the potential q of the Sturm–Liouville operator S such that the relation $(y_-' - \sigma y_-)(1,\mu_{k^*}) = 0$ holds, and determine the corresponding Jacobi matrix J giving the representation (1.1) of $\mathcal{A}$. We denote by $\mu_{n,\mathrm{N}}$ and $\nu_{n,\mathrm{J}}$ the eigenvalues of S_{N} and J and observe that the above choice of σ makes μ_{k^*} an eigenvalue of S_{N}. Due to the construction of β_n and formula (2.2) for α_n, we have the equality

$$\frac{\psi(\lambda_n)}{\chi(\lambda_n)} = \sqrt{\lambda_n}\prod_{k=1}^\infty\frac{\mu_{k,\mathrm{N}}-\lambda_n}{\pi^2(k-\frac12)^2}\Big/\prod_{k=1}^N(\lambda_n-\nu_{k,\mathrm{J}})$$

for all $n \in B_\Lambda$. Recalling that $\psi(\lambda_n) = \chi(\lambda_n) = 0$ for $n \in A_\Lambda$, we see that

$$\psi(\lambda_n)\prod_{k=1}^N(\lambda_n-\nu_{k,\mathrm{J}}) = \sqrt{\lambda_n}\chi(\lambda_n)\prod_{k=1}^\infty\frac{\mu_{k,\mathrm{N}}-\lambda_n}{\pi^2(k-\frac12)^2}$$

for all $n \in \mathbb{N}$.

Put

$$\Phi_1(z) := \frac{\psi(z)}{\sqrt{z}} \prod_{k=1}^{N} (z - \nu_{k,\mathrm{J}}), \quad \Phi_2(z) := \chi(z) \prod_{k=1}^{\infty} \frac{\mu_{k,\mathrm{N}} - z}{\pi^2 (k - \frac{1}{2})^2};$$

then $\Phi_1(\lambda_n) = \Phi_2(\lambda_n)$ for all $n \in \mathbb{N}$, and also $\Phi_1(\mu_{k^*}) = \Phi_2(\mu_{k^*}) = 0$ (the latter relation follows from the fact that μ_{k^*} is among $\mu_{n,\mathrm{N}}$ by the construction of S_{N}). In view of Proposition 2.8 the functions Φ_j have the form

$$\Phi_j(z) = p_j(z)\Big(\cos\sqrt{z} + \int_0^1 g_j(t) \cos\sqrt{z}t\, dt\Big)$$

for some monic polynomials p_j of degree N and some functions $g_j \in L_2(0,1)$, $j = 1, 2$. It follows that $\Phi := \Phi_1 - \Phi_2$ is an entire function of exponential type $\frac{1}{2}$ with zeros $\{\lambda_n\}_{n\in\mathbb{N}} \cup \{\mu_{k^*}\}$ such that

$$\Phi(z) = \mathrm{o}\big(z^N \,\mathrm{e}^{|\operatorname{Im}\sqrt{z}|}\big) \tag{4.2}$$

as $|z| \to \infty$. Next we show as in the proof of Theorem 3.5 that $\Phi \equiv 0$ by noticing that otherwise Φ would have no zeros other than λ_n, $n \in \mathbb{N}$, and μ_{k^*}, and that the canonical product for Φ then would contradict the estimate (4.2).

Thus $\Phi_1 \equiv \Phi_2$, which together with the weak interlacing property of (λ_n) and (λ'_n) as well as of (λ_n) and $(\mu_{n,\mathrm{N}}) \amalg (\nu_{n,\mathrm{J}})$ shows that $\mu_n = \mu_{n,\mathrm{N}}$ for all $n \in \mathbb{N}$ and that $\nu_k = \nu_{k,\mathrm{J}}$ for $k = 1, \dots, N$. Uniqueness statement follows from Proposition 2.3, and the proof is complete. □

We remark that the set of Λ-isospectral operators $\mathcal{A}$ is again a manifold of dimension equal to the cardinality of the set A_Λ.

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Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D–53115, Bonn, Germany; SFB 611, Bonn, Germany; BiBoS, Bielefeld, Germany; IZKS; CERFIM, Locarno, Switzerland; and Accademia di Architettura, Mendrisio, Switzerland
E-mail address: `albeverio@uni-bonn.de`

Institute for Applied Problems of Mechanics and Mathematics, 3b Naukova, Lviv, 79601, Ukraine
E-mail address: `rhryniv@iapmm.lviv.ua`

Lviv National University, 1 Universytetska, Lviv, 79602, Ukraine
E-mail address: `yamykytyuk@yahoo.com`

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