Methods of Functional Analysis and Topology
Vol. 13 (2007), no. 1, pp. 67–82

ON MODELS OF FUNCTION TYPE FOR A SPECIAL CLASS OF NORMAL OPERATORS IN KREIN SPACES AND THEIR POLAR REPRESENTATION

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Abstract. The paper is devoted to a function model representation of a normal operator N acting in a Krein space. We assume that N and its adjoint operator $N^{\#}$ have a common invariant subspace L_+ which is a maximal nonnegative subspace and has a representation as a sum of a finite-dimensional neutral subspace and a uniformly positive subspace. For N we construct a model representation as the multiplication operator by a scalar function acting in a suitable function space. This representation is applied to the problem of existence of a polar representation for normal operators of D^+_κ-class.

0. Introduction

This work has a direct connection with the papers [20], [3], [4] and [19]. It is assumed the reader is familiar with the elements of Krein space geometry and operator theory (see [7], [1], [13], [15]). In this paper the terminology introduced in [2] will be used. In Section 1 one can find terminology, including the notion of normal operators of D^+_κ-class (Definition 1.5), and known results that are used in the next sections. In particular, Theorem 1.8 describes a spectral decomposition of a J-normal operator of D^+_κ-class that will play a key role in the course of the work. In Section 2 there are some results on model representation for J-unitary, J-self-adjoint and J-normal operators (Theorems 2.6, 2.7, 2.8 and 2.9).

In Section 3 these results are applied to the problem of polar representation for J-normal operators (Theorems 3.6, 3.8 and 3.11, Examples 3.3, 3.9, 3.10 and 3.12).

1. Preliminaries

1.1. **Basic definitions.** Let $\mathcal{H}$ be a Krein space with an indefinite sesquilinear form $[\cdot,\cdot]$. Everywhere below $[\cdot,\cdot]$ is assumed to be fixed and is called the Krein form. At the same time let us note that in the problem we consider the concrete choice of the Hilbert scalar product on $\mathcal{H}$ is not really essential. One needs only to fix the topology (defined by the above mentioned scalar product) and the structure of the Gram operator $J\colon [x,y] = (Jx,y)$. According to the tradition, we employ the term "canonical scalar product" for any scalar product $(\cdot,\cdot)_1$ if it generates on $\mathcal{H}$ the same input topology and if the corresponding Gram operator J_1 is of the Krein form (i.e., $[x,y] = (J_1x,y)_1$) is unitary with respect to $(\cdot,\cdot)_1$. The Gram operator of the Krein form in the case of the canonical scalar product is called a canonical symmetry.

We use the terms "positive vector", "neutral vector", "non-negative subspace", "maximal non-negative subspace", etc., in the usual way; they are defined with respect to

2000 *Mathematics Subject Classification.* Primary 46C20, 47B50; Secondary 47B40, 47A60.
Key words and phrases. Indefinite metric, spectral resolution, function model, polar representation.
This work was completed with the support of project FONACIT (Venezuela) N 97000668.

the sign of the Krein form (see [2]). Analogously by the symbol $[\perp]$ we denote the orthogonality of vectors or sets with respect to the Krein form.

The following definition was introduced in [10].

Definition 1.1. A subspace $\mathcal{L}$ is called *pseudo-regular* if

$$\mathcal{L} = \hat{\mathcal{L}}_+ \dot{+} \mathcal{L}_1, \tag{1}$$

where $\hat{\mathcal{L}}_+$ is a regular (=projectionally complete) subspace and $\mathcal{L}_1$ is a neutral subspace (i.e. $\mathcal{L}_1$ is an isotropic part of $\mathcal{L}$).

Below we shall use the following result concerning some special choice of canonical scalar products.

Proposition 1.2. ([3]). *Let:*

- $\mathcal{L}_+$ *be a pseudo-regular maximal nonnegative subspace;*
- $\mathcal{L}_1$ *be an isotropic subspace of* $\mathcal{L}_+$*;*
- $(\cdot,\cdot)'$ *be a scalar product on* $\mathcal{L}_1$*, such that the norm* $\sqrt{(x,x)'}$ *is equivalent to the original one;*
- $\mathcal{L}_- = \mathcal{L}_+^{[\perp]}$*;*

and let

$$\mathcal{L}_+ = \hat{\mathcal{L}}_+ \dot{+} \mathcal{L}_1, \quad \mathcal{L}_- = \hat{\mathcal{L}}_- \dot{+} \mathcal{L}_1, \tag{2}$$

where $\hat{\mathcal{L}}_+$ *and* $\hat{\mathcal{L}}_-$ *are uniformly definite subspaces. Then one can define on* $\mathcal{H}$ *a canonical scalar product* $(\cdot,\cdot)$ *such that:*

$$\left\{\begin{array}{ll} \text{a)} & \text{on } \mathcal{L}_1 \ : \ (\cdot,\cdot) \equiv (\cdot,\cdot)', \\ \text{b)} & \mathcal{L}_1 \perp \hat{\mathcal{L}}_+, \ \mathcal{L}_1 \perp \hat{\mathcal{L}}_-, \\ \text{c)} & \text{on } \hat{\mathcal{L}}_+ \ : \ (\cdot,\cdot) = [\cdot,\cdot], \\ \text{d)} & \text{on } \hat{\mathcal{L}}_- \ : \ (\cdot,\cdot) = -[\cdot,\cdot]. \end{array}\right. \tag{3}$$

Definition 1.3. If a canonical scalar product of a Krein space $\mathcal{H}$ has the properties (3), it is said to be *compatible* with Decomposition (2) and the choice of the scalar product $(\cdot,\cdot)'$ on $\mathcal{L}_1$.

Let A be an operator. Then $A^{\#}$ means the operator adjoint in the sense of the Krein form (briefly J*-adjoint*) to A. If $\mathfrak{Y}$ is an operator family, the symbol $\operatorname{Alg}\mathfrak{Y}$ means the minimal closed (in the weak topology) algebra which contains $\mathfrak{Y}$ and the identity operator.

Definition 1.4. An operator N is called J*-normal* (=J-n.) if $NN^{\#} = N^{\#}N$.

By definition the *real part* of an operator C is the operator $(C + C^{\#})/2$ and the *imaginary part* is the operator $(C - C^{\#})/2i$. It is clear that the real part and the imaginary part of an arbitrary bounded operator are J-self-adjoint (=J-s.a.).

Definition 1.5. Let N be a J-n. operator and let A and B be its real part and its imaginary part respectively. The operator N belongs to the class D_κ^+ if in there is a subspace $\mathcal{L}_+ \subset \mathcal{H}$ such that

- $\mathcal{L}_+$ is A-invariant and B-invariant,
- $\mathcal{L}_+$ is a maximal non-negative subspace,
- $\mathcal{L}_+$ is a pseudo regular subspace,
- $\dim(\mathcal{L}_+ \cap \mathcal{L}_+^{[\perp]}) = \kappa$.

Proposition 1.6. ([3]) . *Let* N *be a* J*-n. operator and let* $N \in D_\kappa^+$*. Then there exists a* J*-orthogonal projection* $P \in \operatorname{Alg}\{N, N^{\#}\}$ *such that*

- *the subspace $(I-P)\mathcal{H}$ has finite dimension;*
- *if $N = A + iB$, $A = A^{\#}$, $B = B^{\#}$, then $\sigma(A|_{P\mathcal{H}}) \subset \mathbb{R}$ and $\sigma(B|_{P\mathcal{H}}) \subset \mathbb{R}$.*

In view of Proposition 1.6 in what follows we shall consider J-n. operators whose real and imaginary parts have real spectra.

1.2. **Some function spaces.** Let us pass to some notation related to direct integrals of Hilbert spaces and corresponding model descriptions of self-adjoint operators (see [17], §41; [6], Chapter 7; [8], Chapter 4.4; [18], Chapter VII). Assume that $\sigma(t)$ is a non-decreasing function defined on the segment $[-1;1]$, continuous in the points -1; 0; 1, continuous (at least) from the left in all other points of the segment and having an infinite number of growth points, where zero is one of these points. The mentioned function generates on $[-1;1]$ the Lebesgue–Stieltjes measure μ_σ and spaces (L^2_σ, L^∞_σ, etc.) of complex-valued functions. We shall consider also some spaces of vector-valued functions so from time to time we shall note after a symbol of a space a symbol of a range for the functions forming this space, for instance, $L^2_\sigma(\mathbb{C})$. Let $\mathcal{E}$ be some separable Hilbert space ($\mathcal{E}$ can be finite-dimensional) Consider a mapping $t \mapsto \mathcal{E}_t$, $t \in [-1;1]$, where $\mathcal{E}_t \subset \mathcal{E}$, $\dim(\mathcal{E}_t)$ is a μ_σ-measurable (but not necessarily finite a.e.) function, and if $\dim(\mathcal{E}_{t_1}) = \dim(\mathcal{E}_{t_2})$, then $\mathcal{E}_{t_1} = \mathcal{E}_{t_2}$. Denote by $M_{\vec{\sigma}}(\mathcal{E})$ the space of the vector-valued functions $f(t)\colon\ t \mapsto \mathcal{E}_t$ μ_σ-measurable in the weak sense, defined a.e. and finite a.e. on the segment $[-1;1]$. Next, the symbol $L^2_{\vec{\sigma}}(\mathcal{E})$ means here a Hilbert space of functions $f(t) \in M_{\vec{\sigma}}(\mathcal{E})$, such that $\int_{-1}^{1} \|f(t)\|^2_{\mathcal{E}}\, d\sigma(t) < \infty$.

We introduce also some notation related to multiplication operators by scalar functions. Everywhere below we assume a scalar function $\varphi(t)$ to be defined a.e. on $[-1;1]$, μ_σ-measurable and a.e. bounded. For $f(t) \in M_{\vec{\sigma}}(\mathcal{E})$ set

$$(\Phi f)(t) = \varphi(t) f(t). \tag{4}$$

It is clear that $(\Phi f)(t) \in M_{\vec{\sigma}}(\mathcal{E})$, so equality (4) defines on $M_{\vec{\sigma}}(\mathcal{E})$ the continuous operator Φ (= the multiplication operator by the function $\varphi(t)$). If $\varphi(t)$ satisfies some additional conditions one can consider the operator Φ as acting simultaneously on different spaces. If, for instance, $\varphi(t)$ is continuous then the operator Φ is well defined on every space $M_\sigma(\mathcal{E})$ independently of $\vec{\sigma}(t)$ and $\mathcal{E}$. If $\varphi(t) \in L^\infty_\sigma(\mathbb{C})$ then $L^2_{\vec{\sigma}}(\mathcal{E})$ can also be taken as a domain of Φ. So, if necessary, we shall mention simultaneously the operator Φ and its domain using the notation $\{\Phi, \mathfrak{D}(\Phi)\}$, say, $\{\Phi, L^2_{\vec{\sigma}}(\mathcal{E})\}$.

Let us introduce an analog of $L^2_{\vec{\sigma}}(\mathcal{E})$ that can be used for a model representation of Krein spaces. Assume that the scalar functions $\sigma_+(t)$ and $\sigma_-(t)$ are such that

$$\sigma_+(t) = \int_{-1}^{t} \rho_+(\lambda)\, d\sigma(\lambda), \quad \sigma_-(t) = \int_{-1}^{t} \rho_-(\lambda)\, d\sigma(\lambda), \quad \rho_+^2(\lambda) = \rho_+(\lambda),$$

$$\rho_-^2(\lambda) = \rho_-(\lambda), \quad \sigma(t) = \int_{-1}^{t} \Big(\rho_+(\lambda) + \rho_-(\lambda) - \rho_+(\lambda)\rho_-(\lambda)\Big)\, d\sigma(\lambda),$$

where $\sigma(\lambda)$ is the same as in the previous subsection, and set

$$\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E}) := L^2_{\vec{\sigma}_+}(\mathcal{E}_+) \oplus L^2_{\vec{\sigma}_-}(\mathcal{E}_-), \quad [f(t), g(t)]\colon = (f_+(t), g_+(t)) - (f_-(t), g_-(t)),$$

where $f(t) = f_+(t) + f_-(t)$, $g(t) = g_+(t) + g_-(t)$, $f_+(t), g_+(t)) \in L^2_{\vec{\sigma}_+}(\mathcal{E}_+)$, $f_-(t), g_-(t) \in L^2_{\vec{\sigma}_-}(\mathcal{E}_-)$. The space $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$ is said to be *a standard Krein space*. As a slight abuse of the previous notation put also $M_{\vec{\sigma}}(\mathcal{E})\colon = M_{\vec{\sigma}_+}(\mathcal{E}_+) \oplus M_{\vec{\sigma}_-}(\mathcal{E}_-)$.

1.3. **Spectral functions with peculiarities.** Let $\Lambda = \{\lambda_k\}_1^n$ be a finite set of real numbers and let $\mathfrak{R}_\Lambda$ be the family $\{X\}$ of all Borel subsets of $\mathbb{R}$ such that $\partial X \cap \Lambda = \emptyset$, where ∂X is the boundary of X in $\mathbb{R}$. Let $E\colon\ X \mapsto E(X)$ be a countably additive (with respect to weak topology) function, that maps $\mathfrak{R}_\Lambda$ to a commutative algebra of projections in a Hilbert space $\mathcal{H}$, $E(\mathbb{R}) = I$. $E(X)$ is called *a spectral function* (*on* $\mathbb{R}$)

with the peculiar spectral set Λ, the mention of Λ can be omitted. The symbol $\mathrm{Supp}(E)$ means the minimal closed subset $S \subset \mathbb{R}$, such that $E(X) = 0$ for every X: $X \subset \mathbb{R}\backslash S$ and $X \in \mathfrak{R}_\Lambda$. Besides the symbol E we shall use also as notation for a spectral function the symbol E_λ, $\lambda \in \mathbb{R}$, where $E_\lambda = E((-\infty, \lambda))$. Note that the notion of peculiar set has no any direct connection with the behavior of the spectral function and it means only that some points on $\mathbb{R}$ are distinguished. See below Definition 1.7 for some explanations. A spectral function E that acts in a Krein space, is said to be *J-orthogonal* (J-orth.sp.f.) if $E(X)$ is a J-ortho-projection for every $X \in \mathfrak{R}_\Lambda$.

Let us recall that an operator A with real spectrum in a Hilbert space is said to be a *scalar spectral operator* ([9]) if there exists a spectral function E with empty peculiar spectral set Λ, such that for every $X \in \mathfrak{R}_\Lambda$: $E(X)A = AE(X)$, $\sigma(A|_{E(X)\mathcal{H}}) \subset \bar{X}$ and $AE(X) = \int_X \xi E(d\xi)$ in the weak sense.

Definition 1.7. Let E be a J-orth.sp.f. with a peculiar spectral set Λ. Let λ be a peculiarity (i.e. $\lambda \in \Lambda$). Fix a set $X \in \mathfrak{R}_\Lambda$: $X \cap \Lambda = \{\lambda\}$. The peculiarity λ is called *regular* if the operator family $\{E(X \cap Y)\}_{Y \in \mathfrak{R}_\Lambda}$ is bounded, otherwise it is called *singular*.

The notion of regular and singular peculiarities is correctly defined since the boundedness of the family $\{E(X \cap Y)\}_{Y \in \mathfrak{R}_\Lambda}$ does not depend on X.

Let E be a spectral function with peculiar spectral set Λ. A scalar function $f(\xi)$ is said to be *defined almost everywhere* (with respect to E), *to have finite value almost everywhere*, etc., if the corresponding property holds almost everywhere in the weak sense for every $X \in \mathfrak{R}_\Lambda$, $X \cap \Lambda = \emptyset$. We'll assume that the function $f(\xi)$ is not defined at Λ. The following theorem follows directly from results announced in [20] and proved in [4].

Theorem 1.8. *Let N be a J-n. operator, let $N \in D^+_\kappa$ and let $N = A + iB$, where $A = A^\#$, $B = B^\#$, $\sigma(A) \subset \mathbb{R}$ and $\sigma(B) \subset \mathbb{R}$. Then there exists a J-orthogonal spectral function E_λ with a finite number of spectral peculiarities Λ (Λ may be the empty set), such that the following conditions hold (*CLin $=$*closed linear span)*

(5)

- *a)* $E_\lambda \in \mathrm{Alg}\{N, N^\#\}$ *for all* $\lambda \in \mathbb{R}\backslash\Lambda$*;*
- *b)* *there is a non-negative subspace $\mathcal{L}_+$, corresponding to Definition 1.5, for which the decomposition $E(\Delta)\mathcal{H} = E(\Delta)\mathcal{L}_+[\dot{+}]E(\Delta)\mathcal{L}_-$ holds, Δ being any closed segment $\Delta \subset \mathbb{R}$ satisfying $\Delta \in \mathfrak{R}_\Lambda$ and $\Delta \cap \Lambda = \emptyset$;*
- *c)* *there exist a defined almost everywhere (uniformly) bounded functions $\phi(\lambda)$ and $\psi(\lambda)$ such that for every interval $\Delta \subset \mathbb{R}$, $\Delta \in \mathfrak{R}_\Lambda$, $\Delta \cap \Lambda = \emptyset$, the decompositions $AE(\Delta) = \int_\Delta \phi(\lambda)E(d\lambda)$ and $BE(\Delta) = \int_\Delta \psi(\lambda)E(d\lambda)$ hold;*
- *d)* *the subspace $\widetilde{\mathcal{H}} = \underset{\Delta \in \mathfrak{R}_\Lambda,\ \Delta \cap \Lambda = \emptyset}{\mathrm{CLin}} \{E(\Delta)\mathcal{H}\}$ is pseudo-regular and its isotropic part has finite dimension;*
- *e)* *if $\lambda_0 \in \Lambda$ and $\mathcal{H}_{\lambda_0} := \bigcap_{\lambda_0 \in \Delta} E(\Delta)\mathcal{H}$, then the sets $\sigma(A|_{\mathcal{H}_{\lambda_0}})$ and $\sigma(B|_{\mathcal{H}_{\lambda_0}})$ are singletons $\{\mu_A\}$ and $\{\mu_B\}$, respectively, and there are positive integers m_A and m_A, such that $(A - \mu_A I)^{m_A}|_{\mathcal{H}_{\lambda_0}} = 0$ and $(B - \mu_B I)^{m_B}|_{\mathcal{H}_{\lambda_0}} = 0$;*
- *f)* *if $\lambda_0 \in \Lambda$, then $\limsup_{\lambda \to \lambda_0} \|E_\lambda\| = \infty$ or at least one of the operators $A|_{\mathcal{H}_{\lambda_0}}$ and $B|_{\mathcal{H}_{\lambda_0}}$ is not a scalar spectral operator.*

A spectral function E with a peculiar spectral set Λ satisfying Conditions (5) are called *a rectified eigen spectral function* (r.e.s.f.) of the operator N.

Remark 1.9. The notion of a r.e.s.f. differs from the notion of the eigen spectral function defined in [3]. Note also that a r.e.s.f. is not uniquely determined by the J-n. operator N.

1.4. **On a model representation for J-orthogonal spectral functions without peculiarities.** A standard Krein space will be used for a model representation of a J-orth.sp.f. E_λ without peculiarities. For simplicity, everywhere below we'll assume that

$$E_{-1} = 0, \quad E_{+1} = I, \quad E_{-1} = E_{-1+0}. \tag{6}$$

Definition 1.10. Let E_λ be a J-orth.sp.f. and let its set of peculiarities be empty. A space $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$ is said to be a *model space* for E_λ if for some canonical scalar product on $\mathcal{H}$ there is an isometric J-isometric operator $W\colon \mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E}) \mapsto \mathcal{H}$, such that for every $\lambda \in [-1;1]$: $E_\lambda = W \mathrm{X}_\lambda W^{-1}$. Here $\mathrm{X}_\lambda = \{\mathrm{X}_\lambda,\ \mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})\}$ is the multiplication operator by the indicator $\chi_{[-1,\lambda)}(t)$ of the interval $[-1,\lambda)$. The operator W is said to be an *operator of similarity.*

1.5. **Unbounded elements in Banach spaces.** Assume that $\mathcal{H}$ is a Hilbert space, P_t is a resolution of the identity (= an orthogonal spectral function with the empty set of peculiar points) defined on the segment $[-1;1]$, continuous in zero (with respect to the w-topology) and

$$\begin{cases} a) & P_{-1} = 0,\ P_1 = I; \\ b) & \text{for every } t \in [-1;1] \text{ the unilateral limits } \mathrm{w}-\lim\limits_{\mu\to t-0} P_\mu \text{ and} \\ & \mathrm{w}-\lim\limits_{\mu\to t+0} P_\mu \text{ exist, where for definiteness } P_{t-0} = P_t. \end{cases} \tag{7}$$

Set $P_{\lambda,\mu} = I + P_\lambda - P_{\mu+0}$, where $\lambda \in [-1;0)$, $\mu \in (0;1]$.

Next, let $x_{\lambda,\mu}$ be a mapping of the numerical set $[-1;0) \times (0;1]$ into $\mathcal{H}$ ($\lambda \in [-1;0)$, $\mu \in (0;1]$). The function $x_{\lambda,\mu}$ is said to be *conformed* with P_t if the following condition is fulfilled: for every $\lambda, \alpha \in [-1;0)$, $\mu, \beta \in (0;1]$ the equality $P_{\lambda,\mu} x_{\alpha,\beta} = x_{\gamma,\delta}$ holds, where $\gamma = \min\{\lambda,\alpha\}$, $\delta = \max\{\mu,\beta\}$.

Note that $x_{\lambda,\mu}$ has the following property

$$\begin{cases} \text{if } \sup\limits_{\substack{\lambda\in[-1;\,0)\\ \mu\in(0;1]}} \{\|x_{\lambda,\mu}\|\} < \infty \text{ then there is an element } x \in \mathcal{H} \\ \text{such that for every } \lambda \in [-1;0),\ \mu \in (0;1] \text{ the equality} \\ x_{\lambda,\mu} = P_{\lambda,\mu} x \text{ holds.} \end{cases} \tag{8}$$

It is clear that the element x from (8) is uniquely defined by $x_{\lambda,\mu}$ and can be found by the formula $x = \mathrm{w}-\lim\limits_{\substack{\lambda\to-0\\ \mu\to+0}} x_{\lambda,\mu}$.

Definition 1.11. A function $x_{\lambda,\mu}$ conformed with P_λ is said to be *an unbounded element conformed with* P_λ (or, if it cannot produce a misunderstanding, an unbounded element) if $\sup\limits_{\substack{\lambda\in[-1;\,0)\\ \mu\in(0;1]}} \{\|x_{\lambda,\mu}\|\} = \infty$.

Note that unbounded elements conformed with P_t exist if and only if zero is a point of growth for P_t, i.e. $P_{+\epsilon} - P_{-\epsilon} \neq 0$ for every $\epsilon > 0$. Everywhere below in this Section this condition for P_t is assumed to be fulfilled.

For brevity in what follows unbounded elements will be denoted by symbols $\widetilde{x}$, $\widetilde{y}$, etc. For $\lambda \in [-1;0)$, $\mu \in (0;1]$ we set $x_{\lambda,\mu}\colon = P_{\lambda,\mu}\widetilde{x}$.

Definition 1.12. Unbounded elements $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_k$, conformed with a (common) resolution of the identity P_λ are said to be *linearly independent modulo* $\mathcal{H}$ if every non-trivial linear combination of them is an unbounded element from $\mathcal{H}$.

Although the inner product $(\widetilde{x}, f)$ is not defined for an unbounded element $\widetilde{x}$ and an arbitrary vector $f \in \mathcal{H}$, at least for some f there is a natural way to define $(\widetilde{x}, f)$. Let $\mathcal{H}_{\lambda,\mu}\colon = P_{\lambda,\mu}\mathcal{H}$, where $\lambda \in [-1;0)$ and $\mu \in (0;1]$. Let $\mathfrak{M}$ be the linear span of the subspaces $\mathcal{H}_{\lambda,\mu}$. If $f \in \mathcal{H}_{\lambda,\mu}$, then we put $(\widetilde{x}, f) \colon = (P_{\lambda,\mu}\widetilde{x}, f) = (x_{\lambda,\mu}, f)$. Thus, for every vector $f \in \mathfrak{M}$ and an unbounded element $\widetilde{x}$ the value $(\widetilde{x}, f)$ is well defined. In what follows $\{\widetilde{x}\}^{\perp}$ means the set of all vectors $f \in \mathfrak{M}$ such that $(\widetilde{x}, f) = 0$.

Lemma 1.13. *Let unbounded elements $\widetilde{x}$, $\widetilde{x}_1$, $\widetilde{x}_2, \dots, \widetilde{x}_m$ be linearly independent modulo $\mathcal{H}$. Then*

$$\sup_{f \in \bigcap_{j=1}^{m} \{\widetilde{X}_j\}^{\perp},\ \|f\|=1} \{|f\widetilde{x}|\} = \infty.$$

Lemma 1.14. *Assume that $\mathcal{H}$ is a separable Hilbert space, $\widetilde{x}_1$, $\widetilde{x}_2, \dots, \widetilde{x}_k$ are a collection of unbounded elements conformed with P_λ and linearly independent modulo $\mathcal{H}$, and that $\{\mathfrak{C}_{\lambda,\mu}\}_{\lambda\in[-1;0),\,\mu\in(0;1]}$ is a family of vector subspaces of $\mathcal{H}$ possessing the following properties*

$$(9)\qquad \begin{cases} a) & P_{\lambda,\mu}\mathfrak{C}_{\lambda,\mu} \subset \mathfrak{C}_{\lambda,\mu}; \\ b) & \text{if } 0 \in (\lambda_2, \mu_2) \subset (\lambda_1, \mu_1), \text{ then } \mathfrak{C}_{\lambda_1,\mu_1} \subset \mathfrak{C}_{\lambda_2,\mu_2}; \\ c) & \text{for every vector } f \in \mathcal{H}_{\lambda,\mu} \text{ there is a sequence } \{f_m\}_1^\infty \subset \\ & \mathfrak{C}_{\lambda,\mu} \text{ with properties } \mathrm{w} - \lim\limits_{m\to\infty} f_m = f,\ \lim\limits_{m\to\infty} \|f_m\| = \|f\|, \\ & \text{where } \mathrm{w} - \lim \text{ is the limit considered in the w-topology.} \end{cases}$$

Then for every vector $f \in \mathcal{H}$ and an arbitrary collection of numbers $\{\alpha_j\}_{j=1}^{k}$ there is a sequence $\{g_m\}_1^\infty \subset \cup_{\lambda,\mu}\mathfrak{C}_{\lambda,\mu}$, such that

$$\begin{cases} a) & \mathrm{w} - \lim\limits_{m\to\infty} g_m = f; \\ b) & \mathrm{w} - \lim\limits_{m\to\infty} g_m\widetilde{x}_j = \alpha_j. \end{cases}$$

We give an additional notation. Let $\{\widetilde{x}_j\}_1^k$ be a fixed family of unbounded elements conformed with P_t and linearly independent modulo $\mathcal{H}$. The linear span of vectors from $\mathcal{H}$ and unbounded elements from $\{\widetilde{x}_j\}_1^k$ consistently taken as functions on $[-1;0) \times (0;1]$ is denoted $\widetilde{\mathcal{H}}$. $\widetilde{\mathcal{H}}$ will be considered as a Hilbert space, where $\mathcal{H}$ is a subspace with the the same scalar product that was given on $\mathcal{H}$ initially and unbounded elements from $\{\widetilde{x}_j\}_1^k$ are mutually orthogonal and orthogonal to $\mathcal{H}$. The space $\widetilde{\mathcal{H}}$ is said to be *the expansion* of $\mathcal{H}$ (generated by $\{\widetilde{x}_j\}_1^k$).

2. Models for a J-orth.sp.f. with a singular peculiarity and J-n. operators

By virtue of Theorem 1.8 it is clear that the general situation of $N \in D_\kappa^+$ can be reduced to the case where its J-orth.sp.f. E_λ has a unique spectral peculiarity in zero. Furthermore, the case of a regular peculiarity is trivial because under this conditions the operator N is spectral in the sense of Dunford and has a finite-dimensional nilpotent part. Thus, we can assume E_λ satisfies:

$$(10)\qquad \begin{cases} \text{a) } E_{-1} = E_{-1+0} = 0,\ E_{+1} = I; \\ \text{b) } \Lambda = \{0\}; \\ \text{c) } \sup\limits_{\lambda\in[-1;1]\setminus\{0\}} \{\|E_\lambda\|\} = \infty. \end{cases}$$

Introduce some notation. Let

$$(11)\qquad \begin{cases} \mathcal{H}_1 = \widetilde{\mathcal{H}} \cap \widetilde{\mathcal{H}}^{[\perp]},\ \mathcal{H}_2 = \mathcal{H}_1^{\perp} \cap \widetilde{\mathcal{H}},\ \mathcal{H}_0 = J\mathcal{H}_1,\ P_j \text{ be an} \\ \text{orthoprojection (in the sense of Hilbert spaces) onto } \mathcal{H}_j, \\ j = 0, 1, 2,\ \widetilde{E}_\lambda \colon = E_\lambda|_{\widetilde{\mathcal{H}}}. \end{cases}$$

Note that by Condition (10c) the inequality $\mathcal{H}_1 \neq \{0\}$ holds.

In addition to (11) set

$$(12)\qquad \widetilde{\mathcal{H}}^{\uparrow} = \mathcal{H}_0 \oplus \mathcal{H}_2, \quad \widetilde{E}_\lambda = E_\lambda|_{\widetilde{\mathcal{H}}}\,, \quad \widetilde{E}_\lambda^{\uparrow} = (P_0 + P_2)E_\lambda|_{\widetilde{\mathcal{H}}^{\uparrow}}.$$

It is necessary to take into account that, generally speaking, the subspace $\mathcal{H}_2$ is indefinite. Since J-orth.sp.f. E_λ belongs to the class D_κ^+, there is an E_λ-invariant pair of J-orthogonal maximal semi-definite pseudo-regular subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ with finite-dimensional isotropic part, moreover by Condition (5b) we can assume that for every closed interval $\Delta \subset [-1;1]\backslash\{0\}$ the subspace $(E(\Delta)\mathcal{H}) \cap \mathcal{L}_+$ is positive and the subspace $(E(\Delta)\mathcal{H}) \cap \mathcal{L}_-$ is negative. Thanks to the last hypothesis the following subspaces are well defined

$$\widetilde{\mathcal{H}}_+ = \underset{\Delta\subset[-1;1]\backslash\{0\}}{\mathrm{CLin}}\{E(\Delta)\mathcal{L}_+\}, \quad \widetilde{\mathcal{H}}_- = \underset{\Delta\subset[-1;1]\backslash\{0\}}{\mathrm{CLin}}\{E(\Delta)\mathcal{L}_-\}. \tag{13}$$

Set

$$\mathcal{H}_2^+ = \mathcal{H}_2 \cap \widetilde{\mathcal{H}}_+, \quad \mathcal{H}_2^- = \mathcal{H}_2 \cap \widetilde{\mathcal{H}}_-, \quad \mathcal{H}_3 = \left(\widetilde{\mathcal{H}} \oplus \mathcal{H}_0\right)^{[\perp]}, \tag{14}$$

and assume that a canonical scalar product on $\mathcal{H}$ is such that, first, $\mathcal{H}_3^{[\perp]} = \mathcal{H}_3^{\perp}$, second, it is also canonical for the subspace $\widetilde{\mathcal{H}} \oplus \mathcal{H}_0$ and, third, on the last space it is compatible (see Definition 1.3) with the given decompositions of the corresponding subspaces

$$\widetilde{\mathcal{H}}_+ = \mathcal{H}_1 \dot{+} \mathcal{H}_2^+ \quad \text{and} \quad \widetilde{\mathcal{H}}_- = \mathcal{H}_1 \dot{+} \mathcal{H}_2^-. \tag{15}$$

Thus, with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \dot{+} \mathcal{H}_1 \dot{+} \mathcal{H}_2 \dot{+} \mathcal{H}_3$

$$J = \begin{pmatrix} 0 & V^{-1} & 0 & 0 \\ V & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 \\ 0 & 0 & 0 & J_3 \end{pmatrix}, \tag{16}$$

where the operator $V\colon \mathcal{H}_0 \mapsto \mathcal{H}_1$ is isometric, J_2 and J_3 are canonical symmetries of the form $[\cdot,\cdot]$ on $\mathcal{H}_2$ and $\mathcal{H}_3$ respectively.

Let $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$ be a standard Krein space (see Subsection 1.2). Let $\{\widetilde{g}_j(t)\}_{j=1}^k \subset M_{\vec{\sigma}}(\mathcal{E})$ be a system of unbounded elements conformed with the operator-valued function X_τ and linearly independent modulo $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$. Denote by $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ the linear span generated by the space $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$ and the system $\{\widetilde{g}_j(t)\}_{j=1}^k$. Define on $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ structures of Hilbert and Krein spaces in the following way: on $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$ both structures coincide with the original structures, functions of the system $\{\widetilde{g}_j(t)\}_{j=1}^k$ are by definition positive (as elements of the Krein space), mutually orthogonal and J-orthogonal, normalized and J-normalized, orthogonal and J-orthogonal to $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$. The space $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ is said to be *the expansion of* $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$ *(generated by the collection* $\{\widetilde{g}_j(t)\}_{j=1}^k$*)*.

If a function $\gamma(t)$ is such that $\gamma(t)f(t) \in \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ for every $f(t) \in \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$, then the multiplication operator $\Gamma = \{\Gamma, \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})\}$ by the function $\gamma(t)$ is well defined. Let us analyze some properties of the operator Γ. First, we note the following evident fact.

Proposition 2.1. *The relation*

$$\Gamma\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E}) \subset \mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E}) \tag{17}$$

holds if and only if

$$\mathrm{EssSup}\{|\gamma(t)|\} < \infty \quad \textit{and} \quad \gamma(t)\widetilde{g}_j(t) \in \mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E}), \quad j = 1, 2, \ldots, k. \tag{18}$$

For future reference we need to re-formulate Proposition 2.1. For this aim we introduce an additional function space. Let $G(t)$ be a μ_σ-measurable function defined a.e. on $[-1;1]$ and such that

- a.e. $G(t) \geq 1$,
- $\int_{-1}^{-\tau} G(t)d\sigma(t) < \infty$, $\int_{\tau}^{1} G(t)d\sigma(t) < \infty$ for every $\tau \in (0;1]$,
- $\int_{-1}^{1} G(t)d\sigma(t) = \infty$.

Set

$$\nu(\tau) \quad := \quad \begin{cases} \int_{-1}^{\tau} G(t)d\sigma(t), & \text{if } \tau \in [-1;0); \\ \\ -\int_{\tau}^{1} G(t)d\sigma(t), & \text{if } \tau \in (0;1]. \end{cases} \tag{19}$$

The function $\nu(t)$ is non-decreasing in both segments $[-1;0)$ and $(0;1]$ but it is unbounded in neighborhoods of zero. Define for it a corresponding function space. Let $f(t)$ and $g(t)$ be arbitrary functions continuous in $[-1;1]$ and vanishing in some neighborhoods (different in the general case for $f(t)$ and $g(t)$) of zero. Then the integral $\int_{-1}^{1} f(t)\overline{g(t)}d\nu(t)$ is well defined and generates a structure of pre-Hilbert space on the set of all such functions. The completion of the space will be denote L^2_ν (or $L^2_\nu(\mathbb{C})$).

Note that by (19) the spaces L^∞_σ and L^2_ν forms a compatible pair or a Banach pair (for details see [5] or [12]). Thus, the space $L^\infty_\sigma \cap L^2_\nu$ is well defined.

Proposition 2.2. *The relation* (17) *holds if and only if*

$$\gamma(t) \in L^\infty_\sigma \cap L^2_\nu, \tag{20}$$

where (see (19)*)* $G(t) = 1 + \sum_{j=1}^{k} \|\widetilde{g}_j(t)\|^2$.

Now we consider the general case, i.e.

$$\Gamma \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E}) \subset \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E}). \tag{21}$$

By (21) every element of the system $\{\Gamma\widetilde{g}_j(t)\}_1^k$ can be uniquely (modulo $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$) represented as a linear combination of the elements of the system $\{\widetilde{g}_j(t)\}_1^k$, i.e.

$$\Gamma\widetilde{g}_j(t) = \sum_{m=1}^{k} \tau_{jm}\widetilde{g}_m(t) (\operatorname{mod} \mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})), \quad j = 1, 2, \ldots, k \tag{22}$$

So, the operator Γ defines the matrix

$$\mathrm{T}_\gamma = (\tau_{jm})_{j,m=1}^{k} \ , \tag{23}$$

Now we consider a relation between $\gamma(t)$ and eigenvalues of the matrix (23).

Proposition 2.3. *Let* $\{\widetilde{g}_j(t)\}_1^k$ *be a system of unbounded elements generating together with* $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E}))$ *the space* $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E}))$, *let a function* $\gamma(t)$ *satisfy Condition* (21) *and let* A *be the set of partial limits of* $\gamma(t)$ *for* $t \to 0$. *Then all eigenvalues of the matrix* (23) *are in* A.

Proof. Let us suppose the contrary, i.e. that the matrix (23) has an eigenvalue $\beta \notin \mathrm{A}$. Then there is a non trivial linear combination $\widetilde{g}(t) = \sum_{j=1}^{k} \xi_j \widetilde{g}_j(t)$, such that $\Gamma\widetilde{g}(t) = \beta\widetilde{g}(t) (\operatorname{mod} \mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E}))$. This implies

$$\sup_{\lambda\in[-1;0),\ \mu\in(0;1]} \left\{ \left(\int_{-1}^{\lambda} + \int_{\mu}^{1} \right) |\gamma(t) - \beta|^2 \|\widetilde{g}(t)\|^2_{\mathcal{E}} \, d\sigma(t) \right\} < \infty. \tag{24}$$

Since for a sufficiently small $\epsilon > 0$ the inequality $0 < |t| < \epsilon$ implies $|\gamma(t) - \beta| \geq \delta > 0$, by (24) we have

$$\int_{-\epsilon}^{\epsilon} \|\widetilde{g}(t)\|^2_{\mathcal{E}} d\sigma(t) < \infty$$

(the integral is treated as improper with a singularity in zero) and, thus, $\widetilde{g}(t) \in \mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$. This is a contradiction! $\square$

Theorem 2.4. *If a J-orth.sp.f. E_λ satisfies Condition* (10) *and a scalar product on $\mathcal{H}$ is compatible with* (15)*, then there are, first, a subspace $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$ and a system of unbounded elements $\{\widetilde{g}_j(t)\}_{j=1}^k$ of this space forming together the space $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ and, second, an isometric J-isometric operator $W\colon \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E}) \mapsto \widetilde{\mathcal{H}}$, $WL^2_{\vec{\sigma}}(\mathcal{E}) = \mathcal{H}_2$, such that for every $\lambda \in [-1;1]$*

$$\widetilde{E}_\lambda = W \cdot \mathrm{X}^{\#}_\lambda \cdot (W)^{-1}, \quad W^{\uparrow} = (I_2 \oplus V)W, \quad \widetilde{E}^{\uparrow}_\lambda = W^{\uparrow} \cdot \mathrm{X}_\lambda \cdot (W^{\uparrow})^{-1}, \tag{25}$$

where $k = \dim \mathcal{H}_0 = \dim \mathcal{H}_1$ and $\mathrm{X}_\lambda = \{\mathrm{X}_\lambda,\, \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})\}$ is the multiplication operator by the indicator $\chi_{[-1,\lambda)}(t)$ of the interval $[-1,\lambda)$.

Definition 2.5. If for the subspaces (11) and (14) a relation between a J-orth.sp.f. E_λ satisfying Condition (10) and a space $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ is given by Formulae (25), then $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ is said to be *a basic model space* for E_λ (compatible with (11), (12), (15)) and the operator W is said to be *an operator of similarity* corresponding to this space.

Theorem 2.6. *Assume that a J-orth.sp.f. E_λ satisfies Condition* (10)*, a scalar product on $\mathcal{H}$ is compatible with* (15) *and $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ is a basic model space for E_λ. If a (bounded) operator C and a function $\gamma(t)$ are such that*

$$CE(\Delta) = \int_\Delta \overline{\gamma}(t)E(dt) \tag{26}$$

for every interval $\Delta \in \mathfrak{R}_{\{0\}}$, $0 \notin \Delta$, then

$$\begin{cases} a) & C\widetilde{\mathcal{H}} \subset \widetilde{\mathcal{H}}\ ; \\ b) & \text{for } \gamma(t) \text{ Condition (21) holds}\ ; \\ c) & \widetilde{C} = W \cdot \Gamma^{\#} \cdot W^{-1}, \end{cases} \tag{27}$$

where $\widetilde{C} = C|_{\widetilde{\mathcal{H}}}$ and W is the operator of similarity from (25)*.*

Proof. First, Condition (27a) follows directly from (26).

Second, let $\widetilde{g}_j(t)$, $j = 1, 2, \dots, k$ be the unbounded elements generating the expansion $\mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ of $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$. If $x \in \mathcal{H}_2$ and according to (25) $f(t) = W^{-1}x$, then

$$W^{-1}E(\Delta)x = \chi_\Delta(t)f(t) + \sum_{j=1}^k \int_\Delta [f(\tau), \overline{\widetilde{g}_j(\tau)}]\, d\tau \cdot \widetilde{g}_j(t) \tag{28}$$

and

$$W^{-1}CE(\Delta)x = \chi_\Delta(t)\overline{\gamma(t)}f(t) + \sum_{j=1}^k \int_\Delta [\overline{\gamma(\tau)}f(\tau), \overline{\widetilde{g}_j(\tau)}]\, d\tau \cdot \widetilde{g}_j(t). \tag{29}$$

We need to show that $\{\gamma(t)\widetilde{g}_j(t)\}_{J=1}^k \subset \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$. Let us suppose the contrary i.e. that the maximal number m of functions linearly independent modulo $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$ in the system $\{\widetilde{g}_j(t),\, \gamma(t)\widetilde{g}_j(t)\}_{j=1}^k$ is greater than k. With no loss of generality we can suppose that the system $\{\widetilde{g}_j(t)\}_{j=1}^k \cup \{\gamma(t)\widetilde{g}_j(t)\}_{j=1}^{m-k}$ is linearly independent modulo $\mathcal{J}\text{-}L^2_{\vec{\sigma}}(\mathcal{E})$, so

$$\gamma(t)\widetilde{g}_l(t) = \sum_{j=1}^k \alpha_j^{(l)}\widetilde{g}_j(t) + \sum_{j=1}^{m-k} \beta_j^{(l)}\gamma(t)\widetilde{g}_j(t), \quad l = m-k+1, \dots, k. \tag{30}$$

By Lemma 1.14 there exists a sequence $\{x_p\}_1^\infty \subset \mathcal{H}_2$ such that

- for every function $f_p(t) = W^{-1}x_p$ there is a neighborhood of zero, where $f_p(t) \equiv 0$,
- $\lim_{p\to\infty} \|f_p(t)\|_{L^2_{\vec{\sigma}}(\mathcal{E})} = 0$,
- $\lim_{p\to\infty} \int_{\mathbb{R}} [f_p(\tau), \overline{\widetilde{g}_j(\tau)}]d\tau = 0$, $j = 1, 2, \dots, k$,
- $\lim_{p\to\infty} \int_{\mathbb{R}} [f_p(\tau), \overline{\gamma(\tau)\widetilde{g}_1(\tau)}]d\tau = 1$,
- $\lim_{p\to\infty} \int_{\mathbb{R}} [f_p(\tau), \overline{\gamma(\tau)\widetilde{g}_j(\tau)}]d\tau = 0$, $j = 2, \dots, m-k$.

Let $e_j = W\widetilde{g}_j(t)$, $j = 1, 2, \dots, k$. Then by (28), (29) and (30) we have

$$\lim_{p\to\infty}\big(x_p + \sum_{j=1}^{k}\int_{\mathbb{R}}[f_p(\tau), \overline{\widetilde{g}_j(\tau)}]\,d\tau\cdot e_j\big) = 0,$$

but

$$\lim_{p\to\infty} C\big(x_p + \sum_{j=1}^{k}\int_{\mathbb{R}}[f_p(\tau), \overline{\widetilde{g}_j(\tau)}]\,d\tau\cdot e_j\big) = e_1 + \sum_{l=m-k+1}^{k}\beta_1^{(l)}e_l.$$

Thus, the operator C is not closable. This is a contradiction.

Third, Condition (27c) follows directly from (27b), (28) and (29). $\square$

Theorem 2.7. *Assume that a J-orth.sp.f. E_λ satisfies Condition* (10)*, a scalar product on $\mathcal{H}$ is compatible with* (15) *and $\mathcal{J}$-$\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ is a basic model space for E_λ. If a J-unitary operator U and a function $\upsilon(t)$ are such that*

$$UE(\Delta) = \int_\Delta \upsilon(t)E(dt) \tag{31}$$

for every interval $\Delta \in \mathfrak{R}_{\{0\}}$, $0 \notin \Delta$, then

$$\begin{cases} a) & \text{a.e. } |\upsilon(t)| = 1;\\ b) & U\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}},\ U^{\#}\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}\ ;\\ c) & \text{for } \upsilon(t) \text{ and } \overline{\upsilon}(t) \text{ Condition (21) holds;}\\ d) & \widetilde{U} = W\cdot\overline{\Upsilon}^{\#}\cdot W^{-1},\ \widetilde{U}^{\uparrow} = W^{\uparrow}\cdot\Upsilon\cdot(W^{\uparrow})^{-1}, \end{cases} \tag{32}$$

where $\widetilde{U} = U|_{\widetilde{\mathcal{H}}}$, $\widetilde{U}^{\uparrow} = (P_0 + P_2)U|_{\widetilde{\mathcal{H}}^{\uparrow}}$, operators W and $W^{\uparrow}$ are from (25)*, $\Upsilon = \{\Upsilon,\ \mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})\}$ and $\overline{\Upsilon} = \{\overline{\Upsilon},\ \mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})\}$ are the multiplication operators by the functions $\upsilon(t)$ and $\overline{\upsilon}(t)$ respectively.*

Proof. First, J-unitary property of U implies (32a). Second, the relations $U\widetilde{\mathcal{H}} \subset \widetilde{\mathcal{H}}$ and $\overline{\Upsilon}\mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E}) \subset \mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ follow from (27a) and (27b) respectively. Third, let $\widetilde{g}_j(t)$, $j = 1, 2, \dots, k$ be the unbounded elements generating the expansion $\mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ of $\mathcal{J}\text{-}L^2_{\widetilde{\sigma}}(\mathcal{E})$. In this case Condition (32a) means that the elements $\upsilon(t)\cdot\widetilde{g}_j(t)$, $j = 1, 2, \dots, k$ are also the unbounded elements linearly independent modulo $\mathcal{J}\text{-}L^2_{\widetilde{\sigma}}(\mathcal{E})$. Thus, $\overline{\Upsilon}\mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E}) = \mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E}) = \Upsilon\mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ and by (27c) $\widetilde{U}\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}$. Since U is a J-unitary operator, the last equality implies $U(\widetilde{\mathcal{H}}^{[\perp]}) \subset \widetilde{\mathcal{H}}^{[\perp]}$.

Thus, with respect to the decomposition $\mathcal{H} = \mathcal{H}_0\dot{+}\mathcal{H}_1\dot{+}\mathcal{H}_2\dot{+}\mathcal{H}_3$ (e.g. (16)),

$$U = \begin{pmatrix} U_{00} & 0 & 0 & 0\\ U_{10} & U_{11} & U_{12} & U_{13}\\ U_{20} & 0 & U_{22} & 0\\ U_{30} & 0 & 0 & U_{33}\end{pmatrix}, \tag{33}$$

and

$$U^{\#} = \begin{pmatrix} V^{-1}U_{11}^*V & 0 & 0 & 0\\ VU_{10}^*V & VU_{00}^*V^{-1} & VU_{20}^*J_2 & VU_{30}^*J_3\\ J_2U_{12}^*V & 0 & J_2U_{22}^*J_2 & 0\\ J_3U_{13}^*V & 0 & 0 & J_3U_{33}^*J_3\end{pmatrix}. \tag{34}$$

Next, (31) and (32a) imply $U\big(E(\Delta)\mathcal{H}\big) = E(\Delta)\mathcal{H}$. Since U is J-unitary, the latter gives $U\big(I - E(\Delta)\big)\mathcal{H} = \big(I - E(\Delta)\big)\mathcal{H}$, so $E(\Delta)U\big(I - E(\Delta)\big) = 0$. The latter implies $UE(\Delta) = E(\Delta)U$. Taking into account (31) one has $U^{\#}E(\Delta) = \int_\Delta \overline{\upsilon(t)}E(dt)$. Now (32d) follows from Theorem 2.6 and a comparison between (33) and (34). $\square$

In the same way one can prove the following theorem.

Theorem 2.8. *Assume that a J-orth.sp.f. E_λ satisfies Condition* (10), *a scalar product on $\mathcal{H}$ is compatible with* (15) *and $\mathcal{J}$-$\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ is a basic model space for E_λ. If a J-s.a. operator C and a function $\gamma(t)$ are such that $CE(\Delta) = \int_\Delta \gamma(t)E(dt)$for every interval $\Delta \in \mathfrak{R}_{\{0\}}$, $0 \notin \Delta$, then (e.g. with* (27)*)*

a) a.e. $\gamma(t) = \overline{\gamma(t)}$;
b) $C\widetilde{\mathcal{H}} \subset \widetilde{\mathcal{H}}$;
c) *for $\gamma(t)$ Condition* (21) *holds;*
d) $\widetilde{C} = W \cdot \Gamma^\# \cdot W^{-1}$, $\widetilde{C}^\uparrow = W^\uparrow \cdot \Gamma \cdot (W^\uparrow)^{-1}$,

where $\widetilde{C} = C|_{\widetilde{\mathcal{H}}}$, $\widetilde{C}^\uparrow = (P_0 + P_2)C|_{\widetilde{\mathcal{H}}^\uparrow}$, operators W and $W^\uparrow$ are from (25).

Applying Theorem 2.8 to J-n. operators one can obtain the following result.

Theorem 2.9. *Assume that $N \in D^+_\kappa$ is a J-n. operator, whose r.e.s.f. E_λ satisfies Condition* (10), *that a canonical scalar product on $\mathcal{H}$ is compatible with* (15), *that $\mathcal{J}$-$\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ is a basic model space for E_λ, and W is a corresponding operator of similarity. Then*

$$\widetilde{N}^\uparrow = W^\uparrow \cdot (\Phi + i\Psi) \cdot (W^\uparrow)^{-1}, \quad \widetilde{N} = W \cdot \left(\Phi^\# + i\Psi^\#\right) \cdot W^{-1} \tag{35}$$

where $\widetilde{N}^\uparrow := (P_0 \oplus P_2)N|_{\widetilde{\mathcal{H}}^\uparrow}$, the space $\widetilde{\mathcal{H}}^\uparrow$ and the operator $W^\uparrow$ are defined via (12), (25), *$\Phi = \{\Phi, \mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})\}$ and $\Psi = \{\Psi, \mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})\}$, the functions $\phi(\lambda)$ and $\psi(\lambda)$ are defined by (5c).*

Corollary 2.10. *Assume that $N \in D^+_\kappa$ is a J-n. operator, whose r.e.s.f. E_λ satisfies Condition* (10), *that a canonical scalar product on $\mathcal{H}$ is compatible with* (15), *that $\mathcal{J}$-$\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ is a basic model space for E_λ, and W is a corresponding operator of similarity. Then*

$$\left(\widetilde{N}^\#\widetilde{N}\right)^\uparrow = W^\uparrow \cdot \left(\Phi^2 + \Psi^2\right) \cdot (W^\uparrow)^{-1}, \qquad \left(\widetilde{N}^\#\widetilde{N}\right) = W \cdot \left((\Phi^\#)^2 + (\Psi^\#)^2\right) \cdot W^{-1} \tag{36}$$

where $\left(\widetilde{N}^\#\widetilde{N}\right)^\uparrow := (P_0 \oplus P_2)\left(N^\# N\right)|_{\widetilde{\mathcal{H}}^\uparrow}$ and the rest of elements in (36) are the same as in (35).

3. On a polar representation for J-n. operator

3.1. **Preliminary remarks.** In this Section we consider the problem of a polar representation for a J-n. operator N from the D^+_κ-class. This problem has been actively studied during the past few years. The present state of investigations in this direction can be found in [16]. As to other operator classes, a J-polar decomposition for so-called strict plus-operators was considered in [14]. For additional references on the subject and some related topics see [1], Subsection II.1.13, and also [11]. Taking into account Proposition 1.6 and Theorem 1.8 we concentrate our study on the J-n. operator whose real and imaginary parts have real spectra and whose spectral peculiarities are reduced to a unique spectral peculiarity in zero. Thus, everywhere below we assume that (10) holds and (see (5))

$$\sigma(A|_{\mathcal{H}_1}) = \sigma(B|_{\mathcal{H}_1}) = \{0\}. \tag{37}$$

Note that (5a) and (37) imply

$$\mu_\sigma(\{t\colon\ \phi^2(t) + \psi^2(t) = 0\}) = 0, \tag{38}$$

where $\sigma(t)$ is the same as in Theorem 2.9 and in Corollary 2.10.

3.2. **Quasi-roots.**

Definition 3.1. Assume that $N \in D^+_\kappa$ is a J-n. operator with r.e.s.f. E_λ. A J-s.a. operator C is called *a (square) quasi-root of $N^\# N$ conformed with E_λ* if there is a scalar function $\gamma(t)$ such that for every $X \in \mathfrak{R}_{\{0\}}$, $0 \notin X$ the equalities $CE(X) = E(X)C = \int_X \gamma(\lambda)E(d\lambda)$ and $C^2E(X) = N^{\#}NE(X)$ hold.

Remark 3.2. Note that spectrum of a quasi-root C of N can contain both positive and negative elements. This approach to the definition of a quasi-root is justified by the fact that even for a J-s.a. operator N a quasi-root C with non-negative spectrum can be nonexistent. Everywhere below the symbol $\widehat{\mathcal{L}}$ means the linear span of subspaces $E(X)\mathcal{H}$, $X \in \mathfrak{R}_{\{0\}}$, $0 \notin X$.

Example 3.3. Let a space $\mathcal{H}$ be spanned by orthonormal vectors g_1, g_2, h_1, h_2 and $\{e_k\}_{k=1}^{\infty}$. Let $Jg_1 := h_1$, $Jg_2 := h_2$, $Jh_1 := g_1$, $Jh_2 := g_2$, $Ag_1 := 0$, $Ag_2 := g_1$, $Ah_1 := h_2$, $Ah_2 := \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e_k$, $Je_k := e_k$, $Ae_k := \frac{(-1)^k}{k} e_k + \frac{1}{k} g_2$, where $k = 1, 2, \ldots$. Then $\frac{(-1)^k}{k}$ is an eigenvalue of A, that corresponds to an eigenvector $e_k + (-1)^k g_2 + k g_1$, where $k = 1, 2, \ldots$. Spectrum of A^2 is simple, so, if there exists a (bounded!) J-s.a. operator C with non-negative spectrum such that $C^2 = A^2$, then the same vector is an eigenvector for C that corresponds to an eigenvalue $\frac{1}{k}$. Let

$$\begin{aligned} x_m &= \sum_{k=1}^{2m} (e_k + (-1)^k g_2 + k g_1) - \frac{2m+1}{6m+1} \sum_{k=2m+1}^{4m} (e_k + (-1)^k g_2 + k g_1) \\ &= \sum_{k=1}^{2m} e_k - \frac{2m+1}{6m+1} \sum_{k=2m+1}^{4m} e_k. \end{aligned}$$

Then $x_m \in \widetilde{\mathcal{H}}$ and

$$\begin{aligned} Cx_m &= \sum_{k=1}^{2m} \frac{1}{k}(e_k + (-1)^k g_2 + k g_1) - \frac{2m+1}{6m+1} \sum_{k=2m+1}^{4m} \frac{1}{k}(e_k + (-1)^k g_2 + k g_1) \\ &= \frac{8m^2}{6m+1} g_1 + \left(\sum_{k=1}^{2m} \frac{(-1)^k}{k} - \frac{2m+1}{6m+1} \sum_{k=2m+1}^{4m} \frac{(-1)^k}{k} \right) g_2 \\ &+ \left(\sum_{k=1}^{2m} \frac{1}{k} e_k - \frac{2m+1}{6m+1} \sum_{k=2m+1}^{4m} \frac{1}{k} e_k \right). \end{aligned}$$

These formulae imply that $\frac{6m+1}{8m^2} x_m \to 0$ and $\frac{6m+1}{8m^2} Cx_m \to g_1$ by $m \to \infty$. Thus, the operator $C|_{\widehat{\mathcal{L}}}$ is nonclosable in $\widehat{\mathcal{L}}$.

Remark 3.4. Example 3.3 is related to the idea expressed by Lemma 1.14 and Theorem 2.8. In particular, here the fact that the sequences $\{k\}_{k=1}^{\infty}$, $\{(-1)^k\}_{k=1}^{\infty}$ and $\{1\}_{k=1}^{\infty}$ are linearly independent modulo $\mathbb{l}_2$ is used.

Proposition 3.5. *Assume that $N \in D_\kappa^+$ is a J-n. operator with r.e.s.f. E_λ. If for all $X \in \mathfrak{R}_{\{0\}}$ with $0 \notin X$ the operator $N^{\#}N|_{E(X)\mathcal{H}}$ has simple spectrum, then every J-s.a. operator C, such that $C^2 = N^{\#}N$, is a quasi-root of $N^{\#}N$ conformed with E_λ.*

Proof. We need only to show that E_λ commutes with C. Let $D := N^{\#}N$ and $\Theta(\lambda) = \phi^2(\lambda) + \psi^2(\lambda)$. Since $CD = C^3 = DC$, the eigen spectral function $E_\mu^{(D)}$ of D commutes with C. Next, the non-zero spectrum of D is simple, so the function $\Theta(\lambda)$ is a one-to-one (up to a set of E-measure equal to zero) mapping $\mathrm{Supp}(E) \mapsto \mathrm{Supp}(E^{(D)})$. Let $X \in \mathfrak{R}_{\{0\}}$ and $0 \notin X$. Set $Y = \Theta(X)$. If $0 \notin \bar{Y}$, then $E(X) = E^{(D)}(Y)$, if $0 \in \bar{Y}$, then $E(X) = \mathrm{s} - \lim_{\mu \to +0}(I - E_\mu^{(D)} + E_{-\mu}^{(D)})E^{(D)}(Y)$. The rest is evident. $\square$

Theorem 3.6. *Assume that $N \in D_\kappa^+$ is a J-n. operator with r.e.s.f. E_λ. If (see* (11)*)*

$$\dim(\mathcal{H}_1) = 1, \tag{39}$$

then for every μ_σ-measurable real function $\omega(t)$ with $|\omega(t)| \equiv 1$ there exists a quasi-root C satisfies the condition

$$CE(X) = \int_X \omega(\lambda)\sqrt{\phi^2(\lambda)+\psi^2(\lambda)}dE_\lambda, \quad X \in \mathfrak{R}_{\{0\}}, \quad 0 \notin X.$$

Proof. Let us fix a basic model space $\mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ for N, where a system of unbounded elements is reduced to a singleton $\{\widetilde{g}_1(t)\}$. Conditions (37) and (39) imply (see Formulae (21), (22) and Theorem 2.9) Condition (17), where Γ is replaced by Φ and Ψ, so by Proposition 2.1 $\int_{-1}^1 |\phi(t)|^2\|\widetilde{g}_1(t)\|^2_{\mathcal{E}}d\sigma(t) < \infty$ and $\int_{-1}^1 |\psi(t)|^2\|\widetilde{g}_1(t)\|^2_{\mathcal{E}}d\sigma(t) < \infty$. Then for $\theta(t){:}= \sqrt{\phi^2(\lambda)+\psi^2(\lambda)}$ these two inequalities yield $\int_{-1}^1 |\theta(t)|^2\|\widetilde{g}_1(t)\|^2_{\mathcal{E}}d\sigma(t) < \infty$. Let us set (see (11))

$$\widetilde{C}^{\uparrow}{:}= W^{\uparrow}\Theta(W^{\uparrow})^{-1}, \quad \widetilde{C}{:}= W\Theta W^{-1}, \quad P_1CP_0 = 0, \quad C|_{\widetilde{\mathcal{H}}^{[\perp]}} = 0, \tag{40}$$

where $\widetilde{C}^{\uparrow}{:}= (P_0 \oplus P_2)C|_{\widetilde{\mathcal{H}}^{\uparrow}}$, $\Theta = \{\Theta, \mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})\}$ is the multiplication operator by $\theta(t)$, and the rest of elements in (40) are the same as in (35). The direct verification shows that C is desired. □

In the same way the following theorem can be proved.

Theorem 3.7. *Assume that $N \in D^+_\kappa$ is a J-n. operator with r.e.s.f. E_λ. If (cf. (37))*

$$A|_{\mathcal{H}_1} = B|_{\mathcal{H}_1} = 0, \tag{41}$$

then for every μ_σ-measurable real function $\omega(t)$ with $|\omega(t)| \equiv 1$ there exists a quasi-root C satisfies the condition

$$CE(X) = \int_X \omega(\lambda)\sqrt{\phi^2(\lambda)+\psi^2(\lambda)}dE_\lambda, \quad X \in \mathfrak{R}_{\{0\}}, \quad 0 \notin X.$$

Example 3.3 shows that if $k > 1$, then Condition 41 in Theorem 3.7 cannot be omitted.

Theorem 3.8. *Assume that $N \in D^+_\kappa$ is a J-n. operator with r.e.s.f. E_λ. If*

$$\dim(\mathcal{H}_1) = 2 \tag{42}$$

and $A|_{\widetilde{\mathcal{H}}} \neq 0$, then for

$$\omega(t){:}= \begin{cases} |\phi(t)|/\phi(t), & \text{if } \phi(t) \neq 0; \\ \\ 1, & \text{if } \phi(t) = 0 \end{cases} \tag{43}$$

there exists a quasi-root C satisfying the condition

$$CE(X) = \int_X \omega(\lambda)\sqrt{\phi^2(\lambda)+\psi^2(\lambda)}dE_\lambda, \quad X \in \mathfrak{R}_{\{0\}}, \quad 0 \notin X.$$

Proof. Let $\mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$ be a basic model space for N. Conditions (37) and (42) imply the following representations for the matrices T_ϕ and T_ψ defined in concordance with (22) and (23)

$$\mathrm{T}_\phi = \alpha \cdot \begin{pmatrix} \xi & -\xi^2 \\ 1 & -\xi \end{pmatrix}, \quad \mathrm{T}_\psi = \beta \cdot \begin{pmatrix} \xi & -\xi^2 \\ 1 & -\xi \end{pmatrix},$$

where $\alpha \neq 0$. The last equalities yield the relation $(\Phi - \frac{\beta}{\alpha}\Psi)\mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E}) \subset \mathcal{J}\text{-}L^2_{\widetilde{\sigma}}(\mathcal{E})$. By Proposition 2.2 we have

$$\zeta(t){:}= (\psi(t) - \frac{\beta}{\alpha}\phi(t)) \in L^\infty_\sigma \cap L^2_\nu. \tag{44}$$

Our next aim is to show that (see (43))

$$\theta(t){:}= \omega(t)\sqrt{\phi^2(t)+\psi^2(t)} = \phi(t)\sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}\left(\operatorname{mod} L^\infty_\sigma \cap L^2_\nu\right). \tag{45}$$

Let $X=\{t\colon\ |\phi(t)|\le|\zeta(t)|\}$, $Y=[-1;1]\backslash(X\cup\{0\})$, and let $\chi^{(X)}(t)$ and $\chi^{(Y)}(t)$ be the indicators of sets X and Y respectively. Then

$$\chi^{(X)}(t)|\theta(t)-\phi(t)|\le\chi^{(X)}(t)|\zeta(t)|\left(\sqrt{1+\left(1+|\frac{\beta}{\alpha}|\right)^2}+\sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}\right).\tag{46}$$

On the other hand, $\chi^{(Y)}(t)\theta(t)=\sqrt{1+\left(\frac{\zeta(t)}{\phi(t)}+(\frac{\beta}{\alpha})\right)^2}\cdot\phi(t)\cdot\chi^{(Y)}(t)$, so

$$\chi^{(Y)}(t)|\theta(t)-\phi(t)|=\frac{|\,2\cdot\frac{\beta}{\alpha}\zeta(t)+\zeta(t)\cdot\frac{\zeta(t)}{\phi(t)}\,|}{\sqrt{1+\left(\frac{\zeta(t)}{\phi(t)}+(\frac{\beta}{\alpha})\right)^2}+\sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}}\cdot\chi^{(Y)}(t).$$

Thus,

$$\chi^{(Y)}(t)\cdot|\,\theta(t)-\phi(t)|\le\chi^{(Y)}(t)\cdot|\zeta(t)|\cdot\frac{2\cdot|\frac{\beta}{\alpha}|+1}{1+\sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}}.\tag{47}$$

Now Equality (45) follows from (44), (46) and (47). Since (45) means that the multiplication operator Θ by the function $\theta(t)$ is well defined on $\mathcal{J}\text{-}\widetilde{L}^2_{\widetilde{\sigma}}(\mathcal{E})$, one can set (cf. with (40))

$$\widetilde{C}^{\uparrow}:=W^{\uparrow}\Theta(W^{\uparrow})^{-1},\quad \widetilde{C}:=W\Theta W^{-1},\quad P_1CP_0=0,\quad C(P_0+P_1+P_2)=C.\tag{48}$$

The rest is trivial. □

Let us show that Condition (42) cannot be omitted.

Example 3.9. Let a space $\mathcal{H}$ be spanned by orthonormal vectors g_1,g_2,g_3,h_1,h_2,h_3 and $\{e_k\}_{k=1}^{\infty}$. Let $Jg_j:=h_j$, $Jh_j:=g_j$, $j=1,2,3$, $Ng_1:=0$, $Ng_2:=ig_1$, $Ng_3:=g_1$, $Nh_1:=ih_2+h_3$, $Nh_2:=\sum_{n=1}^{\infty}(n^{-4/5}+in^{-3/5})e_n$, $Nh_3:=\sum_{n=1}^{\infty}(n^{-1}+n^{-4/5})e_n$, $Je_n:=e_n$, $Ae_n:=(n^{-1}+in^{-4/5})e_n+(n^{-4/5}+in^{-3/5})g_2+(n^{-1}+in^{-4/5})g_3$, where $n=1,2,\dots$. Then $(n^{-1}+in^{-4/5})$ is an eigenvalue of N, that corresponds to an eigenvector $(e_n+g_3+n^{1/5}g_2+ng_1)$, where $n=1,2,\dots$. The spectrum of $N^{\#}N$ is simple, so, if there exists a (bounded!) J-s.a. operator C such that $C^2=N^{\#}N$, then the same vector is an eigenvector for C that corresponds to an eigenvalue $\omega_n\cdot\sqrt{n^{-8/5}+n^{-2}}$, where $\omega_n=\pm1$. A basic model space $\widetilde{\mathbb{l}}_2$ for N can be formed as the linear span of $\mathbb{l}_2$ and three sequences $\{n\}$, $\{n^{1/5}\}$ and $\{1\}$. Then C has the following model representation $\{\alpha_n\}\mapsto\{\omega_n\cdot\sqrt{n^{-8/5}+n^{-2}}\cdot\alpha_n\}$. At the same time it is easy to check that $\{\omega_n\cdot\sqrt{n^{-8/5}+n^{-2}}\cdot n\}\notin\widetilde{\mathbb{l}}_2$ and by Theorem 2.8 the operator C does not exist.

3.3. **Polar representation.** Here we shall discuss the existence of a representation

$$N=UC,\tag{49}$$

for a J-n. operator $N\in D_\kappa^+$, where C is a J-s.a. operator and U is a J-unitary operator. (49) implies that $N^{\#}N=C^2$. We assume additionally that C in (49) is a quasi-root conformed with r.e.s.f. E_λ. Note that it is not a restriction if for all $X\in\mathfrak{R}_{\{0\}}$ with $0\notin X$ the operator $N^{\#}N|_{E(X)\mathcal{H}}$ has simple spectrum.

If C is a quasi-root, we have

$$\int_X(\phi(\lambda)+i\psi(\lambda))E(d\lambda)=U\int_X\gamma(\lambda)E(d\lambda)\tag{50}$$

with $\gamma^2(\lambda)=(\phi^2(\lambda)+\psi^2(\lambda))$ and $X\in\mathfrak{R}_{\{0\}}$, $0\notin X$, so

$$UE(X)=\int_X\frac{(\phi(\lambda)+i\psi(\lambda))}{\gamma(\lambda)}E(d\lambda).\tag{51}$$

Formula (51) uniquely defined $U|_{\widehat{\mathcal{L}}}$. On the other hand the operator (maybe unbounded) $\widehat{U}\colon \widehat{\mathcal{L}} \mapsto \widehat{\mathcal{L}}$,

$$\widehat{U}x = \int_{-1}^{1} \frac{(\phi(\lambda) + i\psi(\lambda))}{\gamma(\lambda)} E(d\lambda)x,$$

where $x \in \widehat{\mathcal{L}}$, is well defined (see (38)) independently of the existence of the operators U and C themselves. If the (bounded!) operator U exists, then the operator $\widehat{U}$ is also bounded and, thus, closable. Let us give an example of nonclosable $\widehat{U}$.

Example 3.10. Let a space $\mathcal{H}$ be spanned by orthonormal vectors g_1, h_1, and $\{e_k\}_{k=1}^{\infty}$. Let $Jg_1\colon = h_1$, $Jh_1\colon = g_1$, $Ng_1\colon = 0$, $Nh_1\colon = \sum_{n=1}^{\infty} i^n n^{-1} e_n$, $Je_n\colon = e_n$, $Ne_n\colon = i^n n^{-1} e_n + i^n n^{-1} g_1$, where $n = 1, 2, \dots$. Then $i^n n^{-1}$ is an eigenvalue of N, that corresponds to an eigenvector $e_n + g_1$, where $n = 1, 2, \dots$. The spectrum of $N^{\#}N$ is simple, so, if there exists a (bounded!) J-s.a. operator C such that $C^2 = N^{\#}N$, then the same vector is an eigenvector for C that corresponds to an eigenvalue $\omega_n \cdot n^{-1}$, where $\omega_n = \pm 1$. A basic model space $\widetilde{\mathbb{l}}_2$ for N can be formed as the linear span of $\mathbb{l}_2$ and the sequence $\{1\}$. Then C has the following basic model representation $\{\alpha_n\} \mapsto \{\omega_n \cdot n^{-1} \cdot \alpha_n\}$ and $\widehat{\mathcal{L}}$ is the set of the vectors in the form $\{\alpha_n\} + (\sum_{n=1}^{\infty} \alpha_n) \cdot g_1$, where $\alpha_n = 0$ for all essentially big n. At the same time it is easy to check that $\{\omega_n \cdot i^n\} \notin \widetilde{\mathbb{l}}_2$ and by Theorem 2.7 (see also Example 3.3) the operator $\widehat{U}$ is nonclosable.

Thus, the above reasoning gives the following result.

Theorem 3.11. *Assume that $N \in D_{\kappa}^{+}$ is a J-n. operator with r.e.s.f. E_λ, such that*

- *functions $\phi(t)$ and $\psi(t)$ are defined by (5c);*
- *$\mathcal{J}$-$\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E})$ is a basic model space for E_λ;*
- *N has Representation (49);*
- *for all $X \in \mathfrak{R}_{\{0\}}$ with $0 \notin X$ the operator $N^{\#}N|_{E(X)\mathcal{H}}$ has simple spectrum.*

Then there is a real scalar function $\gamma(t)$ such that

$$\left\{\begin{array}{ll} a) & \text{a.e. } \gamma^2(t) = \phi^2(t) + \psi^2(t); \\ c) & \text{for } \gamma(t) \text{ Condition (21) holds;} \\ c) & \text{for } \upsilon(t) = \frac{(\phi(t)+i\psi(t))}{\gamma(t)} \text{ the condition} \\ & \Upsilon \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E}) = \mathcal{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathcal{E}) \text{ holds;} \end{array}\right. \tag{52}$$

Let us show that Conditions (52) are not sufficient for existence of (49).

Example 3.12. Let a space $\mathcal{H}$ be spanned by orthonormal vectors g_1, h_1, and $\{e_n\}_{n=1}^{\infty}$. Let $Jg_1\colon = h_1$, $Jh_1\colon = g_1$, $Ng_1\colon = 0$, $Nh_1\colon = ig_1 + \sum_{n=1}^{\infty} n^{-1} e_n$, $Je_n\colon = e_n$, $Ne_n\colon = n^{-1}(e_n + g_1)$, where $n = 1, 2, \dots$. Then n^{-1} is an eigenvalue of N, that corresponds to an eigenvector $e_n + g_1$, where $n = 1, 2, \dots$. The spectrum of $N^{\#}N$ is simple and for every (bounded!) J-s.a. operator C, such that $C^2 = N^{\#}N$, the same vector is also an eigenvector that corresponds to an eigenvalue $\omega_n \cdot n^{-1}$, where $\omega_n = \pm 1$. A basic model space $\widetilde{\mathbb{l}}_2$ for N can be formed as the linear span of $\mathbb{l}_2$ and the sequence $\{1\}$. Then C has the following basic model representation $\{\alpha_n\} \mapsto \{\omega_n \cdot n^{-1} \cdot \alpha_n\}$. Let us suppose that U from (49) exists. Then $\{\omega_n \cdot \alpha_n\} \in \widetilde{\mathbb{l}}_2$ for every $\{\alpha_n\} \in \widetilde{\mathbb{l}}_2$, therefore $\{\omega_n\} = \omega \cdot \{1\} + \{\zeta_n\}$, where $\{\zeta_n\} \in \mathbb{l}_2$. The latter implies $\lim_{n\to\infty} \omega_n = w$. Since $\omega_n = \pm 1$, $\omega = \pm 1$ too. With no loss of generality one can assume $\omega = 1$. In this case $Ug_1 = g_1$ and there is no more than finite number of $\omega_n = -1$. By Theorem 2.8 $Ch_1 = \delta \cdot g_1 + \sum_{n=1}^{\infty} \omega_n \cdot n^{-1} e_n$, where $\delta = [Ch_1, h_1] \in \mathbb{R}$. On the other hand by Theorem 2.7 (or by direct calculations) one has $Ue_n = \omega_n \cdot e_n + (\omega_n - 1) \cdot g_1$, so $UCh_1 = \big(\delta + \sum_{n=1}^{\infty} (\omega_n - 1) n^{-1}\big) g_1 + \sum_{n=1}^{\infty} \omega_n \cdot n^{-1} e_n$. Since $\big(\delta + \sum_{n=1}^{\infty} (\omega_n - 1) n^{-1}\big) \in \mathbb{R}$, $Nh_1 \neq UCh_1$. This is a contradiction.

Acknowledgments. The author would like to thank Prof. L. Rodman who attracted his attention to the problem of polar representation in Krein spaces. He also is very grateful to Prof. T. Berry for his assistance to making the paper more readable.

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Received 24/02/2006