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SUPERSTABLE CRITERION AND SUPERSTABLE BOUNDS FOR INFINITE RANGE INTERACTION I: TWO-BODY POTENTIALS

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Abstract. A continuous infinite system of point particles interacting via two-body infinite-range potential is considered in the framework of classical statistical mechanics. We propose some new criterion for interaction potentials to be superstable and give a very transparent proof of the Ruelle's uniform bounds for a family of finite volume correlation functions. It gives a possibility to prove that for any temperature and chemical activity there exists at least one Gibbs state. This article is a generalization of the work [12] for the case of infinite range interaction potential.

1. Introduction

In the article [12] one of the author has proposed a new method to prove the uniform boundedness of the family of the finite volume correlation functions for classical system of point particles which interact by means of pair potential of superstable type with a finite range interaction. This method enables to simplify Ruelle's proof considerably and to improve the estimate of the work [14]. Besides in the article [13] the method of the work [12] was generalized for the case of many-body potentials with finite range interaction. However, till now it was not clear how to extend this method for the case of potentials with infinite range interaction. In the present paper the construction of the work [12] is modified for such kind of potentials. Furthermore this paper offers a simple criterion (condition for interaction potential), which easily allows to prove superstability of considered particles system. The short content of this article is the following. In Section 2 we give some notations, define the system and formulate the main result. In Section 3 we construct a Poisson integral cluster expansion over densities of the configurations and give all needed estimates to prove the main theorems.

2. Definitions and main result

2.1. **Configuration space.** Let $\mathbb{R}^d$ be a d-dimensional Euclidean space. By $\mathcal{O}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all open and Borel sets, respectively. $\mathcal{O}_c(\mathbb{R}^d)$, $\mathcal{B}_c(\mathbb{R}^d)$ denote the systems of all sets in $\mathcal{O}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$, respectively, which are bounded.

The set of positions $\{x_i\}_{i\in\mathbb{N}}$ of identical particles is considered to be a locally finite subset in $\mathbb{R}^d$ and the set of all such subsets creates the configuration space:

$$\Gamma = \Gamma_{\mathbb{R}^d} := \left\{ \gamma \subset \mathbb{R}^d \middle|\ |\gamma\cap\Lambda| < \infty,\ \text{for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\right\},$$

where $|A|$ denotes the cardinality of the set A. The symbol $|\cdot|$ may also represent the Lebesgue measure of the set, but the meaning will always be clear from the context. For any $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ we denote by γ_Λ the projection of γ on Λ and the corresponding configuration space by Γ_Λ. We also need to define the space of finite configurations Γ_0,

$$\Gamma_0 = \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}, \quad \Gamma^{(n)} := \{\eta \subset \mathbb{R}^d \mid |\eta| = n\}, \quad \mathbb{N}_0 = \mathbb{N}\cup\{0\}.$$

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For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a mapping $N_\Lambda : \Gamma \to \mathbb{N}_0$ of the form

$$N_\Lambda(\eta) := |\eta \cap \Lambda|.$$

The Borel σ-algebra $\mathfrak{B}(\Gamma)$ is equal to $\sigma(N_\Lambda \,\big|\, \Lambda \in \mathfrak{B}_c(\mathbb{R}^d))$ and additionally one may introduce the following filtration:

$$\mathfrak{B}_\Lambda(\Gamma) := \sigma(N_{\Lambda'} \,\big|\, \Lambda' \in \mathcal{B}_c(\mathbb{R}^d),\ \Lambda' \subset \Lambda),$$

see [6], [7] for details.

By $\mathfrak{B}(\Gamma_\Lambda)$ we denote the corresponding σ-algebras on Γ_Λ and $\Gamma_{0,\Lambda}$. For a given intensity measure $\sigma = zdx$ $(z > 0)$ on $\mathcal{B}(\mathbb{R}^d)$ and any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered as a measure on

$$\widetilde{(\mathbb{R}^d)^n} = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \big|\ x_k \neq x_l \text{ if } k \neq l \right\}$$

and hence as a measure $\sigma^{(n)}$ on $\Gamma^{(n)}$ through the map

$$\mathrm{sym}_n : \widetilde{(\mathbb{R}^d)^n} \ni (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\} \in \Gamma^{(n)},$$

c.f. [5]. For simplicity we will write $(x)_n$ instead of $\{x_1, ..., x_n\} \in \Gamma^{(n)}$.

Define the Lebesgue-Poisson measure λ_σ on $\mathfrak{B}(\Gamma_0)$ by the formula

$$\lambda_\sigma := \sum_{n \geq 0} \frac{1}{n!} \sigma^{(n)}. \tag{1}$$

The restriction of λ_σ to $\mathfrak{B}(\Gamma_\Lambda)$ we also denote by λ_σ. For a more detailed structure of the configuration spaces Γ, Γ_0, Γ_Λ see [1].

2.2. **Definition of the system.** Let $\lambda \in \mathbb{R}_+$ be arbitrary. For each $r \in \mathbb{Z}^d$ we define (following [14]) an elementary cube $\Delta_\lambda(r)$with rib λ and center r by the formula

$$\Delta_\lambda(r) := \{x \in \mathbb{R}^d \mid \lambda(r^i - 1/2) \leq x^i < \lambda(r^i + 1/2)\}. \tag{2}$$

We will sometimes write Δ instead of $\Delta_\lambda(r)$, if a cube Δ is considered to be arbitrary and there is no reason to emphasize that it is centered at a particular point $r \in \mathbb{Z}^d$. Let $\overline{\Delta}$ be the partition of $\mathbb{R}^d$ into cubes $\Delta_\lambda(r)$. Without any restriction in general case, we consider only that $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ which is a union of cubes $\Delta_\lambda(r)$.

Define configuration spaces in which we will work in this paper

$$\Gamma_\Lambda := \left\{ \gamma \in \Gamma \big|\ \gamma_{\mathbb{R}^d \setminus \Lambda} = \emptyset \right\},$$

for any bounded fixed set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

For any $\Delta \in \overline{\Delta}$ introduce a space of *dilute* configuration

$$\Gamma_\Delta^{\mathrm{dil}} := \{ \gamma \in \Gamma_\Delta \big|\ |\gamma| = 0 \vee 1 \} \tag{3}$$

and a space of *dense* configuration,

$$\Gamma_\Delta^{\mathrm{den}} := \{ \gamma \in \Gamma_\Delta \big|\ |\gamma| \geq 2 \}. \tag{4}$$

For any $\Delta \in \overline{\Delta}$ and any fixed configuration $\eta \in \Gamma_\Lambda$ we split the space of *dense* configuration $\Gamma_\Delta^{\mathrm{den}}$ in two subspaces,

$$\Gamma_\Delta^{(>)}(\eta) = \Gamma_\Delta^{(>)} := \left\{ \gamma \in \Gamma_\Delta^{\mathrm{den}} \big|\ |\gamma| > d_\eta^\varepsilon(\Delta) \right\} \tag{5}$$

and

$$\Gamma_\Delta^{(<)}(\eta) = \Gamma_\Delta^{(<)} := \left\{ \gamma \in \Gamma_\Delta^{\mathrm{den}} \big|\ |\gamma| \leq d_\eta^\varepsilon(\Delta) \right\}, \tag{6}$$

where $\Delta \equiv \Delta_\lambda(r)$, $0 < \varepsilon < 1$ and $d_\eta(\Delta) = \mathrm{dist}(\eta, \Delta)$.

It's obviously that $\Gamma_\Delta^{\mathrm{den}} = \Gamma_\Delta^{(>)} \cup \Gamma_\Delta^{(<)}$.

We consider a general type of two-body interaction potential $\phi(x, y) = \varphi(|x - y|)$, where $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$.

Define an energy functional as

$$U_\varphi(\eta) = U(\eta) := \sum_{\{x,y\}\subset\eta} \varphi(|x-y|), \quad \eta\in\Gamma_0, \tag{7}$$

where $\{\cdot,\cdot\}$ means summation over all possible different pairs of particles from the configuration η. For a given $\gamma\in\Gamma_\Lambda$ define the interaction energy between $\eta\in\Gamma_\Lambda$, $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ and γ by

$$W(\eta|\gamma) := \sum_{\substack{x\in\eta\\ y\in\gamma}} \varphi(|x-y|), \tag{8}$$

and introduce the following notation:

$$U(\eta|\gamma) := U(\eta) + W(\eta|\gamma). \tag{9}$$

Following [14] let us define three important characteristics of the interaction U.

Definition 2.1. An interaction U is *stable* (S), if there exists $B\geq 0$ such that

$$U(\eta) \geq -|\eta|B, \quad \text{for all} \quad \eta\in\Gamma_0. \tag{10}$$

Definition 2.2. An interaction U is *superstable* (SS), if there exist $A>0$ and $B\geq 0$ such that

$$U(\eta) \geq \sum_{\Delta\in\overline{\Delta}} (|\eta_\Delta|^2 A - |\eta_\Delta| B), \quad \text{for all} \quad \eta\in\Gamma_0. \tag{11}$$

Definition 2.3. An interaction U is *lower regular* (LR) if there exists a decreasing function Ψ on $\mathbb{N}_0$ such that

$$\sum_{r\in\mathbb{Z}^d} \Psi(|r|) \;<\; \infty \tag{12}$$

and the interaction energy $W(\eta|\gamma)$ satisfies the following inequality:

$$W(\eta|\gamma) \;\geq\; -\frac{1}{2}\sum_{r,s\in\mathbb{Z}^d} \Psi(|r-s|)(|\eta_{\Delta_\lambda(r)}|^2 + |\gamma_{\Delta_\lambda(s)}|^2) \tag{13}$$

for all $\eta,\gamma\in\Gamma_0$.

Conditions (10)–(13) are rather general and guarantee a uniform estimate for the family of finite volume correlation functions and the existence of Gibbs measure [13] (see, also, [8]). A separate problem is to establish a condition on the potential φ, which ensures (10)–(13). See [8] for a discussion of this problem. Consider decomposition of the potential $\varphi(|x|)$ into two parts

$$\varphi(|x|) = \varphi^+(|x|) - \varphi^-(|x|), \tag{14}$$

where $\varphi^+(|x|) := \max\{0,\varphi(|x|)\}$, $\varphi^-(|x|) := -\min\{0,\varphi(|x|)\}$.

Using (14), for any fixed $\Delta_0\subset\mathbb{R}^d$ define the values

$$\upsilon_\varepsilon(\lambda,\Delta_0) := \sum_{\Delta\in\overline{\Delta}} \sup_{x\in\Delta}\sup_{y\in\Delta_0} \varphi^-(|x-y|)|x-y|^\varepsilon \quad \text{for all} \quad \varepsilon\in[0,1] \tag{15}$$

and

$$b(\lambda,\Delta_0) := \inf_{\{x,y\}\subset\Delta_0} \varphi^+(|x-y|). \tag{16}$$

It's clear from the definition that υ_ε and b do not depend on the position of Δ_0. So we will write

$$\upsilon_\varepsilon(\lambda,\Delta_0) \;=\; \upsilon_\varepsilon(\lambda) \;=\; \upsilon_\varepsilon,$$
$$b(\lambda,\Delta_0) \;=\; b(\lambda) \;=\; b.$$

To prove the property (10)–(13) for the potential φ we assume the following.

There exists a partition of $\overline{\Delta}$ into cubes (2) with fixed $\lambda > 0$ such that

A1.

$$0 \, < \, \upsilon_0 \, < \, +\infty, \tag{17}$$

A2.

$$\frac{1}{2}\, b > \upsilon_0. \tag{18}$$

These assumptions are very similar to those proposed by A. Ya. Povzner and discussed in [15] or in the integral form by R. L. Dobrushin in [2].

In this article we propose a vary transparent construction for proving existence of uniform bounds for the family of finite volume correlation functions ρ^{Λ}, which we define in the next subsection. For this purpose we need a little bit stronger assumption instead of (18),

A3. There exists a constant $\delta \in (0,1)$ such that

$$\frac{1}{2}\,(1-\delta) b > \upsilon_0 \tag{19}$$

and the potential

$$\varphi_\delta^{\rm st} := \delta\varphi^+(|x|) - \varphi^-(|x|) \tag{20}$$

is stable,

$$U_\delta^{\rm st} := U_{\varphi_\delta^{\rm st}}(\gamma) \;\ge\; -B_\delta |\gamma| \, , \quad \gamma \in \Gamma_0. \tag{21}$$

Remark 2.1. It will be clear from the proof of the Theorem 2.1 that if for $\delta \ge \frac{1}{2}$ the inequality (19) is true then $\varphi_\delta^{\rm st}$ is superstable.

A4. There exists some constant $\varepsilon \, > \, 0$ such that

$$\upsilon_\varepsilon \, < \, \infty \, . \tag{22}$$

Remark 2.2. It is clear from the definition of υ_ε in (15) that the potential φ, for which φ^- has an asymptotic behavior like $|x|^{-d-\varepsilon'}$ ($\varepsilon' > 0$) at large $|x|$, satisfies (22) with $\varepsilon < \varepsilon'$.

Remark 2.3. To satisfy (17)–(18) or (17), (19) it's sufficient, for example, to have non-integrability of φ at the origin, because for small λ, $\upsilon_0 \backsim \lambda^{-d}\|\varphi^-(|x|)\|_1$, where $\|\cdot\|_1$ is the $\mathrm{L}^1(\mathbb{R}^d)$-norm and in the case of the behavior $\varphi^+(|x|) \backsim \frac{\varphi_0}{|x|^\mu}$, $\mu \ge d$, the inequalities (18), (19) are true for sufficiently small λ. In the case $\mu < d$, in order to satisfy (18), (19), we have to chose λ small but fixed and φ_0 sufficiently large.

At the end of this subsection we introduce the following notations (see also (20) and definition (7)):

$$U_\delta^+ := U_{\varphi_\delta^+}, \quad \varphi_\delta^+(|x|) := (1-\delta)\, \varphi^+(|x|). \tag{23}$$

So from (20) and (23), we have a decomposition for any $\delta \in (0,1)$,

$$\varphi(|x|) \;=\; \varphi_\delta^+(|x|) + \varphi_\delta^{\rm st}(|x|) \tag{24}$$

and the corresponding decomposition for the energy (7),

$$U(\gamma) \;=\; U_\delta^+(\gamma) + U_\delta^{\rm st}(\gamma). \tag{25}$$

2.3. **Gibbs specification and correlation functions.** Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\Lambda^c := \mathbb{R}^d \setminus \Lambda$, and $\gamma \in \Gamma$. The finite volume Gibbs state with a fixed boundary configuration $\overline{\gamma} := \gamma \cap \Lambda^c$ for U, $z > 0$ and $\beta > 0$ is

$$\mu_\Lambda(d\eta|\,\overline{\gamma}) = \frac{\exp\{-\beta U(\eta|\,\overline{\gamma})\}}{Z_\Lambda(\overline{\gamma})}\lambda_\sigma(d\eta).$$

Under assumptions **A1**–**A2**, the finite volume Gibbs state is well defined as $Z_\Lambda(\overline{\gamma}) < \infty$. For $\overline{\gamma} = \emptyset$ let us write $\mu_\Lambda\,(d\eta\,|\,\emptyset) \equiv \mu_\Lambda(d\eta)$.

The corresponding finite-volume correlation functions with boundary configuration $\overline{\gamma} \in \Gamma$ have the following form:

$$\rho^\Lambda(\eta \mid \overline{\gamma}) = \frac{1}{Z_\Lambda(\overline{\gamma})}\int_{\Gamma_\Lambda} e^{-\beta U(\eta\cup\gamma|\overline{\gamma})}\lambda_\sigma(d\gamma), \quad \eta \in \Gamma_\Lambda. \tag{26}$$

$$Z_\Lambda(\overline{\gamma}) = \int_{\Gamma_\Lambda} e^{-\beta U(\gamma|\overline{\gamma})}\lambda_\sigma(d\gamma), \tag{27}$$

with $U(\cdot,\cdot)$ defined by (9).

Let $\{\pi_\Lambda\}$ denote the specification associated with z, β and the Hamiltonian U (see [10]), which is defined on Γ by

$$\pi_\Lambda(A|\,\overline{\gamma}) = \int_{A'} \mu_\Lambda(d\eta|\,\overline{\gamma}),$$

where $A' = \{\eta \in \Gamma_\Lambda :\ \eta \cup \overline{\gamma}_{\Lambda^c} \subset A\}$, $A \in \mathfrak{B}(\Gamma)$.

A probability measure μ on Γ is called a Gibbs state for U, β, and z if

$$\mu(\pi_\Lambda(A|\,\overline{\gamma})) = \mu(A)$$

for every $A \in \mathfrak{B}(\Gamma)$ and every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

This relation is the well known (DLR)-equation (Dobrushin-Lanford-Ruelle equation), see [4] for more details. The class of all Gibbs states which correspond to the specifications $\{\pi_\Lambda\}_{\Lambda\in\mathcal{B}_c(\mathbb{R}^d)}$ we denote by $\mathcal{G}(U,z,\beta)$.

2.4. **Main results.**

Theorem 2.1. *Let $\varphi(|x|)$ satisfy* **A1**–**A2**. *Then the potential $\varphi(|x|)$ is superstable.*

Theorem 2.2. *Suppose that the interaction potential $\varphi(|x|)$ satisfies the assumptions* **A3**–**A4**. *Then, for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and any $\beta, z \geq 0$ there exists a constant $\xi = \xi(\beta,z)$ (independent of Λ) such that the finite volume correlation function $\rho^\Lambda(\eta) = \rho^\Lambda(\eta \mid \emptyset)$ satisfies the following inequality:*

$$\rho^\Lambda(\eta) \leq \xi^{|\eta|} e^{-\beta U_\delta^+(\eta)}, \quad \eta \in \Gamma_\Lambda. \tag{28}$$

Remark 2.4. The estimate (28) without the exponent factor in the right-hand side is the well-known Ruelle bound [14]. We call (28) a generalized Ruelle bound. This result is a generalization of the work [12] to the case of an infinite range interaction.

Remark 2.5. Theorem 2.2 is also valid for potentials that satisfy the weaker assumption **A1**–**A2**. But for our method of the proof, we need assumption **A3**–**A4**.

As a corollary of Theorem 2.2 we have the following theorem.

Theorem 2.3. *Let the interaction potential $\varphi(|x|)$ satisfy* **A3**–**A4**. *Then for any $z \geq 0$ and $\beta \geq 0$,*

$$\mathcal{G}(U,z,\beta) \neq \emptyset.$$

Remark 2.6. The proof of Theorem 2.3 can be found in [13].

3. The proof of Theorems

3.1. **The proof of Theorem 2.1.** This theorem gives a new criterion for potentials to be superstable and extends the class of superstable potentials.

Proof. For any $\gamma \in \Gamma_0$,

$$
\begin{aligned}
U(\gamma) = \sum_{\{x,y\}\subset\gamma} \varphi(|x-y|) &= \sum_{\Delta\in\overline{\Delta}} \sum_{\{x,y\}\subset\gamma_\Delta} \varphi(|x-y|) + \sum_{\{\Delta,\Delta'\}\subset\overline{\Delta}} \sum_{\substack{x\in\gamma_\Delta \\ y\in\gamma_{\Delta'}}} \varphi(|x-y|) \\
&\geq \sum_{\Delta\in\overline{\Delta}} \frac{1}{2}|\gamma_\Delta|(|\gamma_\Delta|-1)b - \sum_{\{\Delta,\Delta'\}\subset\overline{\Delta}} |\gamma_\Delta||\gamma_{\Delta'}| \sup_{x\in\gamma_\Delta} \sup_{y\in\gamma_{\Delta'}} \varphi^-(|x-y|) \\
&\geq \sum_{\Delta\in\overline{\Delta}} \frac{1}{2}|\gamma_\Delta|(|\gamma_\Delta|-1)b - \frac{1}{2} \sum_{\{\Delta,\Delta'\}\subset\overline{\Delta}} (|\gamma_\Delta|^2 + |\gamma_{\Delta'}|^2) \sup_{x\in\gamma_\Delta} \sup_{y\in\gamma_{\Delta'}} \varphi^-(|x-y|) \\
&\geq \sum_{\Delta\in\overline{\Delta}} \frac{1}{2}|\gamma_\Delta|(|\gamma_\Delta|-1)b - 2\cdot\frac{1}{2} \sum_{\Delta\in\overline{\Delta}} |\gamma_\Delta|^2 \sum_{\Delta'\in\overline{\Delta}} \sup_{x\in\gamma_\Delta} \sup_{y\in\gamma_{\Delta'}} \varphi^-(|x-y|) \\
&\geq \sum_{\Delta\in\overline{\Delta}} \frac{1}{2}(|\gamma_\Delta|^2 - |\gamma_\Delta|)b - \sum_{\Delta\in\overline{\Delta}} |\gamma_\Delta|^2 v_0 \;=\; \sum_{\Delta\in\overline{\Delta}} \left(|\gamma_\Delta|^2\left(\frac{b}{2} - v_0\right) - \frac{b}{2}|\gamma_\Delta|\right).
\end{aligned}
$$

In the second line we used the inequality

$$
|\gamma_\Delta||\gamma_{\Delta'}| \leq \frac{1}{2}(|\gamma_\Delta|^2 + |\gamma_{\Delta'}|^2).
$$

□

3.2. **Cluster expansion in densities of configurations.** The proof is based on the cluster expansion of the Lebesgue-Poisson integral for the correlation functions (26)–(27) into series over dense configurations (cf. [12]).

The main technical idea consists in separation of the *dilute* parts of configurations from the *dense* parts. In order to do this, we define an indicator function for the configuration γ_Λ, $\Lambda \in \mathcal{J}_\lambda(\mathbb{R}^d)$ in the cube Δ, where $\mathcal{J}_\lambda(\mathbb{R}^d)$ are all finite unions of cubes of the form $\Delta_\lambda(r)$ (such sets are used in the construction of the Jordan measure). The indicator for *dilute* configurations is defined by

$$
\chi_-^\Delta(\gamma_\Delta) \;=\; \begin{cases} 1, & \text{for } |\gamma_\Delta| = 0 \vee 1, \\ 0, & \text{otherwise} \end{cases}
$$

and, for *dense* configurations, by

$$
\chi_+^\Delta(\gamma_\Delta) \;=\; 1 - \chi_-^\Delta(\gamma_\Delta).
$$

To obtain an expansion we use the following partition of unity for any $\gamma \in \Gamma_\Lambda$:

$$
\begin{aligned}
1 &= \prod_{\Delta\subset\Lambda} \left[\chi_-^\Delta(\gamma_\Delta) + \chi_+^\Delta(\gamma_\Delta)\right] \\
&= \sum_{n=0}^{N_\Lambda} \sum_{\{\Delta_1,\dots,\Delta_n\}\subset\Lambda} \prod_{i=1}^{n} \chi_+^{\Delta_i}(\gamma) \prod_{\Delta\subset\Lambda\setminus\cup_{i=1}^n\Delta_i} \chi_-^{\Delta}(\gamma) \;=\; \sum_{\emptyset\subseteq X\subseteq\Lambda} \widetilde{\chi}_+^X(\gamma)\widetilde{\chi}_-^{X^c}(\gamma),
\end{aligned}
\tag{29}
$$

where $N_\Lambda :=\; |\Lambda|/|\Delta|$ (here the symbol $|\cdot|$ denotes the Lebesgue measure of the set) is the number of cubes Δ in the volume Λ, X is a union of cubes Δ for which $|\gamma_\Delta| \geq 2$ and for convenience we denoted it by

$$
\widetilde{\chi}_\pm^X(\gamma) \;=\; \prod_{\Delta\subset X} \chi_\pm^\Delta(\gamma_\Delta), \quad X^c := \Lambda\setminus X.
$$

Inserting (29) into (26) for $\overline{\gamma} = \emptyset$ we get

$$\rho^{\Lambda}(\eta) \;=\; \frac{1}{Z_{\Lambda}} \sum_{\emptyset\subseteq X\subseteq\Lambda} \int_{\Gamma_{\Lambda}} \widetilde{\chi}_{+}^{X}(\gamma)\widetilde{\chi}_{-}^{X^{c}}(\gamma) e^{-\beta U(\eta\cup\gamma)} \lambda_{\sigma}(d\gamma). \tag{30}$$

Define a hard-core potential by

$$\chi^{\mathrm{cor}}(\Delta_1,\ldots,\Delta_n) \;=\; \begin{cases} 1, & \text{if } \;\Delta_i\cap\Delta_j=\emptyset \;\text{ far all }\; i\neq j,\\ 0, & \text{otherwise.}\end{cases} \tag{31}$$

Then (30) can be rewritten as

$$\begin{aligned}\rho^{\Lambda}(\eta) = {} & \frac{1}{Z_{\Lambda}} \sum_{n\geq 0} \frac{1}{n!} \sum_{(\Delta_1,\ldots,\Delta_n)\subset\Lambda} \chi^{\mathrm{cor}}(\Delta_1,\ldots,\Delta_n)\\ & \times \int_{\Gamma_{\Lambda}} \chi_{+}^{X}(\gamma)\chi_{-}^{X^{c}}(\gamma) e^{-\beta U(\eta\cup\gamma)} \lambda_{\sigma}(d\gamma),\end{aligned} \tag{32}$$

where $(\Delta_1,\ldots,\Delta_n)$ is a sequence of cubes unlike the set of these cubes in (29). The summation is taken independently over every Δ_i in (32).

The next step is to split the exponent in (32) using (7) and (25),

$$\begin{aligned} e^{-\beta U(\eta\cup\gamma)} = {} & e^{-\beta U_{\delta}^{+}(\eta)} e^{-\beta U_{\delta}^{\mathrm{st}}(\eta)} \prod_{i=1}^{n} e^{-\beta U(\gamma_{\Delta_i})-\beta W(\eta|\gamma_{\Delta_i})} \prod_{1\leq i<j\leq n} e^{-\beta W(\gamma_{\Delta_i}|\gamma_{\Delta_j})}\\ & \times e^{-\beta W(\eta|\gamma_{X_n^c})} e^{-\beta U(\gamma_{X_n^c})} \prod_{i=1}^{n} e^{-\beta W(\gamma_{\Delta_i}|\gamma_{X_n^c})}. \end{aligned} \tag{33}$$

Then, using decomposition (33) and infinite divisible property of the Lebesgue-Poisson measure (see, for example, (2.5) in [11]) we have

$$\rho^{\Lambda}(\eta) \;=\; \frac{e^{-\beta U_{\delta}^{+}(\eta)}}{Z_{\Lambda}} \sum_{n\geq 0} \frac{1}{n!}\, \widetilde{\rho}_{n}^{\Lambda}(\eta), \tag{34}$$

where

$$\begin{aligned} \widetilde{\rho}_{n}^{\Lambda}(\eta) = {} & \sum_{(\Delta_1,\ldots,\Delta_n)\subset\Lambda} \chi^{\mathrm{cor}}(\Delta_1,\ldots,\Delta_n) e^{-\beta U_{\delta}^{\mathrm{st}}(\eta)}\\ & \times \prod_{i=1}^{n} \left(\int_{\Gamma_{\Delta_i}^{\mathrm{den}}} \lambda_{\sigma}(d\gamma_{\Delta_i}) e^{-\beta U(\gamma_{\Delta_i})-\beta W(\eta|\gamma_{\Delta_i})} \right)\\ & \times \prod_{1\leq i<j\leq n} e^{-\beta W(\gamma_{\Delta_i}|\gamma_{\Delta_j})} \int_{\Gamma_{X_n^c}^{\mathrm{dil}}} \lambda_{\sigma}(d\gamma_{X_n^c}) e^{-\beta W(\eta|\gamma_{X_n^c})}\, e^{-\beta U(\gamma_{X_n^c})}\\ & \times \prod_{i=1}^{n} e^{-\beta W(\gamma_{\Delta_i}|\gamma_{X_n^c})}. \end{aligned} \tag{35}$$

Note that

$$\sum_{n\geq 0} \frac{1}{n!}\, \widetilde{\rho}_{n}^{\Lambda}(\emptyset) \;=\; Z_{\Lambda}. \tag{36}$$

Taking into account that $\Gamma_{\Delta}^{\mathrm{den}} = \Gamma_{\Delta}^{(>)} \cup \Gamma_{\Delta}^{(<)}$, each integral in the first product in formula (35) subdivides into two parts,

$$\begin{aligned} & \int_{\Gamma_{\Delta_i}^{\mathrm{den}}} \lambda_{\sigma}(d\gamma_{\Delta_i}) e^{-\beta U(\gamma_{\Delta_i})-\beta W(\eta|\gamma_{\Delta_i})}\\ & \quad = \int_{\Gamma_{\Delta_i}^{(>)}} \lambda_{\sigma}(d\gamma_{\Delta_i}) e^{-\beta U(\gamma_{\Delta_i})-\beta W(\eta|\gamma_{\Delta_i})} + \int_{\Gamma_{\Delta_i}^{(<)}} \lambda_{\sigma}(d\gamma_{\Delta_i}) e^{-\beta U(\gamma_{\Delta_i})-\beta W(\eta|\gamma_{\Delta_i})}. \end{aligned} \tag{37}$$

It follows from (36)–(37) that the sum over all possible $(\Delta_1, \ldots, \Delta_n) \subset \Lambda$ can be subdivided into

$$2^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}$$

items, in each of them the sum over $(\Delta_1, \ldots, \Delta_n)$ splits into the sum over $(\Delta_1, \ldots, \Delta_k) \subset \Lambda$, where the integration is taken over the configuration $\gamma_{\Delta_i} \in \Gamma_{\Delta_i}^{(<)}$ and the summation is taken over $(\Delta_1^{'}, \ldots, \Delta_{n-k}^{'}) \subset \Lambda$, where the integration is taken over the configuration $\gamma_{\Delta_i^{'}} \in \Gamma_{\Delta_i^{'}}^{(>)}$.

Then the expression (34) can be rewritten as

$$\rho^{\Lambda}(\eta) = \frac{e^{-\beta U_\delta^{+}(\eta)}}{Z_\Lambda} \sum_{n\geq 0} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!}\, \widetilde{\rho}^{\Lambda}_{n;k}(\eta), \tag{38}$$

where

$$\begin{aligned} \widetilde{\rho}^{\Lambda}_{n;k}(\eta) = & \sum_{(\Delta_1,\ldots,\Delta_k)\subset\Lambda} \ \sum_{(\Delta_1^{'},\ldots,\Delta_{n-k}^{'})\subset\Lambda} \chi^{\mathrm{cor}}(\Delta_1, \ldots, \Delta_k, \Delta_1^{'}, \ldots, \Delta_{n-k}^{'}) \\ & \times \prod_{i=1}^{k} \left(\int_{\Gamma_{\Delta_i}^{(<)}} \lambda_\sigma(d\gamma_{\Delta_i}) e^{-\beta U(\gamma_{\Delta_i}) - \beta W(\eta|\gamma_{\Delta_i})} \right) \prod_{1\leq i<j\leq k} e^{-\beta W(\gamma_{\Delta_i}|\gamma_{\Delta_j})} \\ & \times \prod_{i=1}^{n-k} \left(\int_{\Gamma_{\Delta_i^{'}}^{(>)}} \lambda_\sigma(d\gamma_{\Delta_i^{'}}) e^{-\beta U(\gamma_{\Delta_i^{'}}) - \beta W(\eta|\gamma_{\Delta_i^{'}})} \right) e^{-\beta U_\delta^{\mathrm{st}}}(\eta) \\ & \times \prod_{1\leq i<j\leq n-k} e^{-\beta W(\gamma_{\Delta_i^{'}}|\gamma_{\Delta_j^{'}})} \prod_{i=1}^{k}\prod_{j=1}^{n-k} e^{-\beta W(\gamma_{\Delta_i}|\gamma_{\Delta_j^{'}})} \int_{\Gamma_{X_n^c}^{\mathrm{dil}}} \lambda_\sigma(d\gamma_{X_n^c}) e^{-\beta W(\eta|\gamma_{X_n^c})} \\ & \times \left(\prod_{j=1}^{n-k} e^{-\beta W(\gamma_{\Delta_j^{'}}|\gamma_{X_n^c})} \right) \left(\prod_{i=1}^{k} e^{-\beta W(\gamma_{\Delta_i}|\gamma_{X_n^c})} \right) e^{-\beta U(\gamma_{X_n^c})} \end{aligned} \tag{39}$$

and $X_n = (\bigcup_{i=1}^{k} \Delta_i) \cup (\bigcup_{j=1}^{n-k} \Delta_j^{'})$.

The condition (36) is rewritten as

$$\sum_{n\geq 0} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!}\, \widetilde{\rho}^{\Lambda}_{n;k}(\emptyset) = Z_\Lambda. \tag{40}$$

3.3. **The proof of Theorem 2.2.** In this subsection we estimate factors from (39). Let $\varphi(x)$ satisfy **A4**. Then from Definition (15), it follows that, for any $k > 0$, the following inequality holds:

$$\sup_{x\in\eta} \sum_{i=1}^{k} \sup_{y\in\gamma_{\Delta_i}} \varphi^{-}_{\mathrm{st}}(|x-y|)|x-y|^{\varepsilon} \leq \upsilon_\varepsilon,$$

which gives the first estimate,

$$\begin{aligned} -\beta \sum_{i=1}^{k} W(\eta|\gamma_{\Delta_i}) \leq \beta \sum_{i=1}^{k} W_{\varphi^-}(\eta|\gamma_{\Delta_i}) &= \beta \sum_{i=1}^{k} \sum_{x\in\eta} \sum_{y\in\gamma_{\Delta_i}} \varphi^{-}(|x-y|) \\ &\leq \beta\,|\eta| \sup_{x\in\eta} \sum_{i=1}^{k} \sup_{y\in\gamma_{\Delta_i}} \varphi^{-}(|x-y|)|\gamma_{\Delta_i}| \leq \beta\,|\eta|\,\upsilon_\varepsilon. \end{aligned} \tag{41}$$

The last inequality in (41) is true, because $\gamma_{\Delta_i} \in \Gamma^{(<)}_{\Delta_i}$, so $|\gamma_{\Delta_i}| \le d^{\varepsilon}_{\eta}(\Delta_i) \le |x-y|^{\varepsilon}$, where $|\gamma_{\Delta_i}| := \mathrm{card}\{\gamma_{\Delta_i}\}$, and $|x-y|$ is the distance between particles in $x \in \eta$ and in $y \in \gamma_{\Delta_i}$.

Using assumption **A3** (see (21)) we get

$$\begin{aligned} \sum_{i=1}^{n-k} \Big(U^{\mathrm{st}}_{\delta}(\gamma_{\Delta'_i}) + W^{\mathrm{st}}_{\delta}(\eta|\gamma_{\Delta'_i}) \Big) + U^{\mathrm{st}}_{\delta}(\eta) + \sum_{1\le i<j\le n-k} W^{\mathrm{st}}_{\delta}(\gamma_{\Delta'_i}|\gamma_{\Delta'_j}) \\ = U^{\mathrm{st}}_{\delta}(\eta \cup \gamma_{\cup_{i=1}^{n-k}\Delta'_i}) \ge -B_{\delta}\,(\,|\eta| + \sum_{i=1}^{n-k} |\gamma_{\Delta'_j}|). \end{aligned} \tag{42}$$

Because of positivity of φ^+,

$$e^{-\beta \sum_{i=1}^{n-k} W^+_{\delta}(\eta|\gamma_{\Delta'_j})} \le 1, \quad e^{-\beta \sum_{1\le i<j\le n-k} W^+_{\delta}(\gamma_{\Delta'_i}|\gamma_{\Delta'_j})} \le 1, \tag{43}$$

and

$$e^{-\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} W_{\varphi^+}(\gamma_{\Delta_i}|\gamma_{\Delta'_j})} \le 1. \tag{44}$$

It is also clear that if condition (19) is true for some $\delta > 0$ then there exists $\delta' > \delta$ for which (19) is also valid. So we can chose, in the decomposition (20) and (23)–(25),

$$\delta = \delta' - \delta''$$

and rewrite $U^+_{\delta}(\Gamma_{\Delta'_i})$ in the form

$$U^+_{\delta}(\gamma_{\Delta'_i}) = (1-\delta')\,U_{\varphi^+}(\gamma_{\Delta'_i}) + \delta''\,U_{\varphi^+}(\gamma_{\Delta'_i}) = U^+_{\delta'}(\gamma_{\Delta'_i}) + \delta'' U_{\varphi^+}(\gamma_{\Delta'_i})\,.$$

Now using the fact that $\gamma_{\Delta'_i} \in \Gamma^{(>)}_{\Delta'_i}$ (see def. (5)) and the definition (16) one can obtain the following inequality:

$$U^+_{\delta}(\gamma_{\Delta'_i}) \ge \frac{1}{2}(1-\delta')\,b\,|\gamma_{\Delta'_i}|\,(|\gamma_{\Delta'_i}| - 1) + \delta''\,b\,d^{\varepsilon}_{\eta}(\Delta'_i). \tag{45}$$

Lemma 3.1. *Let $\varphi(|x|)$ satisfy* **A4**, *then the next inequality holds:*

$$-\beta \sum_{i=1}^{k}\sum_{j=1}^{n-k} W_{\varphi^-}(\gamma_{\Delta_i}|\gamma_{\Delta'_j}) \le \beta \sum_{j=1}^{n-k} |\gamma_{\Delta'_j}|(\upsilon_{\varepsilon} + (|\gamma_{\Delta'_j}|+1)\upsilon_0). \tag{46}$$

Proof.

$$\begin{aligned} -\beta \sum_{i=1}^{k}\sum_{j=1}^{n-k} W_{\varphi^-}(\gamma_{\Delta_i}|\gamma_{\Delta'_j}) &= \beta \sum_{i=1}^{k}\sum_{j=1}^{n-k} \sum_{x\in\gamma_{\Delta_i}} \sum_{y\in\gamma_{\Delta'_j}} \varphi^-(|x-y|) \\ &\le \beta \sum_{i=1}^{k}\sum_{j=1}^{n-k} |\gamma_{\Delta_i}||\gamma_{\Delta'_j}| \sup_{x\in\Delta_i}\sup_{y\in\Delta'_j} \varphi^-(|x-y|) \\ &\le \beta \sum_{i=1}^{k}\sum_{j=1}^{n-k} |\gamma_{\Delta'_j}| \sup_{x\in\Delta_i}\sup_{y\in\Delta'_j} \varphi^-(|x-y|)|x-p|^{\varepsilon} \\ &\le \beta \sum_{j=1}^{n-k} |\gamma_{\Delta'_j}| \sum_{i=1}^{k} \sup_{x\in\Delta_i}\sup_{y\in\Delta'_j} \varphi^-(|x-y|)|x-y|^{\varepsilon} \end{aligned}$$

$$
\begin{aligned}
&+\beta\sum_{j=1}^{n-k}|\gamma_{\Delta_j'}|\sum_{i=1}^{k}\sup_{x\in\Delta_i}\sup_{y\in\Delta_j'}\varphi^-(|x-y|)|y-p|^\varepsilon\\
&\le \beta\sum_{j=1}^{n-k}|\gamma_{\Delta_j'}|\sum_{i=1}^{k}\sup_{x\in\Delta_i}\sup_{y\in\Delta_j'}\varphi^-(|x-y|)|x-y|^\varepsilon\\
&+\beta\sum_{j=1}^{n-k}|\gamma_{\Delta_j'}|\sum_{i=1}^{k}\sup_{x\in\Delta_i}\sup_{y\in\Delta_j'}\varphi^-(|x-y|)(|\gamma_{\Delta_j'}|+1)\\
&\le \beta\sum_{j=1}^{n-k}|\gamma_{\Delta_j'}|(\upsilon_\varepsilon+(|\gamma_{\Delta_j'}|+1)\upsilon_0).
\end{aligned}
$$

The second inequality holds, because $\gamma_{\Delta_i}\in\Gamma^{(<)}_{\Delta_i}$ (see def. (6)), and so $|\gamma_{\Delta_i}|\le d^\varepsilon_\eta(\Delta_i)\le |x-z|^\varepsilon$, where z is the coordinate of the particle from η, lying nearest to Δ_i, $|\gamma_{\Delta_i}|:=\operatorname{card}\{\gamma_{\Delta_i}\}$, $|x-z|$ is the distance between the particles with coordinate $x\in\Delta_i$ and $z\in\eta\;\Rightarrow\;|x-z|\le|x-p|$, where $p\in\Delta_\eta$, and such that it is situated nearest to $\Delta_j^{'}$.

In the third inequality we use the modified triangle inequality, namely, $\forall\,\varepsilon\in[0;1]$, $\forall\,x,y,p\in\mathbb{R}^d$,

$$|x-p|^\varepsilon\le|x-y|^\varepsilon+|y-p|^\varepsilon.$$

Because $y\in\gamma_{\Delta_j'}$ and $p\in\eta$ is specially chosen (see above), for a sufficiently small λ, namely $\lambda<\frac14$, the next inequality holds:

$$|y-p|^\varepsilon\le d_\eta(\Delta_j^{'})+2\sqrt{\lambda}\le|\gamma_{\Delta_j'}|+1.$$

This inequality concludes the proof. □

Using the fact that $|\gamma_{\Delta_i}|\le 1,\ \forall\,\Delta\in\Lambda\setminus X_n$, we get

$$-\beta W(\eta\,|\,\gamma_{X_n^c})\le\beta\,|\eta|\,\upsilon_0 \tag{47}$$

and

$$-\beta W(\gamma_{\Delta_j'}\,|\,\gamma_{X_n^c})\le\beta|\gamma_{\Delta_j'}|\upsilon_0. \tag{48}$$

Now, using an elementary estimate (see def. (31)),

$$\chi^{\rm cor}(\Delta_1,\dots,\Delta_k,\Delta_1^{'},\dots,\Delta_{n-k}^{'})\le\chi^{\rm cor}(\Delta_1,\dots,\Delta_k), \tag{49}$$

and estimates (41)–(49), $\widetilde{\rho}^{\Lambda}_{n;k}(\eta)$ can be estimated (see (39)) in the following way:

$$
\begin{aligned}
\widetilde{\rho}^{\Lambda}_{n;k}(\eta)\le\; & e^{\beta(B_\delta+\upsilon_0+\upsilon_\varepsilon)\,|\eta|}\sum_{(\Delta_1,\dots,\Delta_k)\subset\Lambda}\chi^{\rm cor}(\Delta_1,\dots,\Delta_k)\\
&\times\int_{\Gamma_{X_k}}\lambda_\sigma(d\gamma_{X_k})\,e^{-\beta U(\gamma_{X_k})}\\
&\times\sum_{(\Delta_1',\dots,\Delta_{n-k}')\subset\Lambda}\int_{\Gamma^{(\rm dil)}_{(\Lambda\setminus X_{n-k}')\setminus X_k}}\lambda_\sigma(d\gamma_{(\Lambda\setminus X_{n-k}')\setminus X_k})\\
&\times e^{-\beta\,W(\gamma_{X_k}|\gamma_{(\Lambda\setminus X_{n-k}')\setminus X_k})}\,e^{-\beta\,U(\gamma_{(\Lambda\setminus X_{n-k}')\setminus X_k})}\\
&\times\prod_{\Delta\subset X_{n-k}'}\Bigg(\int_{\Gamma_\Delta^{\rm den}}\lambda_\sigma(d\gamma_\Delta)\,e^{-\frac12(1-\delta')\beta\,b\,|\gamma_\Delta|(|\gamma_\Delta|-1)}\\
&\times e^{\beta\,(B_\delta+\upsilon_0\,(2+|\gamma_\Delta|)+\upsilon_\varepsilon)|\gamma_\Delta|-\delta''\beta\,b\,d^\varepsilon_\eta(\Delta)}\Bigg),
\end{aligned}
\tag{50}
$$

where $X'_{n-k} = \bigcup_{i=1}^{n-k} \Delta'_i$, $X_k = \bigcup_{j=1}^{k} \Delta_j$.

Lemma 3.2.

$$\begin{aligned}\int_{\Gamma_\Delta^{\mathrm{den}}} \lambda_\sigma(d\gamma_\Delta)\, e^{-\frac{1}{2}(1-\delta')\beta\, b\,|\gamma_\Delta|(|\gamma_\Delta|-1)+\beta\,(B_\delta+v_0\,(2+|\gamma_\Delta|)+v_\varepsilon)|\gamma_\Delta|} \\ = K_1(\lambda, z, \beta, \varphi) = K_1,\end{aligned} \tag{51}$$

where K_1 is a constant that depends on $\lambda, z, \beta, \varphi$ and independent of Λ.

Proof. From the definition of the measure λ_σ (see (1)), we have

$$\begin{aligned}&\int_{\Gamma_\Delta^{\mathrm{den}}} \lambda_\sigma(d\gamma_\Delta)\, e^{-\frac{1}{2}(1-\delta')\beta\, b\,|\gamma_\Delta|(|\gamma_\Delta|-1)+\beta\,(B_\delta+v_0\,(2+|\gamma_\Delta|)+v_\varepsilon)|\gamma_\Delta|} \\ &= \sum_{n=2}^{\infty} \frac{(\lambda^d\, z)^n}{n!} e^{-\frac{1}{4}\beta\, b\, n(n-1)+\beta\,(B_\delta+v_0\,(2+n)+v_\varepsilon)n} \\ &= \sum_{n=2}^{\infty} \frac{(\lambda^d\, z)^n}{n!} \left(e^{-\frac{1}{2}(1-\delta')\beta\, b+\beta v_0}\right)^{n^2} \left(e^{-\frac{1}{2}(1-\delta')\beta\, b+\beta(B_\delta+2v_0+v_\varepsilon)}\right)^{n}.\end{aligned}$$

The convergence of the series follows from **A3**. □

Lemma 3.3.

$$\sum_{\Delta'\subset\Lambda} e^{-\delta''\beta\, d^\varepsilon_\eta(\Delta')} = |\eta|\, K_2(\lambda, \beta, \varepsilon) = |\eta|\, K_2, \tag{52}$$

where K_2 is a constant that depends on $\lambda, \beta, \varepsilon$ and is independent of Λ.

Proof. Let $\eta = \{x_1, \ldots, x_m\}$. Split Λ into domains $\Lambda_1, \ldots, \Lambda_m$ in such a way that if $\Delta' \subset \Lambda_k$, then $d^\varepsilon_\eta(\Delta') = d^\varepsilon_{x_k}(\Delta')$. Then

$$\begin{aligned}\sum_{\Delta'\subset\Lambda} e^{-\delta''\beta\, d^\varepsilon_\eta(\Delta')} &= \sum_{\Delta'\subset\Lambda_1} e^{-\delta''\beta\, d^\varepsilon_{x_1}(\Delta')} + \cdots + \sum_{\Delta'\subset\Lambda_m} e^{-\delta''\beta\, d^\varepsilon_{x_m}(\Delta')} \\ &\le |\eta| \sum_{\Delta'\subset\Lambda} e^{-\delta''\beta\, d^\varepsilon_{x_0}(\Delta')} = |\eta|\, K_2(\lambda, \beta, \varepsilon),\end{aligned}$$

where $d^\varepsilon_{x_0}(\Delta')$ is the distance from any fixed point $x_0 \in \mathbb{R}^d$ and Δ'. □

Denote by $\overline{X}_{n-k}$ the unions of all cubes $\Delta'_1 \cup \ldots \cup \Delta'_{n-k} = X'_{n-k}$ on which the integral (50) over configurations in $(\Lambda \setminus X'_{n-k}) \setminus X_k$ takes a maximal value. We have

$$\widetilde{\rho}^{\Lambda}_{n;k}(\eta) = e^{\beta(B_\delta+v_0+v_\varepsilon)\,|\eta|}\,(|\eta|\, K)^{n-k}\, \widetilde{\rho}_k^{\Lambda\setminus\overline{X}_{n-k}}(\emptyset), \tag{53}$$

where $K = K_1\, K_2$ (see (51), (52)), and $\widetilde{\rho}_k^{\Lambda\setminus\overline{X}_{n-k}}(\emptyset)$ which is defined by formula (35). Denote $\bar\xi := e^{\beta(B_\delta+v_0+v_\varepsilon)}$, then, inserting (53) into (38), we get (using def. (40)) that

$$\begin{aligned}\rho^\Lambda(\eta) &\le \frac{1}{Z_\Lambda}\, e^{-\beta\, U_\delta^+}\, \bar\xi^{|\eta|} \sum_{n=0}^{|\Lambda|}\sum_{k=0}^{n} \frac{(|\eta|K)^{n-k}}{k!(n-k)!}\, \widetilde{\rho}_k^{\Lambda\setminus\overline{X}_{n-k}}(\emptyset) \\ &= \frac{1}{Z_\Lambda}\, e^{-\beta\, U_\delta^+}\, \bar\xi^{|\eta|} \sum_{l=0}^{|\Lambda|} \frac{(|\eta|K)^{l}}{l!} \sum_{k=0}^{|\Lambda|-l} \frac{1}{k!} \widetilde{\rho}_k^{\Lambda\setminus\overline{X}_{n-k}}(\emptyset) \\ &= e^{-\beta\, U_\delta^+}\, \bar\xi^{|\eta|} \sum_{l=0}^{|\Lambda|} \frac{(|\eta|K)^{l}}{l!} \frac{Z_{\Lambda\setminus\overline{X}_l}}{Z_\Lambda}.\end{aligned} \tag{54}$$

The fact that $Z_{\Lambda_1} \le Z_{\Lambda_2}$ for $\Lambda_1 \subset \Lambda_2$ gives the inequality

$$\rho^{\Lambda}(\eta) \le e^{-\beta U_{\delta}^{+}(\eta)}\, e^{|\eta|\,(\beta(B_{\delta}+\upsilon_0+\upsilon_{\varepsilon})+K)} \tag{55}$$

which is (28) with $\xi = \bar{\xi} e^{K|\eta|}$. $\square$

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