THE EFIMOV EFFECT FOR A MODEL OPERATOR ASSOCIATED WITH THE HAMILTONIAN OF A NON CONSERVED NUMBER OF PARTICLES

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ABSTRACT. A model operator associated with the energy operator of a system of three non conserved number of particles is considered. The essential spectrum of the operator is described by the spectrum of a family of the generalized Friedrichs model. It is shown that there are infinitely many eigenvalues lying below the bottom of the essential spectrum, if a generalized Friedrichs model has a zero energy resonance.

1. INTRODUCTION

The main goal of the paper is to give a complete proof of the existence of infinitely many eigenvalues (the Efimov effect) for the model operator H associated with the Hamiltonian of a non conserved number of particles on lattices, announced in [18].

Roughly speaking the Efimov effect consists in the following: if in a system of three-particles, interacting by means of short-range pair potentials, none of the three two-particle subsystems has bound states with negative energy, but at least two of them have a resonance with zero energy, then this three-particle system has an infinite number of three-particle bound states with negative energies, accumulating at zero.

This effect was first discovered by Efimov [7]. Since then this problem has been studied in many works [1, 2, 5, 6, 8, 25, 28, 29, 30, 31]. A rigorous mathematical proof of the existence of the Efimov effect was originally carried out by Yafaev in [31] and then in [25, 28, 29, 30].

In a systems of three-particles on three-dimensional lattices, due to the fact that the discrete analogue of the Laplacian or its generalizations are not rotationally invariant, the Hamiltonian of a system does not separate into two parts, one relating to the center-of-mass motion and the other one to the internal degrees of freedom. In particular, in this case the Efimov effect exists only for the zero value of the three-particle quasi-momentum $K \in \mathbb{T}^3 = (-\pi, \pi]^3$ (see, e.g., [3, 4, 15, 17, 21] for relevant discussions and [9, 12, 13, 21, 23, 24, 26] for the general study of the low-lying excitation spectrum for quantum systems on lattices).

In the theory of solid-state physics [21, 24], quantum field theory [11] and statistical physics [20, 22] some important problems arise where the number of quasi-particles is not fixed. The study of systems with a non conserved, but bounded, number of particles is reduced to the study of the spectral properties of self-adjoint operators acting in "the cut" subspace $\mathcal{H}^{(n)}$, consisting of one particle, two particle and n-particle subspaces of the Fock space [22, 24].

In the present paper a model operator H acting in $\mathcal{H}^{(3)}$ associated to a system describing three-particles in interactions without conservation of the number of particles on lattices is considered.

The essential spectrum of H is described by the spectrum of a family of generalized Friedrichs models.

The presence of an infinite number of eigenvalues below the bottom of the essential spectrum of H is proved, if the corresponding generalized Friedrichs model has a zero energy resonance. The result can be considered as a first step towards a general proof of the Efimov effect for the Hamiltonians of the systems of solid state physics with a non conserved number of particles. Mathematically the results require new techniques in addition to those used in the proof of the Efimov effect for three-particle lattice system [3].

The organization of this paper is as following: Section 1 is an introduction to the whole work. In Section 2 the model operator H is described as bounded and self-adjoint operator in $\mathcal{H}^{(3)}$ and the main result is formulated. Some spectral properties of a family of generalized Friedrichs model $h(p), p \in \mathbb{T}^3$ is studied in Section 3. In Section 4 we introduce the "channel operator" and describe its spectrum. In Section 5 we obtain an analogue of the Faddeev-Newton type system of integral equations for the eigenfunctions of H. In Section 6 we prove the main result of the present paper. Some technical material is collected in Appendix A.

Throughout the present paper we adopt the following conventions: Denote by $\mathbb{T}^3$ the three-dimensional torus, the cube $(-\pi, \pi]^3$ with appropriately identified sides. For each $\delta > 0$ the notation $U_\delta(0) = \{p \in \mathbb{T}^3 : |p| < \delta\}$ stands for a δ-neighborhood of the origin.

2. PRELIMINARY INFORMATION AND STATEMENTS OF THE MAIN RESULT

Let $\mathbb{C} = \mathbb{C}^1$ be the field of complex numbers and let $L_2(\mathbb{T}^3)$ be the Hilbert space of square-integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^s((\mathbb{T}^3)^2)$ be the Hilbert space of square-integrable symmetric (complex) functions on $(\mathbb{T}^3)^2$.

Denote by $\mathcal{H}^{(3)}$ the direct sum of spaces $\mathcal{H}_0 = \mathbb{C}^1$, $\mathcal{H}_1 = L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 = L_2^s((\mathbb{T}^3)^2)$, that is, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

Let the operator H act in the Hilbert space $\mathcal{H}^{(3)}$ as a matrix operator

$$(2.1) \qquad H = \begin{pmatrix} H_{00} & H_{01} & 0 \\ H_{10} & H_{11} & H_{12} \\ 0 & H_{21} & H_{22} \end{pmatrix},$$

with the entries $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i, i,j = 0,1,2$ defined by

$$(H_{00}f_0)_0 = u_0 f_0, \quad (H_{01}f_1)_0 = \int_{\mathbb{T}^3} b(q')f_1(q')\,dq', \quad (H_{10}f_0)_1(p) = b(p)f_0,$$

$$(H_{11}f_1)_1(p) = u(p)f_1(p), \quad (H_{12}f_2)_1(p) = \int_{\mathbb{T}^3} b(q')f_2(p,q')\,dq',$$

$$(H_{21}f_1)_2(p,q) = \frac{1}{2}(b(p)f_1(q) + b(q)f_1(p)), \quad (H_{22}f_2)_2(p,q) = w(p,q)f_2(p,q).$$

Here $f_i \in \mathcal{H}_i$, $i = 0,1,2$, u_0 is a real number, u and b are real-analytic even functions on $\mathbb{T}^3$ and w is defined by the equality

$$w(p,q) = \varepsilon(p) + \varepsilon(p+q) + \varepsilon(q),$$

where

$$(2.2) \qquad \varepsilon(p) = 3 - \cos p_1 - \cos p_2 - \cos p_3, \quad p = (p_1, p_2, p_3) \in \mathbb{T}^3.$$

Under these assumptions the operator H defined by (2.1) is bounded and self-adjoint.

We remark that the operators H_{10} and H_{21} resp. H_{01} and H_{12} defined in the Fock space are called creation resp. annihilation operators.

To formulate the results we introduce a family of generalized Friedrichs model $h(p)$, $p \in \mathbb{T}^3$, which acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

(2.3)
$$h(p) \begin{pmatrix} f_0 \\ f_1(q) \end{pmatrix} = \begin{pmatrix} u(p)f_0 + \frac{1}{\sqrt{2}} \int_{\mathbb{T}^3} b(q')f_1(q')dq' \\ \frac{1}{\sqrt{2}}b(q)f_0 + w(p,q)f_1(q) \end{pmatrix}.$$

Let the operator $h_0(p)$ act in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$h_0(p) \begin{pmatrix} f_0 \\ f_1(q) \end{pmatrix} = \begin{pmatrix} 0 \\ w_p(q)f_1(q) \end{pmatrix}.$$

The perturbation $h(p) - h_0(p)$ of the operator $h_0(p)$ is a self-adjoint operator of rank 2. Therefore in accordance with the invariance of the essential spectrum under the finite rank perturbations the essential spectrum $\sigma_{\mathrm{ess}}(h(p))$ of $h(p)$ fills the following interval on the real axis:
$$\sigma_{\mathrm{ess}}(h(p)) = [m(p), M(p)],$$
where the numbers $m(p)$ and $M(p)$ are defined by
$$m(p) = \min_{q \in \mathbb{T}^3} w(p,q) \quad \text{and} \quad M(p) = \max_{q \in \mathbb{T}^3} w(p,q).$$

Remark 2.1. For some $p \in \mathbb{T}^3$ (for example $p = (\pi, \pi, \pi) \in \mathbb{T}^3$) the essential spectrum of $h(p)$ can be degenerated to the set consisting of unique point $[m(p), m(p)]$. Because of we can not state that the essential spectrum of $h(p)$ is absolutely continuous for any $p \in \mathbb{T}^3$.

The following theorem describes the essential spectrum of the operator H by the spectrum of the family $h(p)$, $p \in \mathbb{T}^3$ of generalized Friedrichs model.

Theorem 2.2. *For the essential spectrum $\sigma_{\mathrm{ess}}(H)$ of the operator H the equality*
$$\sigma_{\mathrm{ess}}(H) = \bigcup_{p \in \mathbb{T}^3} \{\sigma_d(h(p)) \cup [m(p), M(p)]\}$$
holds, where $\sigma_d(h(p))$ is the discrete spectrum of $h(p), p \in \mathbb{T}^3$.

For any $p \in \mathbb{T}^3$ we define an analytic function $\Delta(p, \cdot)$ (the Fredholm determinant associated to the operator $h(p)$) in $\mathbb{C} \setminus [m(p), M(p)]$ by

(2.4)
$$\Delta(p, z) = u(p) - z - \frac{1}{2} \int_{\mathbb{T}^3} \frac{b^2(q') \, dq'}{w(p, q') - z}.$$

Let σ be the set of complex numbers $z \in \mathbb{C} \setminus [m(p), M(p)]$ such that the equality $\Delta(p, z) = 0$ holds for some $p \in \mathbb{T}^3$.

Remark 2.3. We remark that in [19] the essential spectrum of the operator H has been described by zeroes of the Fredholm determinant defined in (2.4) and the spectrum of the multiplication operator H_{22} as follows:
$$\sigma_{\mathrm{ess}}(H) = \sigma \cup [0, M],$$
where
$$M = \max_{p,q \in \mathbb{T}^3} w(p,q).$$
We notice that the equality
$$\sigma = \bigcup_{p \in \mathbb{T}^3} \sigma_d(h(p))$$
holds.

Definition 2.4. The set σ resp. $[0, M]$ is called two- resp. three-particle branch of the essential spectrum $\sigma_{\mathrm{ess}}(H)$ of H, which will be denote by $\sigma_{\mathrm{two}}(H)$ resp. $\sigma_{\mathrm{three}}(H)$.

The function $w(0, \cdot)$ has a unique non-degenerate minimum at $q = 0$ (see Lemma A.1) and hence by dominated convergence theorem the finite limit

$$\Delta(0, 0) = \lim_{z \to -0} \Delta(0, z)$$

exists.

Let $C(\mathbb{T}^3)$ be the Banach space of continuous functions on $\mathbb{T}^3$.

Definition 2.5. Let $u(0) \neq 0$. The operator $h(0)$ is said to have a zero energy resonance, if the number 1 is an eigenvalue of the integral operator given by

$$(\mathrm{G}\psi)(q) = \frac{b(q)}{2u(0)} \int_{\mathbb{T}^3} \frac{b(t)\psi(t)}{w(0, t)} \, dt, \quad \psi \in C(\mathbb{T}^3)$$

and the associated eigenfunction ψ (up to constant factor) satisfies the condition $\psi(0) \neq 0$.

Remark 2.6. *a)* If $u(0) \leq 0$, then the equation $h(0)f = 0$ has only the trivial solution $f \in C^1 \oplus L_1(\mathbb{T}^3)$.
b) Assume that $u(0) > 0$ and $\Delta(0, 0) = 0$.
   (i) If $b(0) \neq 0$, then the operator $h(0)$ has a zero energy resonance and the vector $f = (f_0, f_1)$ obeys the equation $h(0)f = 0$, where

(2.5) $$f_0 = \mathrm{const} \neq 0, \quad f_1(q) = -\frac{b(q)f_0}{\sqrt{2}w(0, q)},$$

and hence $f_1 \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$ (see Lemma 3.2).
   (ii) If $b(0) = 0$, then the operator $h(0)$ has a zero eigenvalue and the vector $f = (f_0, f_1)$, where f_0 and f_1 are defined by (2.5), obeys the equation $h(0)f = 0$ and hence $f_1 \in L_2(\mathbb{T}^3)$ (see Lemma 3.3).

Throughout the paper we assume the following additional assumption.

Assumption 2.7. *The function u has a unique minimum at $0 \in \mathbb{T}^3$.*

The main result of the paper is the following.

Theorem 2.8. *Let Assumption 2.7 be fulfilled and the operator $h(0)$ have a zero energy resonance. Then the operator H has infinitely many negative eigenvalues $(E_n)_{n \in N}$ such that $\lim_{n \to \infty} E_n = 0$.*

Remark 2.9. We remark that if Assumption 2.7 is fulfilled and the operator $h(0)$ has either a zero energy resonance or a zero eigenvalue, then the operator $h(p)$, $p \in \mathbb{T}^3$ has no negative eigenvalues and hence $\inf \sigma_{\mathrm{ess}}(H) = 0$ (see Lemma 3.4).

3. Spectral properties of the operators $h(p)$, $p \in \mathbb{T}^3$

In this section we study some spectral properties of the family of the generalized Friedrichs model $h(p)$, $p \in \mathbb{T}^3$ given by (2.3), which plays an important role in the study of the spectral properties of H. We notice that the spectrum and resonances of the generalized Friedrichs model have been studied in detail in [14].

Lemma 3.1. *For any $p \in \mathbb{T}^3$ the operator $h(p)$ has an eigenvalue $z \in \mathbb{C} \setminus [m(p), M(p)]$ if and only if $\Delta(p, z) = 0$.*

Proof. If $u(p) \in \mathbb{R} \setminus [m(p), M(p)]$ for any $p \in \mathbb{T}^3$, then the equation $h(p)f = m(p)f$, $f \in \mathcal{H}_0 \oplus \mathcal{H}_1$ has only a trivial solution and hence the value $u(p) \in \mathbb{R} \setminus [m(p), M(p)]$ can not be an eigenvalue of the operator $h(p)$. The number $z \in (\mathbb{C} \setminus [m(p), M(p)]) \cup \{u(p)\}$

is an eigenvalue of the operator $h(p)$, $p \in \mathbb{T}^3$ if and only if (by the Birman-Schwinger principle) the number 1 is an eigenvalue of the integral operator

$$(\mathrm{G}(p,z)\psi)(q) = \frac{b(q)}{2(u(p)-z)} \int_{\mathbb{T}^3} \frac{b(t)\psi(t)\,dt}{w(p,t)-z}, \quad \psi \in L_2(\mathbb{T}^3).$$

According to the Fredholm theorem the number $\lambda = 1$ is an eigenvalue of the operator $G(p,z)$ if and only if $\Delta(p,z) = 0$. $\qquad \square$

Lemma 3.2. *The operator $h(0)$ has a zero energy resonance iff $\Delta(0,0) = 0$ and $b(0) \neq 0$.*

Proof. "Only If Part". Let the operator $h(0)$ have a zero energy resonance. Then by Definition 2.5 we have that $u(0) \neq 0$ and the equation

$$(3.1) \qquad \psi(q) = \frac{b(q)}{2u(0)} \int_{\mathbb{T}^3} \frac{b(t)\psi(t)dt}{w(0,t)}, \quad \psi \in C(\mathbb{T}^3)$$

has a simple solution $\varphi \in C(\mathbb{T}^3)$ and $\varphi(0) \neq 0$.
   This solution is equal to the function b (up to a constant factor) and hence $\Delta(0,0) = 0$.
   "If Part". Let the equality $\Delta(0,0) = 0$ hold and $b(0) \neq 0$. Then the function $b \in C(\mathbb{T}^3)$ is a solution of the equation (3.1), that is, the operator $h(0)$ has a zero energy resonance. $\qquad \square$

Lemma 3.3. *The operator $h(0)$ has a zero eigenvalue iff $\Delta(0,0) = 0$ and $b(0) = 0$.*

Proof. "Only If Part". Suppose $f = (f_0, f_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1$ is an eigenvector of the operator $h(0)$ associated with the zero eigenvalue. Then f_0 and f_1 satisfy the system of equations

$$(3.2) \qquad \begin{cases} u(0)f_0 + \frac{1}{\sqrt{2}} \int_{\mathbb{T}^3} b(q')f_1(q')\,dq' = 0 \\ \frac{1}{\sqrt{2}}b(q)f_0 + w(0,q)f_1(q) = 0. \end{cases}$$

From (3.2) we find that f_0 and f_1 are given by (2.5) and from the first equation of (3.2) we derive the equality $\Delta(0,0) = 0$.
   Since $w(0,\cdot)$ and b are even analytic functions on $\mathbb{T}^3$ and the function $w(0,\cdot)$ has a unique non-degenerate minimum at the origin we can conclude that $f_1 \in L_2(\mathbb{T}^3)$ if and only if $b(0) = 0$.
   "If Part". Let $b(0) = 0$ and $\Delta(0,0) = 0$. Then the vector $f = (f_0, f_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1$, where f_0 and f_1 are defined by (2.5), obeys the equation $h(0)f = 0$ and $f_1 \in L_2(\mathbb{T}^3)$. $\qquad \square$

Lemma 3.4. *If Assumption 2.7 is fulfilled and the operator $h(0)$ has either a zero energy resonance or a zero eigenvalue, then the operator $h(p)$, $p \in \mathbb{T}^3$ has no negative eigenvalues.*

Proof. Let the function $\Lambda(\cdot, z)$, $z \leq 0$ be defined in $\mathbb{T}^3$ as

$$\Lambda(p,z) = \int_{\mathbb{T}^3} \frac{b^2(t)\,dt}{w(p,t)-z}.$$

First we prove that the inequality $\Lambda(p,0) < \Lambda(0,0)$ holds for any nonzero $p \in \mathbb{T}^3$.
   Since w and b are even the function $\Lambda(\cdot)$ is also even. Then we get

$$(3.3) \qquad \begin{aligned} &\Lambda(p,0) - \Lambda(0,0) \\ &= \frac{1}{4} \int_{\mathbb{T}^3} \frac{2w(0,t)-(w(p,t)+w(-p,t))}{w(p,t)w(-p,t)w(0,t)} [w(p,t)+w(-p,t)]b^2(t)\,dt \\ &\quad - \frac{1}{4} \int_{\mathbb{T}^3} \frac{[w(p,t)-w(-p,t)]^2}{w(p,t)w(-p,t)w(0,t)} b^2(t)\,dt. \end{aligned}$$

From the equality

$$w(0,t) - \frac{w(p,t) + w(p,-t)}{2} = \sum_{i=1}^{3} (\cos p_i - 1)(1 + \cos t_i)$$

and (3.3) we get the inequality $\Lambda(p,0) - \Lambda(0,0) < 0$ for all nonzero $p \in \mathbb{T}^3$, that is, the function $\Lambda(\cdot)$ has a unique maximum at $p = 0$.

Since the function $\Delta(p,\cdot)$ is decreasing on $(-\infty, 0)$ and the function $u(\cdot)$ (resp. $\Lambda(\cdot, 0)$) has a unique minimum (resp. maximum) at $p = 0$ we have

$$(3.4) \qquad \Delta(p,z) = u(p) - z - \frac{1}{2}\Lambda(p,z) > u(0) - \frac{1}{2}\Lambda(0,0)$$

for all $z < 0$ and $p \in \mathbb{T}^3$.

If the operator $h(0)$ has either a zero energy resonance or a zero eigenvalue, then by Lemmas 3.2 and 3.3 we have $\Delta(0,0) = 0$. Hence by inequality (3.4) we have $\Delta(p,z) > 0$ for all $p \in \mathbb{T}^3$ and $z < 0$. By Lemma 3.1 the operator $h(p)$, $p \in \mathbb{T}^3$ has no negative eigenvalues. $\qquad \square$

Set

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}, \quad \mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}, \quad \mathbb{R}_+^0 = \mathbb{R}_+ \cup \{0\}.$$

Let $V(0)$ be the ball of radius $\gamma > 0$ with the center at $\zeta = 0 \in \mathbb{C}$.

Let $w_0(\cdot, \cdot)$ be the function defined on $U_\delta(0) \times \mathbb{T}^3$, $\delta > 0$ – sufficiently small, as

$$(3.5) \qquad w_0(p,q) = w_p(q + q_0(p)) - m(p),$$

where $q_0(\cdot)$ is analytic function on $U_\delta(0)$ and for any $p \in U_\delta(0)$ the point $q_0(p)$ is the non-degenerate minimum of the function $w_p(\cdot)$ (see Lemma A.1).

For any $p \in \mathbb{T}^3$ we define an analytic function $D(\cdot, \zeta)$ in $\mathbb{C}_+$ by

$$D(p,\zeta) = u(p) - m(p) + \zeta^2 - \frac{1}{2}\int_{\mathbb{T}^3} \frac{b^2(q + q_0(p))\,dq}{w_0(p,q) + \zeta^2}.$$

Lemma 3.5. *Then there exist a number $\delta > 0$ such that*

i) For any $\zeta \in \mathbb{C}_+$ the function $D(\cdot, \zeta)$ is analytic in $U_\delta(0)$ and the following decomposition holds

$$(3.6) \qquad D(p,\zeta) = D(0,\zeta) + D^{\mathrm{res}}(p,\zeta),$$

where $D^{\mathrm{res}}(p,\zeta) = O(p^2)$ as $p \to 0$ uniformly in $\zeta \in \mathbb{R}_+^0$.

ii) The derivative of $D(0,\cdot)$ at $\zeta = 0$ exists and the decomposition

$$(3.7) \qquad D(0,\zeta) = D(0,0) + \frac{1}{2}\pi^2 b^2(0)\zeta + D^{\mathrm{res}}(\zeta)$$

holds, where $D^{\mathrm{res}}(\zeta) = O(\zeta^2)$, $\zeta \in \mathbb{C}_+ \cup V(0)$.

Proof. i) Since $m(\cdot)$ is analytic in $U_\delta(0)$ (i.e. is analytic in a complex neighborhood of $U_\delta(0)$) by definition of the function D and Assumption 2.7 we obtain that the function $D(\cdot, \zeta)$ is also analytic in $U_\delta(0)$ for any $\zeta \in \mathbb{C}_+$.

Using

$$w_0(p,q) = q^2 + O(|p|^2|q|^2) + O(|q|^4) \quad \text{as} \quad |p|, |q| \to 0$$

we obtain that there exists $C > 0$ such that for any $\zeta \in \mathbb{R}_+^0$ and $i, j = 1, 2, 3$ the inequalities

$$(3.8) \qquad \left| \frac{\partial^2}{\partial p_i \partial p_j} \frac{b^2(q + q_0(p))}{w_0(p,q) + \zeta^2} \right| \le \frac{C}{q^2}, \quad p, q \in U_\delta(0)$$

and

$$(3.9) \qquad \left| \frac{\partial^2}{\partial p_i \partial p_j} \frac{b^2(q + q_0(p))}{w_0(p,q) + \zeta^2} \right| \le C, \quad p \in U_\delta(0), \quad q \in \mathbb{T}^3 \setminus U_\delta(0)$$

hold.

The dominated convergence theorem implies that

$$\frac{\partial^2}{\partial p_i \partial p_j} D(p,0) = \lim_{\zeta \to 0+} \frac{\partial^2}{\partial p_i \partial p_j} D(p,\zeta).$$

Repeating the application of the Hadamard lemma (see [32] V.1, p. 512) we obtain

$$D(p,\zeta) = D(0,\zeta) + \sum_{i=1}^3 \frac{\partial}{\partial p_i} D(0,\zeta) p_i + \sum_{i,j=1}^3 H_{ij}(p,\zeta) p_i p_j,$$

where for any $\zeta \in \mathbb{R}_+^0$ the functions $H_{ij}(\cdot,\zeta)$, $i,j = 1,2,3$ are continuous in $U_\delta(0)$ and

$$H_{ij}(p,\zeta) = \frac{1}{2} \int_0^1 \int_0^1 \frac{\partial^2}{\partial p_i \partial p_j} D(x_1 x_2 p, \zeta) \, dx_1 dx_2.$$

The estimates (3.8) and (3.9) give

$$|H_{i,j}(p,\zeta)| \le \frac{1}{2} \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial p_i \partial p_j} D(x_1 x_2 p, \zeta) \right| dx_1 dx_2 \le C \left(1 + \int_{U_\delta(0)} \frac{dq}{q^2} \right)$$

for any $p \in U_\delta(0)$ uniformly in $\zeta \in \mathbb{R}_+^0$.

Since for any $\zeta \in \mathbb{R}_+^0$ the function $D(\cdot,\zeta)$ is even in $U_\delta(0)$ we have

$$\left[\frac{\partial}{\partial p_i} D(p,\zeta) \right]_{p=0} = 0, \quad i = 1,2,3.$$

$ii)$ The function $D(0,\cdot)$ can be analytically continued (see [16]) to $\mathbb{C}_+ \cup V(0)$. Denote by $D^*(0,\cdot)$ this analytic continuation. Then the representation

$$D^*(0,\zeta) = D(0,0) + \frac{\partial}{\partial \zeta} D(0,0) \zeta + D^{*,\mathrm{res}}(\zeta)$$

holds, where $D^{*,\mathrm{res}}(\zeta) = O(\zeta^2)$ as $\zeta \to 0$.

It is easy to compute that (see Lemma A.2)

$$\frac{\partial}{\partial \zeta} D(0,0) = \frac{1}{2} \pi^2 b^2(0).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following decomposition plays a crucial role in the proof of the main result.

Lemma 3.6. *Let Assumption 2.7 be fulfilled and the operator $h(0)$ have a zero energy resonance. Then there exist a number $\delta > 0$ such that for all $p \in U_\delta(0)$ the asymptotics*

$$(3.10) \qquad \Delta(p,0) = \frac{\sqrt{3}}{4} \pi^2 b^2(0) |p| + O(|p|^2) \quad as \quad p \to 0$$

holds.

Proof. By Lemma 3.5 the asymptotics $m(p) = \frac{3}{4} p^2 + O(|p|^4)$ as $p \to 0$ (see (A.1)) and the equality $\Delta(p,0) = D(p, \sqrt{m(p)})$ yields (3.10).

Corollary 3.7. *Let Assumption 2.7 be fulfilled and the operator $h(0)$ have a zero energy resonance. Then there exist numbers $C_1, C_2 > 0$ and $\delta > 0$ such that the following inequalities hold:*
(i) $C_1|p| \le \Delta(p,0) \le C_2|p|$ for all $p \in U_\delta(0)$,
(ii) $0 < C_1 \le \Delta(p,0) \le C_2$ for all $p \in \mathbb{T}^3 \setminus U_\delta(0)$.

Proof. From the representation (3.10) we have (i) for some positive numbers C_1 and C_2.

Assertion (ii) follows from the positivity (see proof of Lemma 3.4) and continuity of the function $\Delta(\cdot, 0)$ on the compact set $\mathbb{T}^3 \setminus U_\delta(0)$.

4. The spectrum of the channel operator $\hat{H}$

In this section we describe the spectrum of the channel operator $\hat{H}$ defined below. Using the decomposition into direct operator integrals (see [26]) we reduce to study the spectral properties of the operator $\hat{H}$ to the investigation of the spectral properties of the family of operators $h(p), p \in \mathbb{T}^3$ defined by (2.3).

Let us consider the channel operator $\hat{H}$ acting in $\hat{\mathcal{H}} = L_2(\mathbb{T}^3) \oplus L_2((\mathbb{T}^3)^2)$ as

$$(4.1) \qquad \hat{H} \begin{pmatrix} f_1(p) \\ f_2(p, q) \end{pmatrix} = \begin{pmatrix} u(p) f_1(p) + \frac{1}{\sqrt{2}} \int_{\mathbb{T}^3} b(q') f_2(p, q') \, dq' \\ \frac{1}{\sqrt{2}} b(q) f_1(p) + w(p, q) f_2(p, q) \end{pmatrix}.$$

It is easy to see that the operator $\hat{H}$ is bounded and self-adjoint in $\hat{\mathcal{H}}$.

The operator $\hat{H}$ commutes with any multiplication operator U_γ by the function $\gamma(\cdot)$ acting in $\hat{\mathcal{H}}$ as

$$U_\gamma \begin{pmatrix} f_1(p) \\ f_2(p, q) \end{pmatrix} = \begin{pmatrix} \gamma(p) f_1(p) \\ \gamma(p) f_2(p, q) \end{pmatrix}, \quad \gamma \in L_2(\mathbb{T}^3).$$

Therefore the decomposition of the space $\hat{\mathcal{H}}$ into the direct integral

$$\hat{\mathcal{H}} = \int_{\mathbb{T}^3} \oplus (\mathcal{H}_0 \oplus \mathcal{H}_1) \, dp$$

yields the decomposition into the direct integral

$$\hat{H} = \int_{\mathbb{T}^3} \oplus h(p) \, dp,$$

where the fiber operators $h(p), p \in \mathbb{T}^3$ are defined by (2.3).

For the spectrum $\sigma(h(p))$ of the operator $h(p)$, $p \in \mathbb{T}^3$ the following equality

$$(4.2) \qquad \sigma(h(p)) = \sigma_d(h(p)) \cup [m(p), M(p)]$$

holds.

The theorem on the spectrum of decomposable operators and the information on the structure (4.2) of the spectrum of $h(p)$ yield the following:

Lemma 4.1. *For the spectrum $\sigma(\hat{H})$ of the operator $\hat{H}$ the equality*

$$\sigma(\hat{H}) = \bigcup_{p \in \mathbb{T}^3} \{ \sigma_d(h(p)) \cup [m(p), M(p)] \}$$

holds.

Theorem 4.2. *The essential spectrum $\sigma_{\text{ess}}(H)$ of the operator H coincides with the spectrum of $\hat{H}$, that is,*

$$(4.3) \qquad \sigma_{\text{ess}}(H) = \sigma(\hat{H}).$$

Proof. In [19] it has been proved that the essential spectrum $\sigma_{\text{ess}}(H)$ of the operator H coincides with $\sigma \cup [0, M]$. By Lemma 3.1 we have

$$\sigma = \bigcup_{p \in \mathbb{T}^3} \sigma_d(h(p))$$

and hence by Lemma 4.1 we obtain (4.3).

5. THE FADDEEV-NEWTON TYPE SYSTEM OF INTEGRAL EQUATIONS

In this section we derive an analogue of the Faddeev-Newton type system of integral equations for the eigenvectors, corresponding to the eigenvalues lying below the bottom $\tau_{\mathrm{ess}}(H)$ of the essential spectrum of the operator H.

Let $T(z)$, $z \leq \tau_{\mathrm{ess}}(H)$ be the self-adjoint operator which acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as follows

$$T(z) = \left(\begin{array}{cc} T_{00}(z) & T_{01}(z) \\ T_{10}(z) & T_{11}(z) \end{array} \right),$$

let its entries $T_{ij}(z) : \mathcal{H}_j \to \mathcal{H}_i, i, j = 0, 1$ be defined by the rule

$$(T_{00}(z)f_0)_0 = (u_0 - z + 1)f_0, \quad (T_{01}(z)f_1)_0 = -\int_{\mathbb{T}^3} \frac{b(q')f(q')\,dq'}{\sqrt{\Delta(q', z)}},$$

$$(T_{10}(z)f_0)_1(p) = -\frac{b(p)f_0}{\sqrt{\Delta(p, z)}},$$

$$(T_{11}(z)f_1)_1(p) = \frac{b(p)}{2\sqrt{\Delta(p, z)}} \int_{\mathbb{T}^3} \frac{b(q')f(q')\,dq'}{\sqrt{\Delta(q', z)}(w(p, q') - z)}.$$

The following lemma establishes a connection between of eigenvalues of H and $T(z)$.

Lemma 5.1. *The number $z < \tau_{\mathrm{ess}}(H)$ is an eigenvalue of the operator H if and only if the number 1 is an eigenvalue of the operator $T(z)$.*

Proof. Let $z < \tau_{\mathrm{ess}}(H)$ be an eigenvalue of the operator H and $f \in \mathcal{H}$ be the corresponding eigenvector, that is, the system of equations

(5.1)
$$\begin{cases} (u_0 - z)f_0 + \int_{\mathbb{T}^3} b(q')f_1(q')\,dq' = 0 \\ b(p)f_0 + (u(p) - z)f_1(p) + \int_{\mathbb{T}^3} b(q')f_2(p, q')\,dq' = 0 \\ \frac{1}{2}(b(p)f_1(q) + b(q)f_1(p)) + (w(p, q) - z)f_2(p, q) = 0 \end{cases}$$

has a nontrivial solution $f = (f_0, f_1, f_2) \in \mathcal{H}$. Since $z \notin [0, M]$, from the third equation of the system (5.1) for f_2 we have

(5.2)
$$f_2(p, q) = -\frac{b(p)f_1(q) + b(q)f_1(p)}{2(w(p, q) - z)}.$$

Substituting the expression (5.2) for f_2 into the second equation of the system (5.1) we obtain that the system of equations

(5.3)
$$\begin{cases} (u_0 - z)f_0 + \int_{\mathbb{T}^3} b(q')f_1(q')\,dq' = 0 \\ \Delta(p, z)f_1(p) - \frac{b(p)}{2} \int_{\mathbb{T}^3} \frac{b(q')f_1(q')\,dq'}{w(p, q') - z} = -b(p)f_0 \end{cases}$$

has a nontrivial solution and this system of equations has a nontrivial solution if and only if the system of equations (5.1) has a nontrivial solution.

By the definition of σ the inequality $\Delta(p, z) > 0$ holds for all $p \in \mathbb{T}^3$ and $z < \tau_{\mathrm{ess}}(H)$.

Therefore, the following system of equations

(5.4)
$$\begin{cases} (u_0 - z)f_0 + \int_{\mathbb{T}^3} \frac{b(q')f_1(q')}{\sqrt{\Delta(q', z)}}\,dq' = 0 \\ f_1(p) - \frac{b(p)}{2\sqrt{\Delta(p, z)}} \int_{\mathbb{T}^3} \frac{b(q')f_1(q')\,dq'}{\sqrt{\Delta(q', z)}(w(p, q') - z)} = -\frac{b(p)f_0}{\sqrt{\Delta(p, z)}} \end{cases}$$

has a nontrivial solution iff the system of equations (5.3) has a nontrivial solution.

Remark 5.2. We point out that the equation $T(z)g = g$ is an analogue of the symmetric version of the Faddeev-Newton type system of integral equations for eigenvectors of the operator H.

Henceforth, we shall denote by C, C_1, C_2, C_3 different positive numbers and set $\mathbb{T}_\delta = \mathbb{T}^3 \setminus U_\delta(0)$.

Lemma 5.3. *There exist numbers $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that the following inequalities hold*
(i) $C_1(|p|^2 + |q|^2) \le w(p, q) \le C_2(|p|^2 + |q|^2)$ for all $p, q \in U_\delta(0)$;
(ii) $w(p, q) \ge C_3 > 0$ for all $(p, q) \in (\mathbb{T}_\delta \times \mathbb{T}^3) \cup (\mathbb{T}^3 \times \mathbb{T}_\delta)$.

Proof. The function ε has a unique non-degenerate minimum at $0 \in \mathbb{T}^3$ and hence
$$w(p, q) = |p|^2 + (p, q) + |q|^2 + O(|p|^4 + |q|^4) \quad \text{as} \quad p, q \to 0.$$
Then there exist positive numbers C_1, C_2 and a $\delta-$neighborhood of $p = 0 \in \mathbb{T}^3$ so that (i) and (ii) hold true.

Lemma 5.4. *Let Assumption 2.7 be fulfilled and the operator $h(0)$ have a zero energy resonance. The operator $T(z), z \le 0$ is bounded in the Hilbert space $L_2(\mathbb{T}^3)$ and the operator $T(z)$ converges to the operator $T(0)$ in the strongly operator topology as $z \to 0$.*

Proof. To check the boundedness of the operator $T(z)$ we show that the operators $T_{ij}(z)$, $i, j = 0, 1$ are bounded. We represent $T_{11}(z)$ as a sum of two operators $T_{11}^{(1)}(z)$ and $T_{11}^{(2)}(z)$ acting in $L_2(\mathbb{T}^3)$, respectively, as

$$T_{11}^{(1)}(z)f_1(p) = \frac{b(p)}{2\sqrt{\Delta(p, z)}} \int_{\mathbb{T}_\delta} \frac{b(q')f_1(q')\, dq'}{\sqrt{\Delta(q', z)}(w(p, q') - z)}, \quad f_1 \in L_2(\mathbb{T}^3)$$

$$T_{11}^{(2)}(z)f_1(p) = \frac{b(p)}{2\sqrt{\Delta(p, z)}} \int_{U_\delta(0)} \frac{b(q')f_1(q')\, dq'}{\sqrt{\Delta(q', z)}(w(p, q') - z)}, \quad f_1 \in L_2(\mathbb{T}^3).$$

We will prove the boundedness of the operators $T_{11}^{(1)}(z)$ and $T_{11}^{(2)}(z)$. First we estimate the norm of $g(p) = T_{11}^{(2)}(z)f_1(p)$. We have

$$
\begin{aligned}
(5.5) \quad \int_{\mathbb{T}^3} |g(p)|^2 dp &\le \frac{B}{4} \int_{\mathbb{T}_\delta} \frac{1}{\Delta(p, z)} \left| \int_{U_\delta(0)} \frac{f_1(q')dq'}{\sqrt{\Delta(q', z)}(w(p, q') - z)} \right|^2 dp \\
&\quad + \frac{B}{4} \int_{U_\delta(0)} \frac{1}{\Delta(p, z)} \left| \int_{U_\delta(0)} \frac{f_1(q')dq'}{\sqrt{\Delta(q', z)}(w(p, q') - z)} \right|^2 dp,
\end{aligned}
$$

where $B = [\max_{t \in \mathbb{T}^3} |b(t)|]^4$.

Now we estimate each summand on the r.h.s. of (5.5).

Applying the Schwartz inequality for the first summand on the r.h.s. of (5.5) and using statements of Corollary 3.7 and Lemma 5.3 we get that the first summand on the r.h.s. of (5.5) does not exceed

$$\int_{U_\delta(0)} \frac{dq'}{|q'|} \int_{U_\delta(0)} |f_1(q')|^2 dq' < C\|f_1\|^2.$$

Applying the Schwartz inequality and using statements (i) of Corollary 3.7 and Lemma 5.3 we get that the second summand on the r.h.s. of (5.5) does not exceed

$$C \int_{U_\delta(0)} \frac{dp}{|p|} \int_{U_\delta(0)} \frac{dq'}{|q'|^2(|q'|^2 + |p|^2)} \int_{U_\delta(0)} \frac{|q'||f_1(q')|^2 dq'}{|q'|^2 + |p|^2}.$$

Passing on to a spherical coordinate system in the second integral and changing the order of integrations we get that the second summand on the r.h.s. of (5.5) does not exceed $C'\|f_1\|^2$ for some $C' > 0$.

Thus,
$$\|T_{11}^{(2)}(z)f_1\| \leq C\|f_1\|.$$
In analogy with the estimate for $\|T_{11}^{(2)}(z)f_1\|$ one proves that
$$\|T_{11}^{(1)}(z)f_1\| \leq C\|f_1\|.$$
Hence, the operator $T_{11}(z)$ is bounded as a sum of bounded operators .

It is easy to see that the operators $T_{00}(z)$, $T_{01}(z)$ and $T_{10}(z)$ are bounded .

Now we show the convergence $T(z)$ to $T(0)$ in the strongly operator topology as $z \to 0$.

Denote by $Q(p, q; z)$ the kernel of the integral operator $T_{11}(z) - T_{11}(0)$, that is,
$$Q(p, q; z) = \frac{b(p)b(q)}{2\sqrt{\Delta(p, z)}\sqrt{\Delta(q, z)}(w(p, q) - z)} - \frac{b(p)b(q)}{2\sqrt{\Delta(p, 0)}\sqrt{\Delta(q, 0)}w(p, q)}.$$
Then for any $\varphi \in L_2(\mathbb{T}^3)$ we have

$$
\begin{aligned}
(5.6) \quad &\|(T_{11}(z) - T_{11}(0))\varphi\|^2 \\
&= \int_{\mathbb{T}_\delta}\Big|\int_{\mathbb{T}_\delta} Q(p, q; z)\varphi(q)\,dq\Big|^2 dp + \int_{\mathbb{T}_\delta}\Big|\int_{U_\delta(0)} Q(p, q; z)\varphi(q)\,dq\Big|^2 dp \\
&+ \int_{U_\delta(0)}\Big|\int_{\mathbb{T}_\delta} Q(p, q; z)\varphi(q)\,dq\Big|^2 dp + \int_{U_\delta(0)}\Big|\int_{U_\delta(0)} Q(p, q; z)\varphi(q)\,dq\Big|^2 dq.
\end{aligned}
$$

Since the kernel of $T_{11}(z)$ is continuous on $\mathbb{T}_\delta^2$ and $Q(p, q; z) \to 0$ as $z \to 0$ for any $p, q \in \mathbb{T}_\delta$, the first summand on the r.h.s. of (5.6) tends to zero as $z \to 0$.

Let us consider the second summand on the r.h.s. of (5.6). According to statements (ii) of Corollary 3.7 and Lemma 5.3 we have
$$\int_{\mathbb{T}_\delta}\Big|\int_{U_\delta(0)} Q(p, q; z)\varphi(q)\,dq\Big|^2 dp < C\Big|\int_{U_\delta(0)} \frac{\varphi(q)\,dq}{\sqrt{\Delta(q, 0)}}\Big|^2.$$
Applying the Schwartz inequality and using statement (i) of Corollary 3.7 we get
$$\int_{\mathbb{T}_\delta}\Big|\int_{U_\delta(0)} Q(p, q; z)\varphi(q)\,dq\Big|^2 dp < C\int_{U_\delta(0)} |\varphi(q)|^2 dq.$$
By virtue of the absolutely continuity of the Lebesgue integral the latter integral tends to zero as $\delta \to 0$.

Similarly one proves that the third summand on the r.h.s. of (5.6) also tends to zero as $\delta \to 0$.

Consider the fourth summand on the r.h.s. of (5.6). We have

$$
\begin{aligned}
(5.7) \quad \int_{U_\delta(0)}\Big|\int_{U_\delta(0)} Q(p, q; z)\varphi(q)\,dq\Big|^2 dp &\leq C\int_{U_\delta(0)} \frac{dp}{\Delta(p, 0)}\Big|\int_{U_\delta(0)} \frac{\varphi(q)\,dq}{\sqrt{\Delta(q, 0)}w(p, q)}\Big|^2 \\
&+ C\int_{U_\delta(0)} \frac{dp}{\Delta(p, 0)}\Big|\int_{U_\delta(0)} \frac{\varphi(q)\,dq}{\sqrt{\Delta(q, 0)}}\Big|^2.
\end{aligned}
$$

Applying the Schwarz inequality to the interior integral of the first summand on the r.h.s. of (5.7) and using the statements (i) of Corollary 3.7 and Lemma 5.3 we obtain that the first summand on the r.h.s. of (5.6) does not exceed

$$(5.8) \quad C\int_{U_\delta(0)} \frac{dp}{|p|}\int_{U_\delta(0)} \frac{|q||\varphi^2(q)\,dq}{|q|^2 + |p|^2}\int_{U_\delta(0)} \frac{|q|^{-2}dq}{|q|^2 + |p|^2}.$$

Passing to a spherical coordinate system in the latter integral and then changing the order of integration we have that the integral (5.8) does not exceed
$$C\int_{U_\delta(0)} |\varphi(q)|^2 dq.$$

Analogously the second summand on the r.h.s. of (5.6) is bounded by

$$C \int_{U_\delta(0)} |\varphi(q)|^2 dq.$$

Hence,

$$\int_{U_\delta(0)} \Big| \int_{U_\delta(0)} Q(p,q;z)\varphi(q)\, dq \Big|^2 dp \le C \int_{U_\delta(0)} |\varphi(q)|^2 dq.$$

By the absolutely continuity of the Lebesgue integral it follows that

$$\int_{U_\delta(0)} |\varphi(q)|^2 dq \to 0 \quad \text{as} \quad \delta \to 0.$$

Thus, $\|(T_{11}(z) - T_{11}(0))\varphi\| \to 0$ as $z \to 0$.

In the same manner we can see that other entries of the operator $T(z) - T(0)$ converges to zero in the strongly operator topology as $z \to 0$.

6. The proof of the main result

In this section we shall prove Theorem 2.8. Throughout the proof of Theorem 2.8 we shall use some facts of [17].

It was proved in Lemma 5.4 that the operator $T(0)$ acting in $\mathcal{H}_0 \oplus \mathcal{H}_1$ is bounded and self-adjoint. Let us show that the essential spectrum of the "limiting" operator $T(0)$ contains a closed interval lying on the r.h.s. of the point 1.

Denote by $\chi_\delta(p)$ the characteristic function of the set $U_\delta(0)$.

Let $T_\delta(0)$ be the operator acting in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$T_\delta(0) = \begin{pmatrix} 0 & 0 \\ 0 & T_\delta^{(0)}(0) \end{pmatrix},$$

where

$$(T_\delta^{(0)}(0)f)(p) = \frac{b(p)\chi_\delta(p)}{2\sqrt{\Delta(p,0)}} \int_{\mathbb{T}^3} \frac{b(q)\chi_\delta(q)f(q)\, dq}{\sqrt{\Delta(q,0)}w(p,q)}.$$

Lemma 6.1. *Let Assumption 2.7 be fulfilled and the operator $h(0)$ have a zero energy resonance. Then the operator $T(0) - T_\delta(0)$ is compact.*

Proof. Denote by $Q(p,q)$ the kernel of the operator $T_{11}(0) - T_\delta^{(0)}(0)$. Then

$$Q(p,q) = 0, \quad \text{for all} \quad p,q \in U_\delta(0)$$

and

$$Q(p,q) = \frac{b(p)b(q)}{2\sqrt{\Delta(p,0)}\sqrt{\Delta(q,0)}w(p,q)} \quad \text{for all} \quad (p,q) \in (\mathbb{T}_\delta \times \mathbb{T}^3) \cup (\mathbb{T}^3 \times \mathbb{T}_\delta).$$

It is sufficient to show that the kernel $Q(p,q)$ is square-integrable. In fact

$$(6.1) \quad \int_{\mathbb{T}^3} \int_{\mathbb{T}^3} |Q(p,q)|^2 dp\, dq = \int_{\mathbb{T}_\delta} \int_{\mathbb{T}_\delta} |Q(p,q)|^2 dp\, dq + 2 \int_{\mathbb{T}_\delta} \int_{U_\delta(0)} |Q(p,q)|^2 dp\, dq.$$

Since $Q(p,q)$ is continuous on $\mathbb{T}_\delta^2 = \mathbb{T}_\delta \times \mathbb{T}_\delta$, the first summand on the r.h.s. of (6.1) is finite. From Corollary 3.7 and Lemma 5.3 it follows that the second summand on the r.h.s. of (6.1) is bounded by

$$C + C_1 \int_{U_\delta(0)} \frac{dp}{|p|} < \infty.$$

Now the assertion of the Lemma follows from the fact that the operator $T_{11}(0) - T_\delta^{(0)}(0)$ is a Hilbert-Schmidt operator and the operators $T_{00}(0)$, $T_{01}(0)$ and $T_{10}(0)$ are of rank 1.

The space of all functions of $L_2(\mathbb{T}^3)$ with support in $U_\delta(0)$ is an invariant subspace of the operator $T_\delta^{(0)}(0)$. Let the operator $T_\delta^{(1)}(0)$ be the restriction of $T_\delta^{(0)}(0)$ to this subspace.

Let the operator $T_\delta^{(2)}(0) : L_2(U_\delta(0)) \to L_2(U_\delta(0))$ act as

$$(T_\delta^{(2)}(0)f)(p) = \frac{b^2(0)}{2\sqrt{\Delta(p,0)}} \int_{U_\delta(0)} \frac{f(q)\,dq}{\sqrt{\Delta(q,0)}w(p,q)}.$$

Lemma 6.2. *Let Assumption 2.7 be fulfilled and the operator $h(0)$ have a zero energy resonance. Then $T_\delta^{(1)}(0) - T_\delta^{(2)}(0)$ is a Hilbert-Schmidt operator.*

Proof. It suffice to prove that the integral

$$J = \int_{U_\delta(0)} \int_{U_\delta(0)} \Big| \frac{b(p)b(q) - b^2(0)}{\sqrt{\Delta(p,0)}\sqrt{\Delta(q,0)}w(p,q)} \Big|^2 dp\,dq$$

is finite.

We represent the function $b(p)b(q)$ as $b(p)b(q) = b^2(0) + \psi(p,q)$, where the function ψ is real-analytic and $\psi(0,0) = 0$. By virtue of the statements (i) of Corollary 3.7 and Lemma 5.3 we have

$$J \le C \int_{U_\delta(0)} \int_{U_\delta(0)} \frac{|\psi(p,q)|^2 dp\,dq}{|p||q|(p^2+q^2)^2}.$$

Passing on to the spherical coordinate system with respect to the variables p and q and then going over the polar coordinate system we obtain

$$J \le C \int_0^\delta \frac{|\psi(r,r)|^2}{r} dr.$$

Since ψ is real-analytic and $\psi(0,0) = 0$, the latter integral converges.

Set

$$\tilde{w}(p,q) = p^2 + (p,q) + q^2.$$

Let $T_\delta^{(3)}(0)$ be the operator acting in $L_2(U_\delta(0))$ by

$$(T_\delta^{(3)}(0)f)(p) = \frac{2}{\sqrt{3}\pi^2} \int_{U_\delta(0)} \frac{f(q)\,dq}{\sqrt{|p|}\sqrt{|q|}\tilde{w}(p,q)}.$$

Lemma 6.3. *Let Assumption 2.7 be fulfilled and the operator $h(0)$ have a zero energy resonance. Then $T_\delta^{(2)}(0) - T_\delta^{(3)}(0)$ is a Hilbert-Schmidt operator.*

Proof. It suffices to show that the integral

$$J = \int_{U_\delta(0)} \int_{U_\delta(0)} \Big| \frac{b^2(0)}{\sqrt{\Delta(p,0)}\sqrt{\Delta(q,0)}w(p,q)} - \frac{2}{\sqrt{3}\pi^2\sqrt{|p|}\sqrt{|q|}\tilde{w}(p,q)} \Big|^2 dp\,dq$$

is finite.

For any $p,q \in U_\delta(0)$ the function w is represented as

$$w(p,q) = p^2 + (p,q) + q^2 + O(p^4) + O(q^4) \quad \text{as} \quad p,q \to 0.$$

From this and the representation (3.10) for the function $\Delta(\cdot, 0)$ it follows that

$$J = \frac{4}{3\pi^4} \int_{U_\delta(0)} \int_{U_\delta(0)} \frac{O(|p|) + O(|q|)}{|p||q|\tilde{w}^2(p,q)} dq\,dp.$$

Now passing on to the polar coordinate system we obtain

$$J < C \int_0^\delta \frac{O(r)}{r} dr < \infty.$$

In [17] it has been proved that the essential spectrum of $T_\delta^{(3)}(0)$ contains a closed interval lying on the r.h.s. of the point 1.

The well known Weyl theorem [27] on the stability of the essential spectrum under compact perturbations and Lemmas 6.1–6.2 imply that the essential spectrum $T(0)$ contains a closed interval lying to the r.h.s. of the point 1.

Now Theorem 2.8 can be proved applying Lemma 5.1 in the same way as Theorem 2.1 in [17].

Appendix A

Lemma A.1. *(i) For any $p \in (-\pi, \pi)^3$ the point $q_0(p) = -p/2$ is a unique non-degenerate minimum of $w(p, \cdot)$.*
(ii) The function $m(\cdot) = w(\cdot, q_0(\cdot))$ is analytic in $(-\pi, \pi)^3$ and has the asymptotics

$$(A.1) \qquad m(p) = \frac{3}{4}p^2 + O(|p|^4) \quad as \quad p \to 0.$$

Proof. (i) The function w can be represented in the form

$$(A.2) \qquad w(p, q) = \varepsilon(p) + \sum_{i=1}^{3}\left(2 - 2\cos\frac{p_i}{2}\cos\left(q_i + \frac{p_i}{2}\right)\right), \quad p, q \in (-\pi, \pi)^3.$$

It follows that for any $p \in (-\pi, \pi)^3$ the point $q_0(p) = -p/2$ is a unique non-degenerate minimum of $w(p, \cdot)$.

(ii) The analyticity of the functions w on $(\mathbb{T}^3)^2$ implies that the function $m(\cdot) = w(\cdot, q_0(\cdot))$ is also analytic in $(-\pi, \pi)^3$. By the Taylor theorem and the representation (A.2) we get the asymptotics (A.1).

Lemma A.2. *The following equality*

$$(A.3) \qquad \frac{\partial}{\partial\zeta}D^*(0, 0) = \frac{1}{2}\pi^2 b^2(0)$$

holds.

Proof. The function $w_0(0, \cdot)$ has a unique non-degenerate minimum at $q = 0$. Therefore, by virtue of the Morse lemma (see [10]) there exists a one-to-one mapping $q = \varphi(t)$ of a certain ball $W_\gamma(0)$ of radius $\gamma > 0$ with the center at $t = 0$ to a neighborhood $\tilde{W}(0)$ of the point $q = 0$ such that:

$$(A.4) \qquad w_0(0, \varphi(t)) = t^2$$

with $\varphi(0) = 0$ and for the Jacobian $J_\varphi(t)$ of the mapping $q = \varphi(t)$ the equality

$$J_\varphi(0) = \frac{1}{2}$$

holds.

For any $\zeta \in \mathbb{C}_+$ the function $\frac{\partial}{\partial\zeta}D(0, \cdot)$ can be represented in the form

$$(A.5) \qquad \frac{\partial}{\partial\zeta}D(0, \zeta) = D_1(\zeta) + D_2(\zeta), \quad \zeta \in \mathbb{C}_+$$

with

$$(A.6) \qquad D_1(\zeta) = \zeta \int_{\mathbb{T}^3\setminus\tilde{W}(0)} \frac{b^2(q)\,dq}{(w_0(0, q) + \zeta^2)^2}, \quad \zeta \in \mathbb{C}_+$$

and

$$(A.7) \qquad D_2(\zeta) = \zeta \int_{\tilde{W}(0)} \frac{b^2(q)\,dq}{(w_0(0, q) + \zeta^2)^2}, \quad \zeta \in \mathbb{C}_+.$$

Since the function $w_0(0, \cdot)$ has a unique minimum at $q = 0$ and is continuous on the compact set $\mathbb{T}^3 \setminus \tilde{W}(0)$ there exists $M = const > 0$ such that $|w_0(0, q)| > M$ for all $q \in \mathbb{T}^3 \setminus \tilde{W}(0)$.

Then we have

$$(A.8) \qquad \int_{\mathbb{T}^3 \setminus \tilde{W}(0)} \frac{b^2(q)\,dq}{(w_0(0, q) + \zeta^2)^2} \to \int_{\mathbb{T}^3 \setminus \tilde{W}(0)} \frac{b^2(q)\,dq}{(w_0(0, q))^2} \quad \text{as} \quad \zeta \to 0.$$

In the integral in (A.7) making a change of variable $q = \varphi(t)$ and using the equality (A.4) we obtain

$$(A.9) \qquad D_2(\zeta) = \zeta \int_{W_\gamma(0)} \frac{b^2(\varphi(t)) J_\varphi(t)}{(t^2 + \zeta^2)^2}\,dt.$$

Going over in the integral in (A.9) to spherical coordinates $t = r\omega$, we reduce it to the form

$$(A.10) \qquad D_2(\zeta) = \zeta \int_0^\gamma \frac{r^2 F(r)}{(r^2 + \zeta^2)^2}\,dr$$

with

$$F(r) = \int_{\Omega_2} b^2(\varphi(r\omega)) J_\varphi(r\omega)\,d\omega,$$

where Ω_2 is the unit sphere in $\mathbb{R}^3$ and $d\omega$ is the element of the unit sphere in this space.

Computing the integrals

$$\int_0^\gamma \frac{\zeta}{r^2 + \zeta^2}\,dr \quad \text{and} \quad \int_0^\gamma \frac{\zeta r^2 (F(r) - F(0))}{(r^2 + \zeta^2)^2}\,dr$$

we obtain

$$(A.11) \qquad \int_0^\gamma \frac{\zeta}{r^2 + \zeta^2}\,dr \to \frac{\pi}{2} \quad \text{and} \quad \int_0^\gamma \frac{\zeta r^2 (F(r) - F(0))}{(r^2 + \zeta^2)^2}\,dr \to 0 \quad \text{as} \quad \zeta \to 0 +.$$

Hence the equality (A.3) holds.

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