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ON COMPLETENESS OF THE SET OF ROOT VECTORS FOR UNBOUNDED OPERATORS

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Dedicated to 80th birthday anniversary of Professor Yu. L. Daletskii.

Abstract. For a closed linear operator A in a Banach space, the notion of a vector accessible in the resolvent sense at infinity is introduced. It is shown that the set of such vectors coincides with the space of exponential type entire vectors of this operator and the linear span of root vectors if, in addition, the resolvent of A is meromorphic. In the latter case, the completeness criteria for the set of root vectors are given in terms of behavior of the resolvent at infinity.

In what follows, we suppose A to be a closed linear operator densely defined in a Banach space $\mathfrak{B}$ with norm $\|\cdot\|$ over the field $\mathbb{C}$ of complex numbers.

1. We say that a vector $x \in \mathfrak{B}$ is accessible in the resolvent sense for the operator A on a set $M \subseteq \mathbb{C}$ if there exists a $\mathfrak{B}$-valued function $f_x(\lambda)$ analytic in a certain neighborhood $O \supseteq M$, such that for any $\lambda \in O$, $f_x(\lambda) \in \mathcal{D}(A)$ and

$$(A - \lambda I) f_x(\lambda) = x \tag{1}$$

($\mathcal{D}(\cdot)$ is the domain of an operator, I is the identity operator). We denote the set of such vectors by $\mathfrak{R}_M(A)$.

It is obvious that $0 \in \mathfrak{R}_{\mathbb{C}}(A)$; in this case $f_x(\lambda) \equiv 0$.

Let x_0 be the eigenvector of the operator A corresponding to an eigenvalue λ_0. Then $x_0 \in \mathfrak{R}_{\mathbb{C}\setminus\{\lambda_0\}}(A)$. In this example,

$$f_{x_0}(\lambda) = \frac{x_0}{\lambda_0 - \lambda}, \quad \lambda \in \mathbb{C} \setminus \{\lambda_0\}.$$

Equality (1) shows that if $x \in \mathfrak{R}_M(A)$ and for any $\lambda \in O$ there exists the inverse $(A - \lambda I)^{-1}$ of the operator $A - \lambda I$, then $f_x(\lambda)$ is uniquely determined as $f_x(\lambda) = (A - \lambda I)^{-1} x$. In particular, if $M \subseteq \rho(A)$ ($\rho(A)$ is the resolvent set of A), then $\mathfrak{R}_M(A) = \mathfrak{B}$, and for each $x \in \mathfrak{B}$, $f_x(\lambda) = R_A(\lambda)x$, where $R_A(\lambda) = (A - \lambda I)^{-1}$ is the resolvent of A.

Let $x \in \mathfrak{R}_M(A)$. Then the the vector-valued function $f_x(\lambda)$ possesses the following properties.

(i) If $x \neq 0$ and M is bounded, then

$$\exists c > 0 \quad \forall \lambda \in M \quad \|f_x(\lambda)\| > c.$$

(Here and below $c > 0$ denotes a constant, own in every concrete situation).

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This assertion is evident if M is finite. Suppose M to be infinite and such that there exists a sequence $\{\lambda_n \in M\}_{n=1}^{\infty}$ for which

$$f_x(\lambda_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Then, by (1),

$$Af_x(\lambda_n) \to x \quad \text{as} \quad n \to \infty.$$

The closedness of A implies the equality $x = 0$, contrary to the assumption.

(ii) If $x \in \mathcal{D}(A^{n-1}),\ n \in \mathbb{N}$, then $f_x(\lambda) \in \mathcal{D}(A^n)$, and

$$f_x(\lambda) = -\sum_{k=0}^{n-1} \frac{A^k x}{\lambda^{k+1}} + \frac{A^n f_x(\lambda)}{\lambda^n}, \quad \lambda \in M. \tag{2}$$

We prove this property by induction.

By the definition, $f_x(\lambda) \in \mathcal{D}(A)$ ($\lambda \in M$), and, according to (1), we have representation (2) for $n = 1$.

Suppose now assertion (ii) to be true for $n = m$, that is,

$$x \in \mathcal{D}(A^{m-1}) \Longrightarrow f_x(\lambda) \in \mathcal{D}(A^m) \text{ and } f_x(\lambda) = -\sum_{k=0}^{m-1} \frac{A^k x}{\lambda^{k+1}} + \frac{A^m f_x(\lambda)}{\lambda^m}. \tag{3}$$

If $x \in \mathcal{D}(A^m)$, then, by (3), $\dfrac{A^m f_x(\lambda)}{\lambda^m} \in \mathcal{D}(A)$, so $f_x(\lambda) \in \mathcal{D}(A^{m+1})$. Moreover,

$$A^{m+1} f_x(\lambda) = A^m(Af_x(\lambda)) = A^m(x + \lambda f_x(\lambda)) = A^m x + \lambda A^m f_x(\lambda),$$

whence

$$A^m f_x(\lambda) = \frac{A^{m+1} f_x(\lambda)}{\lambda} - \frac{A^m x}{\lambda}.$$

Substituting this expression into the equality in (3), we obtain for $f_x(\lambda)$ representation (2) in the case where $n = m + 1$.

We call a vector $x \in \mathfrak{B}$ accessible in the resolvent sense for the operator A at infinity if there exists a function $f_x(\lambda)$ with values in $\mathcal{D}(A)$, analytic in the domain $D_\alpha = \{\lambda \in \mathbb{C} : |\lambda| > \alpha\}$ with some $\alpha = \alpha(x) > 0,\ f_x(\lambda) \to 0$ as $|\lambda| \to \infty$, and for $\lambda \in D_\alpha$, equality (1) is fulfilled. Denote by $\mathfrak{R}_\infty(A)$ the set of all such vectors.

It is clear that $0 \in \mathfrak{R}_\infty(A)$ for any operator A. It is also not difficult to see that if $\overline{\mathcal{D}(A)} = \mathfrak{B}$, then $\mathfrak{R}_\infty(A) = \mathfrak{B}$. But this is not the case when the operator A is unbounded. To see this, we introduce the following notation.

Denote by $\mathcal{E}(A)$ the set of all exponential type entire vectors of the operator A (see [1]), that is,

$$\mathcal{E}(A) = \bigcup_{\alpha \geq 0} \mathcal{E}^\alpha(A),$$

where

$$\mathcal{E}^\alpha(A) = \{x \in C^\infty(A) \big| \exists c > 0 \quad \forall k \in \mathbb{N}_0 \quad \|A^k x\| \leq c\alpha^k\},$$

$C^\infty(A) = \bigcap\limits_{n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}} \mathcal{D}(A^n)$ is the space of infinitely differentiable vectors of A, $0 < c = c(x, \alpha) = \text{const}$. Obviously, $\mathcal{E}^\alpha(A) \subseteq \mathcal{E}^{\alpha'}(A)$ as $\alpha < \alpha'$. By the type of a vector $x \in \mathcal{E}(A)$ we mean the number

$$\sigma(x, A) = \inf\{\alpha \geq 0 : x \in \mathcal{E}^\alpha(A)\}.$$

Lemma 1. *Suppose that for the operator A, there exists a closed rectifiable contour $\Gamma \in \rho(A)$, and let $f(\lambda)$ be a function analytic in the domain G_Γ bounded by Γ, and continuous on $\overline{G_\Gamma}$. Then for any $x \in \mathfrak{B}$,*

$$y = \int_\Gamma f(\lambda) R_A(\lambda) x \, d\lambda \in \mathcal{E}(A).$$

Proof. By virtue of closedness of A,

$$Ay = \int_\Gamma f(\lambda) A R_A(\lambda) x \, d\lambda = \int_\Gamma f(\lambda)(I + \lambda R_A(\lambda)) x \, d\lambda = \int_\Gamma \lambda f(\lambda) R_A(\lambda) x \, d\lambda.$$

It follows from this that $Ay \in \mathcal{D}(A)$, and

$$A^2 y = \int_\Gamma \lambda f(\lambda) A R_A(\lambda) x \, d\lambda = \int_\Gamma \lambda^2 f(\lambda) R_A(\lambda) x \, d\lambda.$$

Repeating such a procedure n times, we get

$$A^n y = \int_\Gamma \lambda^n f(\lambda) R_A(\lambda) x \, d\lambda.$$

Hence,

$$\|A^n y\| \le c r^n,$$

where $r = \max_{\lambda \in \Gamma} |\lambda|$, which implies $y \in \mathcal{E}(A)$. □

Theorem 1. *A vector $x \in \mathfrak{B}$ is accessible in the resolvent sense for the operator A at infinity if and only if $x \in \mathcal{E}(A)$. In other words,*

$$\mathfrak{R}_\infty(A) = \mathcal{E}(A).$$

For $x \in \mathfrak{R}_\infty(A)$, the $\mathcal{D}(A)$-valued function $f_x(\lambda)$ is uniquely determined by x as

$$f_x(\lambda) = -\sum_{k=0}^{\infty} \frac{A^k x}{\lambda^{k+1}}. \tag{4}$$

Proof. Let $x \in \mathcal{E}(A)$ and $\alpha > \sigma(x, A)$. Then

$$\exists c = c(x,\alpha) > 0 \quad \forall n \in \mathbb{N}_0 \quad \|A^n x\| \le c\alpha^n.$$

For $|\lambda| > \alpha$, we have

$$\sum_{n=0}^{\infty} \frac{\|A^n x\|}{|\lambda|^{n+1}} \le \frac{c}{|\lambda|} \sum_{n=0}^{\infty} \left(\frac{\alpha}{|\lambda|}\right)^n = \frac{c}{|\lambda| - \alpha} < \infty.$$

Thus the $\mathfrak{B}$-valued function $f_x(\lambda)$ appearing in (4) is analytic in D_α. Moreover,

$$f_x(\lambda) \to 0 \quad \text{as} \quad |\lambda| \to \infty.$$

Since for $|\lambda| > \alpha$,

$$-(A - \lambda I) \sum_{k=0}^{n} \frac{A^k x}{\lambda^{k+1}} = x - \frac{A^{n+1} x}{\lambda^{n+1}} \to x \quad \text{as} \quad n \to \infty,$$

and the operator A is closed, we may conclude that

$$f_x(\lambda) \in \mathcal{D}(A) \quad \text{and} \quad (A - \lambda I) f_x(\lambda) = x.$$

So, $x \in \mathfrak{R}_\infty(A)$, and the function $f_x(\lambda) = -\sum_{n=0}^{\infty} \frac{A^n x}{\lambda^{n+1}}$ is analytic in the domain D_α with an arbitrary $\alpha > \sigma(x, A)$.

Conversely, let $x \in \mathfrak{R}_\infty$. Then there exists a $\mathcal{D}(A)$-valued function $f_x(\lambda)$ analytic in the domain D_α with a certain $\alpha > 0$, $f_x(\lambda) \to 0$ as $|\lambda| \to \infty$, and $(A - \lambda I) f_x(\lambda) = x$ ($|\lambda| > \alpha$. Therefore, $f_x(\lambda)$ admits a representation

$$f_x(\lambda) = \sum_{k=0}^{\infty} \frac{c_k}{\lambda^{k+1}}, \tag{5}$$

where

$$c_k = \frac{1}{2\pi i} \int_{|\xi| = r} f_x(\xi) \xi^k \, d\xi, \quad r > \alpha.$$

Since the operator A is closed and the integral

$$\int\limits_{|\xi|=r} Af_x(\xi)\xi^k\,d\xi = \int\limits_{|\xi\|=r} (x+\xi f_x(\xi))\xi^k\,d\xi$$

exists, $c_k \in \mathcal{D}(A)$. Moreover,

$$Af_x(\lambda) = \lambda f_x(\lambda) + x = \sum_{k=0}^{\infty} \frac{c_k}{\lambda^k} + x \to c_0 + x \quad \text{as} \quad |\lambda| \to \infty.$$

Taking into account that $f_x(\lambda) \to 0$ as $|\lambda| \to \infty$ and the closedness of A, we arrive at the conclusion that $\mathcal{D}(A) \ni c_0 = -x$.

Further,

$$\begin{aligned}\mathcal{D}(A) \ni c_1 = \frac{1}{2\pi i} \int\limits_{|\xi|=r} \xi f_x(\xi)\,d\xi &= \frac{1}{2\pi i} \int\limits_{|\xi|=r} (\xi I - A)f_x(\xi)\,d\xi + \frac{1}{2\pi i} \int\limits_{|\xi|=r} Af_x(\xi)\,d\xi \\ &= \frac{1}{2\pi i} A \int\limits_{|\xi|=r} f_x(\xi)\,d\xi = Ac_0 = -Ax,\end{aligned}$$

that is, $x \in \mathcal{D}(A^2)$.

Suppose, by induction, $c_n = -A^n x$. Then

$$\begin{aligned}c_{n+1} = \frac{1}{2\pi i} \int\limits_{|\xi|=r} \xi^{n+1} f_x(\xi)\,d\xi &= \frac{1}{2\pi i} \int\limits_{|\xi|=r} \xi^n (\xi I - A) f_x(\xi)\,d\xi + \frac{1}{2\pi i} A \int\limits_{|\xi|=r} \xi^n f_x(\xi)\,d\xi \\ &= -\frac{1}{2\pi i} \int\limits_{|\xi|=r} \xi^n\,d\xi x + Ac_n = Ac_n = -A^{n+1}x.\end{aligned}$$

Thus, $x \in C^\infty(A)$ and $-A^n x = c_n$ ($n \in \mathbb{N}_0$). It follows from (5) that $f_x(\lambda)$ is represented in the form (4), so, for any $\lambda_0 : |\lambda_0| > \alpha$, we have

$$\|A^n\| \le c|\lambda_0|^n, \quad c = c(x, \lambda_0) > 0,$$

which means that $x \in \mathcal{E}(A)$.

The explicit form of the vector-valued function $f_x(\lambda)$, established in the course of the proof of this theorem, verifies its uniqueness. □

The next statement follows immediately from the proof of Theorem 1.

Corollary 1. *If $x \in \mathfrak{R}_\infty(A)$, then*

$$\exists \alpha > 0 \quad \exists c = c(x) \quad \|f_x(\lambda)\| < \frac{c}{|\lambda|}, \quad \lambda \in D_\alpha.$$

For $x \in \mathfrak{R}_\infty(A)$, denote by $r(x, A)$ the radius of the least circle outside which the function $f_x(\lambda)$ is analytic. It is evident that $\dfrac{1}{r(x,A)}$ coincides with the convergence radius of the power series $\sum\limits_{n=0}^{\infty} \lambda^n A^n x$. So,

$$r(x, A) = \overline{\lim_{n\to\infty}} \sqrt[n]{\|A^n x\|} = \sigma(x, A).$$

Corollary 2. *If $\mathfrak{R}_\infty(A) = \mathfrak{B}$, then the operator A is bounded.*

Corollary 2 implies, in particular, a well-known fact (see [2]) asserting that the boundedness of spectrum $\sigma(A)$ of the operator A and the estimate

$$\|R_A(\lambda)\| \le c\frac{1}{|\lambda|}$$

near the point at infinity have as a consequence the boundedness of A.

Corollary 3. *If $\sigma(A) = \emptyset$, then $\mathfrak{R}_\infty(A) = \mathcal{E}(A) = \{0\}$.*

Really, for a vector $x \in \mathcal{E}(A)$, under the condition that $\sigma(A) = \emptyset$, the vector-valued function $f_x(\lambda) = R_A(\lambda)x$ is entire, and $R_A(\lambda)x \to 0$ as $|\lambda| \to \infty$. By the Liouville theorem, $x = 0$.

2. Consider the Cauchy problem

$$\left\{ \begin{array}{rcl} \dfrac{dy(t)}{dt} & = & Ay(t), \quad t \in [0, \infty), \\ y(0) & = & x, \end{array} \right. \tag{6}$$

where the operator A is as before, $x \in \mathfrak{B}$.

If A is bounded, then for any $x \in \mathfrak{B}$, problem (6) is solvable, and its solution $y(t)$ has the form

$$y(t) = e^{At}x = \sum_{n=0}^{\infty} \frac{t^n A^n x}{n!}.$$

Moreover, the solution $y(t)$ is an exponential type entire $\mathfrak{B}$-valued function (that is, $y(t)$ may be extended to an exponential type entire vector-valued function $y(\lambda)$ taking its values in $\mathfrak{B}$). This is not, generally, the case if the operator A is unbounded: there exist closed operators in $\mathfrak{B}$ for which problem (6) has no nontrivial solutions in the class of exponential type entire vector-valued functions. Here, by the type of an exponential type entire $\mathfrak{B}$-valued function $g(\lambda)$ we mean the number

$$s(g) = \overline{\lim_{|\lambda| \to \infty}} \frac{\ln \|g(\lambda)\|}{|\lambda|}.$$

Denote by $\mathfrak{B}_{exp}(A)$ the set of all $x \in \mathfrak{B}$ such that problem (6) is solvable in the class of exponential type entire $\mathfrak{B}$-valued functions.

Theorem 2. (see [3]). *The following equality is valid:*

$$\mathfrak{B}_{exp}(A) = \mathcal{E}(A).$$

Moreover, if $x \in \mathfrak{B}_{exp}(A)$ and $y(t)$ is the corresponding solution of problem (6), *then*

$$s(y) = \sigma(x, A).$$

Proof. Let $x \in \mathcal{E}(A)$. Then the vector-valued function

$$y(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n A^n x}{n!}$$

is entire, and for any $\varepsilon > 0$,

$$\|y(\lambda)\| \le c \sum_{n=0}^{\infty} \frac{(\sigma(x, A) + \varepsilon)^n |\lambda|^n}{n!} = ce^{(\sigma(x,A)+\varepsilon)|\lambda|},$$

so,

$$s(y) \le \sigma(x, A).$$

It is not hard to verify that $y(t)$ satisfies (6).

Conversely, let $y(t)$ be an entire solution of problem (6) of exponential type $s = s(y)$. The closedness of A implies the equality

$$A^n x = y^{(n)}(0) = \frac{n!}{2\pi i} \int\limits_{|\lambda|=r} \frac{y(\lambda)}{\lambda^{n+1}} \, d\lambda,$$

whence

$$\|A^n x\| \le c \frac{n! e^{(s+\varepsilon)r}}{r^n}$$

for an arbitrary $r > 0$. Taking into account the equality

$$\min_{r>0} \frac{e^{(s+\varepsilon)r}}{r^n} = \frac{e^n (s+\varepsilon)^n}{n^n},$$

and the well-known Stirling formula

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + O\left(\frac{1}{n}\right)\right),$$

we conclude that

$$\|A^n x\| \le c\frac{n! e^n (s+\varepsilon)^n}{n^n} = c(s+\varepsilon)^n\sqrt{2\pi n}\left(1 + O\left(\frac{1}{n}\right)\right) \le c_1(s+\varepsilon_1)^n, \tag{7}$$

where $\varepsilon_1 \to 0$ as $\varepsilon \to 0, 0 < c_1 = \text{const}$. Estimate (7) shows that $x \in \mathcal{E}(A)$ and $\sigma(x, A) \le s = s(y)$. □

3. We say that A is an operator with meromorphic resolvent if the spectrum $\sigma(A)$ of this operator consists of isolated eigenvalues λ_k, which are poles of $R_A(\lambda)$, with the only possible accumulation point at infinity. Suppose A to be such an operator.

For each λ_k, we choose r_k so that

$$\{\lambda : |\lambda - \lambda_k| \le r_k\} \cap \sigma(A) = \{\lambda_k\}.$$

As is known [4,5], the projector

$$P_{\lambda_k}(A) = -\frac{1}{2\pi i}\int\limits_{|\lambda-\lambda_k|=r_k} R_A(\lambda)\, d\lambda$$

(the integral is taken along the contour in the counter-clockwise direction) maps $\mathfrak{B}$ onto the root subspace

$$\mathfrak{L}_k(A) = \{x \in C^\infty(A) \big| (A - \lambda_k I)^{p_k} x = 0\}$$

of the operator A, corresponding to λ_k (p_k is the multiplicity of the pole λ_k), and

$$\dot{+}_{k:|\lambda_k|<r}\mathfrak{L}_k(A) = -\frac{1}{2\pi i}\int\limits_{|\lambda|=r} R_A(\lambda)\, d\lambda\, \mathfrak{B},$$

where r is chosen so that

$$\{\lambda : |\lambda| = r\} \cap \sigma(A) = \emptyset. \tag{8}$$

It follows from here and Lemma 1 that

$$\dot{+}_{k:|\lambda_k|<r}\mathfrak{L}_k(A) \subset \mathcal{E}(A).$$

Conversely, let $x \in \mathcal{E}(A)$. By Theorem 1, $x \in \mathfrak{R}_\infty(A)$. Taking into account the uniqueness principle for analytic functions, we get

$$f_x(\lambda) = R_A(\lambda)x = -\sum_{k=0}^{\infty}\frac{A^k x}{\lambda^{k+1}} \quad \text{for} \quad |\lambda| > \sigma(x, A),$$

where $\sigma(x, A)$ is the type of x. If $r > \sigma(x, A)$ satisfies (8), then

$$x = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int\limits_{|\lambda|=r}\frac{A^k x}{\lambda^{k+1}}\, d\lambda = -\frac{1}{2\pi i}\int\limits_{|\lambda|=r} R_A(\lambda)x\, d\lambda \in \dot{+}_{k:|\lambda_k|<r}\mathfrak{L}_k(A).$$

Thus, we have proved the following theorem.

Theorem 3. *Let A be an operator with meromorphic resolvent in $\mathfrak{B}$ and $\mathfrak{L}(A)$ the linear span of all root vectors of A . Then*

$$\mathfrak{L}(A) = \mathcal{E}(A).$$

Corollary 4. *Suppose that A is an operator with meromorphic resolvent in $\mathfrak{B}$. In order that the set of all root vectors of the operator A be complete in $\mathfrak{B}$, it is necessary and sufficient that $\overline{\mathcal{E}(A)} = \mathfrak{B}$.*

(Recall that the completeness of a set of vectors in $\mathfrak{B}$ means the density in $\mathfrak{B}$ of the linear span of these vectors.)

In the special case where the resolvent $R_A(\lambda)$ is compact for some $\lambda \in \rho(A)$, this assertion was established in [6].

Theorem 4. *Let A be an operator with meromorphic resolvent in $\mathfrak{B}$. The set $\mathfrak{L}(A)$ is dense in $\mathfrak{B}$ if and only if there exist a set $\mathfrak{M}$ complete in $\mathfrak{B}$ and a number $m \geq 0$ such that*

$$\forall x \in \mathfrak{M} \quad \exists \Gamma_n(x) = \{\lambda : |\lambda| = r_n = r_n(x)\} \subset \rho(A) \ (n \in \mathbb{N}, \ \lim_{n\to\infty} r_n = \infty):$$

$$\sup_{n\in\mathbb{N}} \int_0^{2\pi} \ln \frac{\|R_A(r_n e^{i\varphi})x\|}{r_n^m}\, d\varphi < \infty. \tag{9}$$

Proof. Suppose $\overline{\mathfrak{L}(A)} = \mathfrak{B}$ and put $\mathfrak{M} = \mathfrak{L}(A)$. By Theorems 1,3, $\mathfrak{M} = \mathfrak{R}_\infty(A)$. It follows from Corollary 1 that if $x \in \mathfrak{M}$, then

$$\exists \alpha > 0 \quad \forall \lambda \in D_\alpha \quad \|f_x(\lambda)\| = \|R_A(\lambda)x\| \leq \frac{c}{|\lambda|}.$$

Thus, for an arbitrary sequence of circles $\Gamma_n(x) = \{\lambda : |\lambda| = r_n\}, r_n > \alpha, r_n \to \infty$ as $n \to \infty$, we have

$$\|R_A(r_n e^{i\varphi})x\| = \|f_x(r_n e^{i\varphi})\| \leq \frac{c}{r_n},$$

whence

$$\sup_{n\in\mathbb{N}} \int_0^{2\pi} \ln \|R_A(r_n e^{i\varphi})x\|\, d\varphi \leq \sup_{n\in\mathbb{N}} \int_0^{2\pi} \ln \frac{c}{r_n}\, d\varphi = \sup_{n\in\mathbb{N}} 2\pi \ln \frac{c}{r_n} < +\infty.$$

Let us prove now the sufficiency. Without loss a generality, we may assume here that $0 \in \rho(A)$.

Starting from the Hilbert resolvent identity, we obtain, by induction,

$$\forall x \in \mathfrak{M} \quad \forall \lambda \in \rho(A) \quad \forall k \in \mathbb{N} \quad R_A(\lambda)A^{-k}x = -\sum_{n=1}^{k} \frac{A^{-n}x}{\lambda^{k-n+1}} + \frac{R_A(\lambda)x}{\lambda^k}.$$

If we multiply this equality by $P^2\left(\frac{\lambda}{r_n}\right)$, where $P(\lambda)$ is a polynomial with the properties $P(0) = 1, \ P^{(i)}(0) = 0 \ (i = 1, \ldots, k-1)$, and integrate along the circle $\Gamma_n(x)$, we arrive at the formula

$$y_n = \frac{1}{2\pi i} \int_{\Gamma_n(x)} P^2\left(\frac{\lambda}{r_n}\right) R_A(\lambda)A^{-k}x\, d\lambda = -A^{-k}x + \frac{1}{2\pi i} \int_{\Gamma_n(x)} \frac{P^2\left(\frac{\lambda}{r_n}\right) R_A(\lambda)x}{\lambda^k}\, d\lambda,$$

whence

$$\|y_n + A^{-k}x\| \leq \frac{1}{2\pi} \int_{\Gamma_n(x)} \frac{\|P^2\left(\frac{\lambda}{r_n}\right) R_A(\lambda)x\|}{|\lambda|^k}\, d\lambda = \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\varphi})|^2 \frac{\|R_A(r_n e^{i\varphi})x\|}{r_n^{k-1}}\, d\varphi$$

$$= \frac{1}{r_n} \inf_P \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\varphi})|^2 \frac{\|R_A(r_n e^{i\varphi})x\|}{r_n^{k-2}}\, d\varphi.$$

By the Szegö theorem (see [7–9]),

$$\inf_P \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\varphi})|^2 \frac{\|R_A(r_n e^{i\varphi})x\|}{r_n^{k-2}}\, d\varphi < \exp\left(\frac{2k-1}{2\pi} \int_0^{2\pi} \ln \left\| \frac{\|R_A(r_n e^{i\varphi})x\|}{r_n^{k-2}} \right\| d\varphi\right).$$

Setting $k = m+2$ and taking into account (9), we conclude that

$$\|y_n + A^{-(m+2)}x\| \to 0 \quad \text{as} \quad n \to \infty.$$

So, each vector of the set $-A^{-(m+2)}\mathfrak{M}$ complete in $\mathfrak{B}$ may be approximated by the vectors y_n which, by Lemma 1, belong to $\mathcal{E}(A) = \mathfrak{L}(A)$. For this reason $\overline{\mathcal{E}(A)} = \mathfrak{B}$. $\square$

Corollary 5. *Let A be an operator with meromorphic resolvent in $\mathfrak{B}$, and there exist a set $\mathfrak{M}$ complete in $\mathfrak{B}$, such that*

$$\forall x \in \mathfrak{M} \quad \|R_A(\lambda)x\| \leq c_x r_{n,x}^m$$

on a certain (own for every x) sequence of circles

$$\Gamma_{n,x} = \{\lambda : |\lambda| = r_{n,x}\} \subset \rho(A), \quad r_{n,x} \to \infty \quad \text{as } n \to \infty,$$

where $c_x > 0$ and $m \in \mathbb{R}$ are constants, m does not depend on x. Then $\overline{\mathfrak{L}(A)} = \mathfrak{B}$.

It should be noted that for $m = -1$, this assertion is contained in [10].

4. In this subsection, $\mathfrak{B} = \mathfrak{H}$ is a Hilbert space with scalar product $(\cdot,\cdot)$. We put also

$$S_{\lambda_0}^{\gamma} = \left\{\lambda \in \mathbb{C} : |\arg(\lambda - \lambda_0)| \leq \gamma\frac{\pi}{2}\right\}.$$

A linear operator A in $\mathfrak{H}$ is called sectorial [4], if its numeric range

$$\theta(A) = \{(Ax,x),\ x \in \mathcal{D}(A),\ \|x\| = 1\}$$

is a subset of the sector $S_{\lambda_0}^{\gamma}$ with some λ_0 and $\gamma < 1$. The values λ_0 and $\gamma\frac{\pi}{2}$ are known as a vertex and a half-angle of a sectorial operator A; they are not uniquely defined.

A closed sectorial operator A is called m-sectorial, if it has not sectorial extensions. For an m-sectorial operator A, $\sigma(A) \subset S_{\lambda_0}^{\gamma}$, and

$$\|R_A(\lambda)\| \leq \frac{1}{\operatorname{dist}(\lambda, \overline{S_{\lambda_0}^{\gamma}})}, \quad \lambda \in \mathbb{C} \setminus S_{\lambda_0}^{\gamma} \subset \rho(A). \tag{10}$$

Theorem 5. *Let A be an m-sectorial operator in $\mathfrak{H}$ with vertex λ_0 and half-angle $\gamma\frac{\pi}{2}$ ($\gamma < 1$), whose resolvent is meromorphic. Suppose also that there is a set $\mathfrak{M}$ complete in $\mathfrak{H}$, which possesses the following property: for any $x \in \mathfrak{M}$, there exists a sequence of circles*

$$\Gamma_{n,x} = \{\lambda \in \mathbb{C} : |\lambda| = r_{n,x}\} \subset \rho(A), \quad r_{n,x} \to \infty \quad \text{as} \quad n \to \infty,$$

such that

$$\forall \lambda \in \Gamma_{n,x} \quad \|R_A(\lambda)x\| \leq c_x e^{a_x r_{n,x}^{1/\delta}} \tag{11}$$

with some constants $c_x > 0$ and $a_x > 0$ depending on x and δ ($\gamma < \delta < 1$) independent of x. Then $\overline{\mathfrak{L}(A)} = \mathfrak{H}$.

Proof. Since for any $\alpha_1 \neq 0,\ \alpha_2 \in \mathbb{C},\ \mathfrak{L}(A) = \mathfrak{L}(\alpha_1 A + \alpha_2 I)$, we may always take $\lambda_0 = 2$.

For a number $\beta : \gamma < \beta < \delta$, we put

$$\tau_\beta(\lambda) = e^{-\lambda^{1/\beta}} = e^{-|\lambda|^{1/\beta}\left(\cos\frac{\varphi}{\beta} + i\sin\frac{\varphi}{\beta}\right)}, \quad 0 \leq \varphi \leq 2\pi.$$

The function $\tau_\beta(\lambda)$ is analytic inside the sector S_0^{β}, continuous on $\overline{S_0^{\beta}}$, and

$$|\tau_\beta(\lambda)| \leq e^{-|\lambda|^{1/\beta}\cos\frac{\gamma\pi}{2\beta}}, \quad \lambda \in \overline{S_0^{\gamma}}. \tag{12}$$

Taking into account that $1 \in \rho(A)$, and the Hilbert resolvent identity, we deduce

$$\frac{R_A(\lambda)x}{(\lambda-1)^2} = \frac{R_A(1)x}{(\lambda-1)^2} + \frac{R_A^2(1)x}{\lambda-1} + R_A^2(1)R_A(\lambda)x,$$

whence

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_\Gamma \frac{\tau_\beta\left(\frac{\lambda}{n}\right)}{(\lambda-1)^2} R_A(\lambda)x\,d\lambda &= \frac{1}{2\pi i}\int\limits_\Gamma \frac{\tau_\beta\left(\frac{\lambda}{n}\right)}{(\lambda-1)^2} R_A(1)x\,d\lambda \\ &+ \frac{1}{2\pi i}\int\limits_\Gamma \frac{\tau_\beta\left(\frac{\lambda}{n}\right)}{\lambda-1} R_A^2(1)x\,d\lambda + \frac{1}{2\pi i}\int\limits_\Gamma \tau_\beta\left(\frac{\lambda}{n}\right) R_A^2(1)R_A(\lambda)x\,d\lambda,\end{aligned} \tag{13}$$

where $n \in \mathbb{N}$, $\Gamma = \{\lambda \in \mathbb{C} : |\arg\lambda| = \gamma\frac{\pi}{2}$. Since the function $\tau_\beta\left(\frac{\lambda}{n}\right)$ is analytic in S_0^γ and continuous on $\overline{S_0^\gamma}$, we have, by virtue of (12),

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\tau_\beta\left(\frac{\lambda}{n}\right)}{(\lambda-1)^2}R_A(1)x\,d\lambda = \frac{1}{n}\tau_\beta'\left(\frac{1}{n}\right)R_A(1)x \to 0 \quad \text{as} \quad n\to\infty, \tag{14}$$

and

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\tau_\beta\left(\frac{\lambda}{n}\right)}{\lambda-1}R_A^2(1)x\,d\lambda = \tau_\beta\left(\frac{1}{n}\right)R_A^2(1)x \to R_A^2(1)x \quad \text{as} \quad n\to\infty. \tag{15}$$

Concentrate now on the integral on the left-hand side in (13). As

$$\operatorname{dist}(\lambda, \overline{S_2^\gamma}) > d > 0 \quad \text{when} \quad \lambda \in \mathbb{C}\setminus S_0^\gamma,$$

we have, in view of (10), (12),

$$\forall\lambda\in\Gamma \qquad \left\|\frac{\tau_\beta\left(\frac{\lambda}{n}\right)R_A(\lambda)x}{(\lambda-1)^2}\right\| < \frac{c}{|\lambda-1|^2}.$$

By the Lebesgue theorem on passage to the limit under the integral sign,

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\tau_\beta\left(\frac{\lambda}{n}\right)}{(\lambda-1)^2}R_A(\lambda)x\,d\lambda = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{R_A(\lambda)x}{(\lambda-1)^2}\,d\lambda = 0. \tag{16}$$

The latter integral in (16) is equal to 0 because of analyticity of the integrand in $\mathbb{C}\setminus S_0^\gamma$ and the estimate

$$\left\|\frac{R_A(\lambda)}{(\lambda-1)^2}\right\| \le \frac{c}{|\lambda-1|^2}.$$

At last, consider the integral

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\tau_\beta\left(\frac{\lambda}{n}\right)R_A(\lambda)R_A^2(1)x\,d\lambda = y_{m,n}+y_{m,n}',$$

where

$$y_{m,n} = \frac{1}{2\pi i}\int\limits_{\Gamma_{(m)}}\tau_\beta\left(\frac{\lambda}{n}\right)R_A(\lambda)R_A^2(1)x\,d\lambda,$$

$$y_{m,n}' = \frac{1}{2\pi i}\int\limits_{\Gamma_{(m)}'}\tau_\beta\left(\frac{\lambda}{n}\right)R_A(\lambda)R_A^2(1)x\,d\lambda,$$

$$\Gamma_{(m)} = \{\lambda\in\Gamma : |\lambda| \le r_{m,x}\} \cup \{\lambda : \lambda = r_{m,x}e^{i\varphi},\ |\varphi| \le \gamma\frac{\pi}{2}\},$$

$$\Gamma_{(m)}' = \{\lambda\in\Gamma : |\lambda| > r_{m,x}\} \cup \{\lambda : \lambda = r_{m,x}e^{i\varphi},\ |\varphi| \le \gamma\frac{\pi}{2}\}.$$

Using (11), (12), we find that

$$\begin{aligned}\|y_{m,n}'\| &\le \frac{1}{\pi}\int\limits_{r_{m,x}}^{\infty}\left|\tau_\beta\left(\frac{re^{i\gamma\frac{\pi}{2}}}{n}\right)\right|\left\|R_A\left(re^{i\gamma\frac{\pi}{2}}\right)R_A^2(1)x\right\|\,dr \\ &+ \frac{r_{m,x}}{2\pi}\int\limits_{-\gamma\frac{\pi}{2}}^{\gamma\frac{\pi}{2}}\left|\tau_\beta\left(\frac{r_{m,x}e^{i\varphi}}{n}\right)\right|\left\|R_A\left(r_{m,x}e^{i\varphi}\right)R_A^2(1)x\right\|\,d\varphi \le \\ &\le c\Bigg[\int\limits_{r_{m,x}}^{\infty}e^{-\left(\frac{r}{n}\right)^{1/\beta}\cos\frac{\gamma\pi}{2\beta}}\,dr + r_{m,x}\int\limits_{-\gamma\frac{\pi}{2}}^{\gamma\frac{\pi}{2}}e^{-\left(\frac{r_{m,x}}{n}\right)^{1/\beta}\cos\frac{\gamma\pi}{2\beta}}e^{a_x r_{m,x}^{1/\delta}}\,d\varphi\Bigg].\end{aligned}$$

Since $\beta<1$, the following relation is fulfilled for an arbitrary fixed $n\in\mathbb{N}$:

$$\int\limits_{r_{m,x}}^{\infty}e^{-\left(\frac{r}{n}\right)^{1/\beta}\cos\frac{\gamma\pi}{2\beta}}\,dr \le \int\limits_{r_{m,x}}^{\infty}e^{-\frac{r}{n}\cos\frac{\gamma\pi}{2\beta}}\,dr = \frac{1}{n}\cos\gamma\frac{\pi}{2\beta}e^{-\frac{r_{m,x}}{n}\cos\gamma\frac{\pi}{2\beta}} \to 0 \quad \text{as} \quad m\to\infty.$$

Taking into account that $\beta < \delta$, we get

$$r_{m,x} \int\limits_{-\gamma\frac{\pi}{2}}^{\gamma\frac{\pi}{2}} e^{-\left(\frac{r_{m,x}}{n}\right)^{1/\beta} \cos\frac{\gamma\pi}{2\beta} + a_x r_{m,x}^{1/\delta}}\, d\varphi = \pi\gamma e^{-\left(\frac{r_{m,x}}{n}\right)^{1/\beta} \cos\frac{\gamma\pi}{2\beta} + a_x r_{m,x}^{1/\delta}} \to 0 \quad \text{as} \quad m \to \infty.$$

So, for a fixed $n \in \mathbb{N}$, we have

$$y'_{m,n} \to 0 \quad \text{as} \quad m \to \infty. \tag{17}$$

Let now $y \in R_A^2(1)\mathfrak{M}$. Note that the set $R_A^2(1)\mathfrak{M}$ is complete in $\mathfrak{H}$. We shall show that for any $\varepsilon > 0$, there exists a vector $y_\varepsilon \in \mathfrak{L}(A) = \mathcal{E}(A)$ such that

$$\|y - y_\varepsilon\| < \varepsilon. \tag{18}$$

For this purpose, return to equality (13). Being based on (14)–(16), we find $n_0 \in \mathbb{N}$ so that for $n > n_0$,

$$\left\| \frac{1}{2\pi i} \int\limits_\Gamma \frac{\tau_\beta\left(\frac{\lambda}{n}\right) R_A(\lambda)x}{(\lambda-1)^2}\, d\lambda \right\| < \frac{\varepsilon}{4}, \quad \left\| \frac{1}{2\pi i} \int\limits_\Gamma \frac{\tau_\beta\left(\frac{\lambda}{n}\right) R_A(1)x}{(\lambda-1)^2}\, d\lambda \right\| < \frac{\varepsilon}{4},$$

$$\left\| \frac{1}{2\pi i} \int\limits_\Gamma \frac{\tau_\beta\left(\frac{\lambda}{n}\right) R_A^2(1)x}{\lambda-1}\, d\lambda - R_A^2(1)x \right\| < \frac{\varepsilon}{4}.$$

Making fixed $n > n_0$ and using (17), choose now $m_0 \in \mathbb{N}$ so that

$$\|y_{m,n} + R_A^2(1)x\| < \varepsilon.$$

By Lemma 1, $y_{m,n} \in \mathcal{E}(A) = \mathfrak{L}(A)$, so in (18) we may put $y_\varepsilon = -y_{m,n}$. Hence,

$$\overline{\mathfrak{L}(A)} = \overline{l.s.R_A^2(1)\mathfrak{M}} = \overline{l.s.\mathfrak{M}} = \mathfrak{H}.$$

□

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