Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 4, pp. 302–314

LÉVY-DIRICHLET FORMS. II

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We dedicate this article to the memory of Yurij Lvovich Dalecky.

Abstract. A Dirichlet form associated with the infinite dimensional symmetrized Lévy-Laplace operator is constructed. It is shown that there exists a natural connection between this form and a Markov process. This correspondence is similar to that studied in a previous paper by the same authors for the non-symmetric Lévy Laplacian.

1. Introduction

Dirichlet form theory is an intensively developing field with deep analytical and probabilistic roots both in finite and infinite dimensional spaces [1]. In our previous paper [2] we have constructed the Dirichlet form associated with the Lévy Laplacian [3] given by

$$\Delta_L F(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (F''_Y(x) f_k, f_k)_H$$

(H is a real separable Hilbert space, $x \in H$, f_k is an orthonormal basis, Y is a dense subspace of H, see below for details).

We recall that the absence of an infinite dimensional Lebesgue type measure prevents any "infinite dimensional Laplacian" and in particular the Lévy Laplacian to be a formally self-adjoint operator. If the infinite dimensional Hilbert space H is equipped with a Gaussian measure μ one can symmetrize the Lévy-Laplacian acting in $\mathfrak{L}_2(H,\mu)$ [4], [5] and construct the Dirichlet form associated with this symmetrized operator.

In this paper we study the Dirichlet forms associated with the symmetrized Lévy Laplacian which is generated by the classical Lévy Laplacian Δ_L itself.

It should be mentioned that the Dirichlet forms associated with Δ_L itself and the corresponding symmetrized Lévy-Laplacian are quite different. In fact, they are not only determined by different expressions but what is even more important they have different domains. At the same time they have some common features since certain distinctions between the Lévy Laplacian and the symmetrized (in $\mathfrak{L}_2(H,\mu)$) Lévy Laplacian disappear in the infinite dimensional case.

Let H be a separable real Hilbert space. Consider the set $C^2(H,Y)$ of scalar functions F defined on H and twice differentiable along a dense subspace Y of it. Recall that F is twice differentiable along Y if the Hessian $F''_Y \in \{Y \to Y'\}$ is a bounded operator, where Y' is dual to Y. If $F \in C^2(H,Y)$ then the Lévy Laplacian Δ_L is determined by

$$\Delta_L F(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (F''_Y(x) e_k, e_k)_H, \quad x \in H$$

2000 *Mathematics Subject Classification.* 35R15, 46G05.

Key words and phrases. Lévy Laplacians, Dirichlet forms, symmetrized Lévy operator, diffusion processes.

under the assumption that the limit exists. Here $\{e_k\}$ is an orthonormal basis in H, $e_k \in Y$ $(F''_H(x) \equiv F''(x))$.

Let μ be a Gaussian measure on H with zero mean and covariance operator K. We assume that K is a trace type positive operator such that $||x||_H \le ||K^{-1/2}x||_H$, $x \in D_{K^{-1/2}}$ where $D_{K^{-1/2}}$ is the domain of $K^{-1/2}$.

Denote by $\mathfrak{L}_2(H,\mu)$ the Hilbert space of square integrable in μ functions F defined on H with the norm

$$||F||_{\mathfrak{L}_2(H,\mu)} = \left[\int_H F^2(x)\,\mu(dx)\right]^{\frac{1}{2}}.$$

Let $H_\alpha \subseteq H_0 \subseteq H_{-\alpha}$, $H_1 \equiv H_+$, $H_0 \equiv H$, $H_{-1} \equiv H_-$, $\alpha > 0$ be a subset of a set of densely embedded Hilbert spaces $\{H_\beta\}$ $(-\infty < \beta < \infty)$ with the embedding operator $T = K^{-1/2}$ and the inner product $(x,y)_{H_\beta} = (T^\beta x, T^\beta y)_H$, $x, y \in H_\beta$. In addition we assume that T^{-1} is a Hilbert-Schmidt operator.

For any $F \in C^2(H, H_\alpha)$ with $\alpha > 0$ we define the symmetrized Lévy Laplacian by

$$\Delta_C F(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \Big[(F''_{H_\alpha}(x)e_k, e_k)_H - (F'_{H_\alpha}(x), e_k)_H (x, e_k)_{H_+}\Big] \tag{1}$$

if the limit exists. Here $F''_{H_\alpha}(x)$ is the Hessian, $F'_{H_\alpha}(x)$ is the gradient of $F(x)$ at the point $x \in H$ and $\{e_k\}_1^\infty$ is an orthonormal basis in H, $e_k \in H_{+2}$, $(F'_H(x) \equiv F'(x))$.

We can rewrite (1) in the form

$$\Delta_C F(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \Big[d^2F(x; e_k) - dF(x; e_k)(x, e_k)_{H_+}\Big].$$

Consider the set of $F(x) \in \mathfrak{L}_2(H,\mu)$ such that there exists $dF(x;h)$ $\forall h \in H_{+2}$ and besides $\exists \hat\varepsilon > 0 : \sup_{|\varepsilon|<\hat\varepsilon} |dF(x+\varepsilon h; h)| \in \mathfrak{L}(H,\mu))$.

By the Cameron-Martin formula we have

$$\int_H dF(x;h)\mu(dx) = \int_H F(x)(h,x)_{H_+}\mu(dx).$$

If U, V posses the above properties then we derive

$$\int_H V(x)dU(x;h)\mu(dx) = \int_H V(x)U(x)(h,x)_{H_+}\mu(dx) - \int_H U(x)dV(x;h)\mu(dx). \tag{2}$$

It results from (2) that

$$\int_H V(x)d^2U(x;e_k)\mu(dx) - \int_H V(x)dU(x;e_k)(x,e_k)_{H_+}\mu(dx)$$
$$= -\int_H dV(x;e_k)dU(x;e_k))\mu(dx).$$

If in addition we can pass to the limit under the integral sign then

$$-\int_H V(x)\Delta_C U(x)\mu(dx) = \lim_{n\to\infty}\int_H \frac{1}{n}\sum_{k=1}^n (V'(x),e_k)_H(U'(x),e_k)_H\mu(dx).$$

Applying (2) once again we get

$$\int_H V(x)d^2U(x;e_k)\mu(dx) - \int_H V(x)dU(x;e_k)(x,e_k)_{H_+}\mu(dx)$$
$$= -\int_H dV(x;e_k)(x,e_k)_{H_+}U(x)\mu(dx) + \int_H d^2V(x;e_k)U(x)\mu(dx).$$

If the passage to the limit under the integral sign is allowed then

$$(V, \Delta_C U)_{\mathfrak{L}_2(H,\mu)} = (\Delta_C V, U)_{\mathfrak{L}_2(H,\mu)}.$$

2. Symmetrized Lévy Laplacian and Dirichlet forms

To simplify the computation of integrals with respect to the Gaussian measure μ we choose for the canonical basis in H the set $\{e_k\}_1^\infty$ where e_k are normalized eigenvectors of the measure μ correlation operator T, corresponding to the eigenvalues λ_k $(k = 1, 2, \dots)$ $Te_k = \lambda_k e_k$. In particular in the sequel we need the following relation.

Let $f(u_1, \dots, u_m)$ be a measurable function of m variables and $F(x) = f((x, e_k)_H, \dots, (x, e_m)_H)$, then

$$\int_H F(x)\mu(dx) = [(2\pi)^m \prod_{j=1}^m \lambda_j]^{-\frac{1}{2}} \int_{R^m} f(u_1, \dots, u_m) e^{-\frac{1}{2}\sum_{i=1}^m \lambda_i^2 u_i^2} du_1 \cdots du_m \tag{3}$$

and the existence of the integral at the right hand side results from the existence of the integral at the left hand side and vice versa.

Denote by $\mathfrak{C}$ the set of cylindrical twice differentiable functions. The set $\mathfrak{C}$ is everywhere dense in $\mathfrak{L}_2(H, \mu)$. Each function $S \in \mathfrak{C}$ admits the representation $S(x) = S(Px)$, $x \in H$ where P is a projection onto an m-dimensional subspace. Besides its gradient $S'(x)$ is in R^m and the Hessian $S''(x)$ is a finite dimensional (m-dimensional) operator. For such a function we get

$$\Delta_C S(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^m [(S''(x)e_k, e_k)_H - (S'(x), e_k)_H (x, e_k)_{H_+}] = 0, \quad x \in H.$$

Thus, the symmetrized Lévy Laplacian annihilates cylindrical functions.

Let φ be a Lipschitz continuous function defined on R^1 with the Lipschitz constant c, $\varphi'(\xi)$ denote the derivative of φ at $\xi \in R^1$ and $S_V(x)$ be a cylindrical twice differentiable function, $x \in H$. Given a scalar function Q defined on H, we set $\varphi_Q(x) = \varphi(Q(x)), x \in H$ and we denote by $\mathfrak{T}$ the set

$$\mathfrak{T} = \big\{V : \ V = \varphi_Q S_V \ \text{ such that both } V(\cdot) \text{ and } \frac{\varphi''(Q(\cdot))}{\varphi_Q(\cdot)} V(\cdot) \in \mathfrak{L}_2(H, \mu)\big\}.$$

Choose $Q(x) = \sum\limits_{k=1}^\infty \zeta_k(x), x \in H$ where

$$\zeta_k(x) = \frac{1}{2\lambda_k} \int_{-\lambda_k x_k}^{\lambda_k x_k} e^{\frac{u^2}{2}} \Big[\int_{u-1}^{u+1} e^{-\frac{\psi_k}{\lambda_k} z - \frac{z^2}{2}} dz\Big] du,$$

$x_k = (x, e_k)_H$ and $\psi_k = \frac{1}{\lambda_k \ln^2 k}$.

We check that the function $Q(x)$ is well defined since the series $\sum\limits_{k=1}^\infty \zeta_k(x)$ converges μ a.e. on H.

To this end recall that a scalar function ϕ on H measurable with respect to its Borel σ-algebra $\mathfrak{A}$ is a random variable defined on the probability space $(H, \mathfrak{A}, \mu)$. Its expectation and variance are given respectively by $\mathrm{E}\phi = \int_H \phi(x)\mu(dx)$, $\mathrm{D}\phi = ||\phi - \mathrm{E}\phi||^2_{\mathfrak{L}_2(H,\mu)}$. We also recall that the μ- everywhere convergence of a sequence of measurable functions defined on H corresponds to the convergence of the sequence of random variables with probability 1.

It results from (3) that

$$\int_H \prod_{i=1}^p g_{m_i}(\zeta_{m_i}(x))\mu(dx) = \prod_{i=1}^p \int_H g_{m_i}(\zeta_{m_i}(x))\mu(dx)$$

for an arbitrary finite set of functions $\zeta_{m_1}(x),\dots,\zeta_{m_p}(x)$, all continuous bounded functions $g_{m_1}(\tau),\dots,g_{m_p}(\tau)$ $(\tau\in R^1)$ and any arbitrary integers m_i $(i=1,\dots,p)$. Hence $\{\zeta_k(x)\}_1^\infty$ is a sequence of independent random variables.

By (3) we get

$$\begin{aligned}\mathrm{E}\zeta_k &= \int_H \zeta_k(x)\mu(dx)\\ &= \frac{\lambda_k}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_k^2x_k^2}\Big\{\frac{1}{2\lambda_k}\int_{-\lambda_kx_k}^{\lambda_kx_k} e^{\frac{u^2}{2}}\Big[\int_{u-1}^{u+1} e^{-\frac{\psi_k}{\lambda_k}z-\frac{z^2}{2}}dz\Big]du\Big\}dx_k = 0\end{aligned}$$

(because the function under the integral sign (in x_k) is odd) and

$$\begin{aligned}\mathrm{D}\zeta_k(x) &= \int_H \zeta_k^2(x)\mu(dx)\\ &= \frac{\lambda_k}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_k^2x_k^2}\Big\{\frac{1}{2\lambda_k}\int_{-\lambda_kx_k}^{\lambda_kx_k} e^{\frac{u^2}{2}}\Big[\int_{u-1}^{u+1} e^{-\frac{\psi_k}{\lambda_k}z-\frac{z^2}{2}}dz\Big]du\Big\}^2dx_k\\ &= \frac{1}{4\lambda_k^2\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{v^2}{2}}\Big\{\int_{-v}^{v} e^{\frac{u^2}{2}}\Big[\int_{u-1}^{u+1} e^{-\frac{\psi_k}{\lambda_k}z-\frac{z^2}{2}}dz\Big]du\Big\}^2dv = O\left(\frac{1}{\lambda_k^2}\right).\end{aligned}$$

Since $T^{-2}=K$ is a trace operator we have

$$\sum_{k=1}^\infty 1/\lambda_k^2 = \mathrm{Sp}T^{-2}<\infty$$

and hence $\sum\limits_{k=1}^\infty \mathrm{D}\zeta_k<\infty$. By Kolmogorov's theorem the series $Q(x)=\sum\limits_{k=1}^\infty \zeta_k(x)$ converges almost surely i.e. for μ-almost all $x\in H$.

The set $\mathfrak{T}-$ is a linear set. We show that it is everywhere dense in $\mathfrak{L}_2(H,\mu)$.

Let $U\in\mathfrak{L}_2(H,\mu)$. The set $\mathfrak{C}$ is known to be everywhere dense in $\mathfrak{L}_2(H,\mu)$ and hence for any $\varepsilon_1>0$ there exists $S_0\in\mathfrak{C}$ such that

$$\|U-S_0\|_{\mathfrak{L}_2(H,\mu)}\le\varepsilon_1.$$

Since $S_0\in\mathfrak{C}$, we have $S_0(x)=S_0(Px)$. Choose $\Phi_0(x)=\varphi_Q(x)S_0(x)/\varphi_Q(Px)$ and notice that $\Phi_0\in\mathfrak{T}$ since $S_0(\cdot)/\varphi_Q(P(\cdot))\in\mathfrak{C}$. For such a function we obtain

$$\|U-\Phi_0\|_{\mathfrak{L}_2(H,\mu)}\le\|U-S_0\|_{\mathfrak{L}_2(H,\mu)}+\|S_0-\Phi_0\|_{\mathfrak{L}_2(H,\mu)}\le\varepsilon_1+\|S_0-\Phi_0\|_{\mathfrak{L}_2(H,\mu)}.$$

If we prove that given $\varepsilon_2>0$ we can find Φ_0 such that $\|S_0-\Phi_0\|^2_{\mathfrak{L}_2(H,\mu)}\le\varepsilon_2$ we get that for given $\varepsilon>0$ we can find $\varepsilon_1,\varepsilon_2>0$ and Φ_0 such that $\|U-\Phi_0\|_{\mathfrak{L}_2(H,\mu)}\le\varepsilon_1+\varepsilon_2=\varepsilon$, and hence $\mathfrak{T}$ is everywhere dense in $\mathfrak{L}_2(H,\mu)$.

Now we prove the required estimate for

$$\begin{aligned}\|S_0-\Phi_0\|^2_{\mathfrak{L}_2(H,\mu)} &= \int_H\Big[S_0(x)-\varphi_Q(x)S_0(x)/\varphi_Q(Px)\Big]^2\mu(dx)\\ &= \int_H[S_0(x)/\varphi_Q(Px)]^2[\varphi_Q(Px)-\varphi_Q(x)]^2\mu(dx)\\ &\le c^2\int_H[S_0(x)/\varphi_Q(Px)]^2[Q(Px)-Q(x)]^2\mu(dx)\\ &= c^2\int_H[S_0(x)/\varphi_Q(Px)]^2\Big[\sum_{k=m+1}^\infty\zeta_k(x)\Big]^2\mu(dx).\end{aligned}$$

Since $S_0(x)/\varphi_Q(Px)$ depends only on $(x,e_1)_H,\dots,(x,e_m)_H$, while

$$\sum_{k=m+1}^\infty\zeta_k(x)$$

depends only on $(x, e_{m+1})_H, (x, e_{m+2})_H, \dots$, by (3) we get

$$\|S_0 - \Phi_0\|^2_{\mathfrak{L}_2(H,\mu)} \le c^2 \int_H \Big[S_0(x)/\varphi_Q(Px)\Big]^2 \mu(dx) \int_H \Big[\sum_{k=m+1}^{\infty} \zeta_k(x)\Big]^2 \mu(dx)$$

$$= c^2 \int_H [S_0(x)/\varphi_Q(Px)]^2 \mu(dx) \sum_{k=m+1}^{\infty} \mathrm{D}\zeta_k.$$

From this the estimate $\|\Phi_0 - S_0\|^2_{\mathfrak{L}_2(H,\mu)} \le \varepsilon_2$ follows since $S_0(\cdot)/\varphi_Q(P\cdot) \in \mathfrak{L}_2(H,\mu)$ and $\sum\limits_{k=1}^{\infty} \mathrm{D}\zeta_k < \infty$.

In the sequel we choose $\varphi(\xi)$ to be a positive solution to the equation

$$\varphi(\xi)\varphi''(\xi) + [\varphi'(\xi)]^2 = 0, \quad \xi \in R. \tag{4}$$

In the solution $\varphi(\xi) = \sqrt{2\xi + C}$ (C is are arbitrary constant) we choose $C > 2|Q(x)|$ for $\mu-$almost surely. Then $\varphi(\xi)$ is a Lipschitz function (for $\xi = Q(x)$).

Set $\kappa(\xi) = \frac{\varphi'(\xi)}{\varphi(\xi)}$ $(\varphi(\xi) \neq 0)$, and notice that κ solves the following Riccati equation

$$\kappa'(\xi) + 2\kappa^2(\xi) = 0. \tag{5}$$

Let us mention that another Riccati equation, $\kappa'(\xi) + 2\kappa^2(\xi) + 2\kappa(\xi) = 0$, has appeared in a similar situation in our previous paper [2]. Below we use the notation $\kappa_Q(x) = \kappa(Q(x))$ for the function κ_Q on H generated by κ and a scalar function Q defined on H.

For $U, V \in \mathfrak{L}_2(H,\mu)$ we determine[1] the form $\mathcal{E}^C(U,V)$ by

$$\mathcal{E}^C(U,V) = \lim_{n\to\infty} \mathcal{E}^C_n(U,V) \tag{6}$$

where

$$\mathcal{E}^C_n(U,V) = \int_H \frac{1}{n} \sum_{k=1}^{n} (U'_{H_+}(x), e_k)_H (V'_{H_+}(x), e_k)_H \mu(dx).$$

Let us set

$$\mathcal{C} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{\xi^2}{2}} \Big[\int_{-1}^{1} e^{-\xi u - \frac{u^2}{2}} du\Big]^2 d\xi. \tag{7}$$

Lemma 1. *The form $\mathcal{E}^C(U,V)$ $U, V \in \mathfrak{T}$ determined by (6) exists and*

$$\mathcal{E}^C(U,V) = \Big(\mathcal{C}\kappa_Q^2 U, V\Big)_{\mathfrak{L}_2(H,\mu)}$$

where $\mathcal{C}$ is given by (7) and κ is a positive solution to (5).

The form (6) is symmetric and densely defined.

Proof. By the definition of the space $\mathfrak{T}$ of functions $U, V \in \mathfrak{T}$, have the form $U(x) = \varphi_Q(x))S_U(x), V(x) = \varphi_Q(x)S_V(x)$. It results

$$dU(x;h) = (U'_{H_+}(x), h)_H = \varphi'(Q(x))(Q'(x), h)_H S_U(x) + \varphi_Q(x))(S'_U(x), h),$$
$$dV(x;h) = (V'_{H_+}(x), h)_H = \varphi'(Q(x))(Q'(x), h)_H S_V(x) + \varphi_Q(x)(S'_V(x), h)$$

[1]Along the whole article we use the notation $\mathcal{E}^C(U,V)$ for Dirichlet forms.

$(h \in H_{+2})$ and

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}(U'_{H_+}(x), e_k)_H (V'_{H_+}(x), e_k)_H \\
&\quad = [\varphi'(Q(x))]^2 S_U(x) S_V(x) \frac{1}{n}\sum_{k=1}^{n}(Q'(x), e_k)_H^2 \\
&\quad + \varphi'(Q(x))\varphi_Q(x)\frac{1}{n}\sum_{k=1}^{m}(Q'(x), e_k)_H (S_U(x)+S_V(x), e_k)_H \\
&\quad + \varphi_Q^2(x)\frac{1}{n}\sum_{k=1}^{m}(S'_U(x), e_k)_H (S'_V(x), e_k)_H.
\end{aligned}$$

But

$$dQ(x;h) = (Q'(x), h)_H = \sum_{i=1}^{\infty} e^{\frac{\lambda_i^2 x_i^2}{2}} \int_{\lambda_i x_i - 1}^{\lambda_i x_i + 1} e^{-\frac{\psi_i}{\lambda_i} z - \frac{z^2}{2}}\, dz \cdot (h, e_i)_H,$$

and we obtain $\eta_k(x) = (Q'(x), e_k)_H^2 = \Big[e^{\frac{\lambda_k^2 x_k^2}{2}} \int_{\lambda_k x_k - 1}^{\lambda_k x_k + 1} e^{-\frac{\psi_k}{\lambda_k} z - z^2/2} dz\Big]^2$.

$\{\eta_k(x)\}_1^\infty$ is a sequence of random variables. By (3) we get

$$\begin{aligned}
\mathrm{E}\eta_k &= \int_H \eta_k(x)\mu(dx) = \frac{\lambda_k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{\lambda_k^2 x_k^2}{2}} \Big[\int_{\lambda_k x_k - 1}^{\lambda_k x_k + 1} e^{-\frac{\psi_k}{\lambda_k} z - \frac{z^2}{2}} dz\Big]^2 dx_k \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{\xi^2}{2}} \Big[\int_{\xi-1}^{\xi+1} e^{-\frac{\psi_k}{\lambda_k} z - \frac{z^2}{2}} dz\Big]^2 d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} \Big[\int_{-1}^{1} e^{-\frac{\psi_k}{\lambda_k}(u+\xi) - u\xi - \frac{u^2}{2}} du\Big]^2 d\xi \\
&\to \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{\xi^2}{2}} \Big[\int_{-1}^{1} e^{-\xi u - \frac{u^2}{2}} du\Big]^2 d\xi = \mathcal{C}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{D}\eta_k &= \mathrm{E}\eta_k^2 - [\mathrm{E}\eta_k]^2, \\
\mathrm{E}\eta_k^2 &= \frac{\lambda_k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{3}{2}\lambda_k^2 x_k^2} \Big[\int_{\lambda_k x_k - 1}^{\lambda_k x_k + 1} e^{-\frac{\psi_k}{\lambda_k} z - \frac{z^2}{2}} dz\Big]^4 dx_k \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} \Big[\int_{-1}^{1} e^{-\frac{\psi_k}{\lambda_k}(u+\xi) - u\xi - \frac{u^2}{2}} du\Big]^4 d\xi \\
&\to \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{\xi^2}{2}} \Big[\int_{-1}^{1} e^{-\xi u - \frac{u^2}{2}} du\Big]^4 d\xi.
\end{aligned}$$

Finally we get

$$\sum_{k=1}^{\infty} \frac{\mathrm{D}\eta_k}{k^2} < \infty.$$

By the Strong Law of Large Numbers for independent random variables we deduce that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \eta_k(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathrm{E}\eta_k = \lim_{n\to\infty} \mathrm{E}\eta_n = \mathcal{C}$$

μ-almost surely.

Since $S_U, S_V \in \mathfrak{C}$ we conclude that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(U'_{H_+}(x), e_k)_H (V'_{H_+}(x), e_k)_H$$
$$= \mathcal{C}[\varphi'_Q(x)]^2 S_U(x) S_V(x) = \mathcal{C}\Big[\frac{\varphi'_Q(x)}{\varphi_Q(x)}\Big]^2 U(x)V(x)$$

for μ-almost all $x \in H$ and $\mathcal{E}^C(U,V) = \Big(\mathcal{C}\Big[\frac{\varphi'_Q}{\varphi_Q}\Big]^2 U, V\Big)_{\mathfrak{L}_2(H,\mu)}$, $U, V \in \mathfrak{T}$. □

Lemma 2. *The symmetrized Lévy Laplacian* Δ_C *is defined on* $\mathfrak{T}$ *and* μ*-almost everywhere on* H *acts as the multiplication operator by the function*

$$\Delta_C V(x) = -\mathcal{C}\kappa_Q^2(x)V(x),$$

where $\mathcal{C}$ *is given by* (7)*,* κ *is a positive solution to* (5)*,* $\kappa_Q = \kappa \circ Q$.

Proof. Since $V \in \mathfrak{T}$, we can choose $V = \varphi_Q S_V$. It results for $x \in H$

$$\begin{aligned} dV(x;h) &= (V'_{H_+}(x), h)_H = \varphi'(Q(x))(Q'(x),h)_H S_V(x) + \varphi_Q(x)(S'_V(x),h)_H,\\ d^2V(x;h) &= (V''_{H_+}(x)h, h)_H = \varphi''(Q(x))(Q'(x),h)^2_H S_V(x)\\ &\quad + \varphi'(Q(x))(Q''(x)h,h)_H S_V(x) + 2\varphi'(Q(x))(Q'(x),h)_H(S'_V(x),h)_H\\ &\quad + \varphi_Q(x)(S''_V(x)h,h)_H. \end{aligned}$$

By (1) we obtain

$$\begin{aligned} \Delta_C V(x) &= \varphi''(Q(x))S_V(x)\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(Q'(x),e_k)^2_H\\ &\quad + \varphi'(Q(x))S_V(x)\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[(Q''(x)e_k,e_k)_H - (Q'(x),e_k)_H(x,e_k)_{H_+}]\\ &\quad + 2\varphi'(Q(x))\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{m}(Q'(x),e_k)_H(S'_V(x),e_k)_H\\ &\quad + \varphi_Q(x)\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{m}[(S''_V(x)e_k,e_k)_H - (S'_V(x),e_k)_H(x,e_k)_{H_+}]\\ &= \varphi''(Q(x))S_V(x)\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(Q'(x),e_k)^2_H\\ &\quad + \varphi'(Q(x))S_V(x)\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[(Q''(x)e_k,e_k)_H - (Q'(x),e_k)_H(T^2x,e_k)_H] \end{aligned}$$

(we recall that $S_V \in \mathfrak{C}$).

In the proof of Lemma 1 we have shown that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(Q'(x),e_k)^2_H = \mathcal{C}$$

μ everywhere on H. Hence

$$\Delta_C V(x) = \mathcal{C}\varphi''(Q(x))S_V(x) + \varphi'(Q(x))S_V(x)\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\xi_k(x),$$

where $\xi_k(x) = (Q''(x)e_k,e_k)_H - (Q'(x),e_k)_H(T^2x,e_k)_H$.

Since for the second differential we have

$$d^2Q(x;h) = (Q''(x)h,h)_H = \sum_{i=1}^{\infty}\Big\{\lambda_i^2 x_i e^{\lambda_i^2 x_i/2}\int_{\lambda_i x_i-1}^{\lambda_i x_i+1} e^{-\frac{\psi_i z}{\lambda_i}-z^2/2}dz + \lambda_i\big[e^{-x_i\psi_i-\frac{\psi_i}{\lambda_i}-\lambda_i x_i-1/2} - e^{-x_i\psi_i+\frac{\psi_i}{\lambda_i}+\lambda_i x_i-1/2}\big]\Big\}(h,e_i)_H^2,$$

then

$$(Q''(x)e_k,e_k)_H = \lambda_k^2 x_k e^{\lambda_k^2 x_k}\int_{\lambda_k x_k-1}^{\lambda_k x_k+1} e^{-\frac{\psi_k z}{\lambda_k}-z^2/2}dz + \lambda_k\big[e^{-x_k\psi_k-\frac{\psi_k}{\lambda_k}-\lambda_k x_k-1/2} - e^{-x_k\psi_k+\frac{\psi_k}{\lambda_k}+\lambda_k x_k-1/2}\big].$$

But at the other hand

$$(Q'(x),e_k) = e^{\lambda_k^2 x_k^2}\int_{\lambda_k x_k-1}^{\lambda_k x_k+1} e^{-\frac{\psi_k z}{\lambda_k}-z^2/2}dz$$

and we deduce that

$$\begin{aligned}\xi_k(x) &= \lambda_k[e^{-x_k\psi_k-\frac{\psi_k}{\lambda_k}-\lambda_k x_k-1/2} - e^{-x_k\psi_k+\frac{\psi_k}{\lambda_k}+\lambda_k x_k-1/2}]\\ &= -2\lambda_k e^{-\frac{1}{2}-x_k\psi_k}\mathrm{sh}\,(\lambda_k x_k+\psi_k/\lambda_k).\end{aligned}$$

Notice that $\mathrm{sh}\,(\lambda_k x_k+\psi_k/\lambda_k)>0$ for $x_k>-\psi_k/\lambda_k^2$, $\mathrm{sh}\,(\lambda_k x_k+\psi_k/\lambda_k)<0$ for $x_k<\psi_k/\lambda_k^2$, . It results that

$$\begin{aligned}\mathrm{E}|\xi_k| &= \int_H|\xi_k(x)|\mu(dx) = \lambda_k e^{-1/2}\Big[\frac{\lambda_k}{\sqrt{2\pi}}\int_{-\psi_k/\lambda_k^2}^{\infty}[e^{\frac{\psi_k}{\lambda_k}+(\lambda_k-\psi_k)x_k}\\ &- e^{-\frac{\psi_k}{\lambda_k}-(\lambda_k+\psi_k)x_k}]e^{-\frac{\lambda_k^2 x_k}{2}}dx_k - \frac{\lambda_k}{\sqrt{2\pi}}\int_{-\infty}^{-\psi_k/\lambda_k^2}[e^{\frac{\psi_k}{\lambda_k}+(\lambda_k-\psi_k)x_k}\\ &- e^{-\frac{\psi_k}{\lambda_k}-(\lambda_k+\psi_k)x_k}]e^{-\frac{\lambda_k^2 x_k}{2}}dx_k = 2e^{\frac{\psi_k^2}{2\lambda_k^2}}\lambda_k\int_{-\psi_k}^{\psi_k}e^{-z^2/2}dz = \mathrm{O}\Big(\frac{1}{\ln^2 k}\Big).\end{aligned}$$

From this it follows that $\sum\limits_{k=1}^{\infty}\frac{\mathrm{E}|\xi_k|}{k}<\infty$ and due to B. Levi's theorem μ -a.e. on H $\sum\limits_{k=1}^{\infty}\frac{\xi_k(x)}{k}<\infty$. Finally by Kronecker's lemma we deduce $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\xi_k(x)=0$ for μ almost all $x\in H$.

By the above considerations we derive

$$\Delta_C V(x) = \mathcal{C}\varphi''(Q(x))S_V(x) = \mathcal{C}\frac{\varphi''(Q(x))}{\varphi_Q(x)}V(x)$$

μ-a.e. on H.

Since φ solves (4), we get

$$\Delta_C V(x) = \,= -\mathcal{C}\left[\frac{\varphi'(Q(x))}{\varphi_Q(x)}\right]^2 V(x) = -\mathcal{C}\kappa_Q^2(x)V(x)$$

μ-a.e on H. □

It follows from Lemma 2 that the operator $L_C U = \Delta_C U$ has the almost everywhere dense domain $\mathrm{D}_{L_C} = \mathfrak{T}$ in $\mathfrak{L}_2(H,\mu)$.

Theorem 1. *The operator L_C is essentially self-adjoint on the domain $\mathfrak{T}$, the operator $-L_C$ is positive. The closure $-\bar{L}_C$, of $-L_C$ — is a self-adjoint positive operator.*

Proof. The operator L_C is symmetric since D_{L_C} is dense in $\mathfrak{L}_2(H,\mu)$ and by lemma 2 for all $V_1, V_2 \in D_{L_C}$ we have

$$(L_C V_1, V_2)_{\mathfrak{L}_2(H,\mu)} = -\int_H \mathcal{C}\kappa_Q^2(x) V_1(x) V_2(x)\mu(dx) = (V_1, L_C V_2)_{\mathfrak{L}_2(H,\mu)}.$$

To show that L_C is essentially self-adjoint we consider the operator L_C^* adjoint to L_C in $\mathfrak{L}_2(H,\mu)$.

Let $Z \in D_{L_C^*}$. Then $\forall V \in D_{L_C}$

$$(L_C V, Z)_{\mathfrak{L}_2(H,\mu)} = (V, L_C^* Z)_{\mathfrak{L}_2(H,\mu)}.$$

In other terms

$$-\int_H \mathcal{C}\kappa_Q^2(x) V(x) Z(x)\mu(dx) = \int_H V(x)(L_C^* Z)(x)\mu(dx).$$

Taking into account that $V = \varphi_Q S_V$ we obtain

$$\int_H \varphi_Q(x) S_V(x)\Big[\mathcal{C}\kappa_Q^2(x) Z(x) + (L_C^* Z)(x)\Big]\mu(dx) = 0.$$

The equality holds for all functions having the form φS, where φ is a positive solution to (4) and $S \in \mathfrak{C}$. Recall that $\mathfrak{C}$ is a set of cylindrical functions, which includes in particular the complete orthonormal system of Fourier- Hermite polynomials. It results that μ-almost everywhere on H

$$L_C^* Z = -\mathcal{C}\kappa_Q^2 Z.$$

Eventually,

$$\forall Z \in D_{L_C^*} \quad \mathcal{C}\kappa_Q^2 Z \in \mathfrak{L}_2(H,\mu) \quad \text{and} \quad L_C^* Z = -\mathcal{C}\kappa_Q^2 Z.$$

By the equality $\int_H \mathcal{C}\kappa_Q^2(x) V(x) Z(x)\mu(dx) = \int_H V(x)[\mathcal{C}\kappa_Q^2(x) Z(x)]\mu(dx)\,)$, one can easily prove that

$$\forall Z \in \mathfrak{L}_2(H,\mu) \quad L_C^* Z = -\mathcal{C}\kappa_Q^2 Z \quad \text{and} \quad \mathcal{C}\kappa_Q^2 Z \in \mathfrak{L}_2(H,\mu).$$

Hence,

$$D_{L_C^*} = \{Z \in \mathfrak{L}_2(H,\mu) : \ \mathcal{C}\kappa_Q^2 Z \in \mathfrak{L}_2(H,\mu)\}.$$

The domain $D_{L_C^*}$ is dense in $\mathfrak{L}_2(H,\mu)$, since $D_{L_C^*} \supset D_{L_C}$. The operator $L_C^* Z = -\mathcal{C}\kappa_Q^2 Z$, $Z \in D_{L_C^*}$ is a self-adjoint operator, being a multiplication operator on its natural domain.

Since L_C^* is self-adjoint it results that L_C is essentially self-adjoint. □

Consider a general solution to (5) having the form $\kappa(\xi) = \frac{1}{2\xi + c}$, where c is an arbitrary constant and $\kappa^2(\xi) > 0$ as $-\infty < \xi < \infty$.

It is easy to check that

$$(-L_C U, U)_{\mathfrak{L}_2(H,\mu)} = (\mathcal{C}\kappa_Q^2(x) U, U)_{\mathfrak{L}_2(H,\mu)} > 0 \quad \forall U \in D_{L_C}.$$

Consider the form

$$\mathcal{E}^C(U,V) = \left(\sqrt{-\bar{L}_C}\, U, \sqrt{-\bar{L}_C}\, V\right)_{\mathfrak{L}_2(H,\mu)}, \quad U, V \in D_{\mathcal{E}^C}, \quad D_{\mathcal{E}^C} = D_{\sqrt{-\bar{L}_C}}. \tag{8}$$

Theorem 2. *The form $\mathcal{E}^C(U,V)$ given by (8) is symmetric densely defined positive and closed. It is a Dirichlet form in $\mathfrak{L}_2(H,\mu)$: that is $\mathcal{E}^C(U,V)$ is a closed bilinear Markov form.*

Proof. The first three statements are a direct consequence of Lemmas 1 and 2 and Theorem 1.

The form $\mathcal{E}^C(U,V)$ is closed since $\sqrt{-\overline{L}_C}$ is a closed operator and ($\sqrt{-\overline{L}_C}$ is positive self-adjoint).

To prove that $\mathcal{E}^C$ is a Dirichlet form we use the sufficient condition from [6]. Let

$$U \in D_{\mathcal{E}^C}, \quad V = (0 \vee U) \wedge 1.$$

Then $V \in D_{\mathcal{E}^C}$, and the estimate $\mathcal{E}^C(V,V) \le \mathcal{E}^C(U,U)$ is obvious since $\mathcal{E}^C(U,V) = (|\kappa_Q|U, |\kappa_Q|V)_{\mathfrak{L}_2(H,\mu)}$. □

By the general theory of Dirichlet forms and symmetric Markov semigroups there is a Markov process associated with $\mathcal{E}^C$ with transition semigroup the one given by $\mathcal{E}^C$. To prove that this process is "nice" (strong Markov with "nice" paths) and is properly associated with $\mathcal{E}^C$ it would be enough to prove that $\mathcal{E}^C$ is quasi regular , see e.g. [1], [7]. By the same theory the Markov process is a diffusion if $\mathcal{E}^C$ is in addition a local Dirichlet form. In the next section we provide an alternative construction of a Markov process associated with $\mathcal{E}^C$, as limit of finite dimensional diffusion processes.

3. The symmetrized Lévy operator and stochastic processes

To construct a stochastic process associated with the symmetrized Lévy operator we start with diffusion processes associated with finite dimensional elliptic operators giving rise to the Lévy operator and prove the existence of a limit Markov process.

Theorem 3. *A Markov process $\xi_x(t)$ with values in H, associated with the form $\mathcal{E}^C$, generated by the symmetrized Lévy Laplacian can be constructed as limit (as $n \to \infty$) of a family of diffusion processes $\xi_{x,n}(t)$, associated with the forms*

$$\mathcal{E}^C_n(U,V) = (\sqrt{\overline{l}_n}\,U, \sqrt{\overline{l}_n}\,V)_{\mathfrak{L}_2(H,\mu)},$$

where

$$l_n U(x) = -\frac{1}{n}\sum_{k=1}^{n}\Big[(U''_{H_+}(x)e_k, e_k)_H - (U'_{H_+}(x), e_k)_H (x, e_k)_{H_+}\Big].$$

Proof. First we notice that $\mathcal{E}^C_n(U,V)$ are symmetric and densely defined. Using (2) we can give a different expression for them

$$\mathcal{E}^C_n(U,V) = \int_H \frac{1}{n}\sum_{k=1}^{n}(U'_{H_+}(x), e_k)_H (V'_{H_+}(x), e_k)_H \mu(dx), \quad U, V \in D_{\mathcal{E}^C}$$

(see Lemma 1).

Since

$$U'_{H_+}(x) = \sum_{k=1}^{\infty}\frac{\partial U}{\partial x_k}\, e_k, \quad U''_{H_+}(x)e_j = \sum_{k=1}^{\infty}\frac{\partial^2 U}{\partial x_j \partial x_k} e_k,$$
$$x_k = (x, e_k)_H, \quad (x, e_k)_{H_+} = \lambda_k^2 x_k,$$

we obtain

$$l_n = -\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\partial^2}{\partial x_k^2} - \lambda_k^2 \frac{\partial}{\partial x_k}\right].$$

For a fixed $n \in N$ the symmetrized n-dimensional Laplacian l_n is positive and essentially self-adjoint on the set $\mathfrak{T}$ dense in $\mathfrak{L}_2(H,\mu)$. It generates the semigroup

$$\mathcal{T}_n(t) = e^{-\bar{l}_n t} \quad (t \geq 0)$$

acting on $\mathfrak{L}_2(H,\mu)$. This is a contraction semigroup which preserves the positivity of a function and possesses the property $\mathcal{T}_n(t) \cdot 1 = 1$. It results that $\mathcal{T}_n(t)$ is a Markov semigroup and hence $\mathcal{E}_n^C(U,V)$ is a Markov form.

There is one to one correspondence between the semigroup $\mathcal{T}_n(t)$ and the transition probability $P(t,x,B) = P\{\xi_{x,n}(t) \in B | \xi_{x,n}(0) = x\}$ of a diffusion process $\xi_{x,n}(t)$ defined on the probability space $(\Omega, \mathfrak{F}, P)$ In addition for any bounded measurable function f

$$(\mathcal{T}_n(t)f)(x) = \int_H f(y)P(t,x,dy) = \mathrm{E}(f(\xi_{x,n}(t))) \tag{9}$$

holds for μ-almost all $x \in H$ $t \geq 0$. There exists a unique solution to the Cauchy problem $\frac{\partial U(t,x)}{\partial t} + l_n U(t,x) = 0 \quad (t > 0), \quad U(0,x) = F(x)$ with a bounded continuous function $F(x)$ on H (here $x_{n+1}, x_{n+2}, \ldots$ are considered as parameters).

We show that $\mathcal{T}_n(t)$ strongly in $\mathfrak{L}_2(H,\mu)$ converges to $\mathcal{T}(t)$ that is $\|\mathcal{T}_n(t)f - \mathcal{T}(t)f\|_{\mathfrak{L}_2(H,\mu)} \to 0$ as $n \to \infty$ $\forall f \in \mathfrak{L}_2(H,\mu)$.

Let $f \in D_{\mathcal{E}^C}$. Then

$$\begin{aligned}\|e^{-t\bar{l}_m} f - e^{-t\bar{l}_n} f\|_{\mathfrak{L}_2(H,\mu)} &\leq \|e^{-t\bar{l}_m} f - e^{-t\frac{n}{m}\bar{l}_n} f\|_{\mathfrak{L}_2(H,\mu)} \\ &\quad + \|e^{-t\frac{n}{m}\bar{l}_n} f - e^{-t\bar{l}_n} f\|_{\mathfrak{L}_2(H,\mu)}.\end{aligned}$$

If $m \geq n$ then $l_m \geq \frac{n}{m} l_n$, because

$$\begin{aligned}(l_m f, f)_{\mathfrak{L}_2(H,\mu)} &= \int_H \frac{1}{m} \sum_{k=1}^m \left(\frac{\partial f}{\partial x_k}\right)^2 \mu(dx) \\ &\geq \int_H \frac{1}{m} \sum_{k=1}^n \left(\frac{\partial f}{\partial x_k}\right)^2 \mu(dx) = \frac{n}{m}(l_n f, f)_{\mathfrak{L}_2(H,\mu)}.\end{aligned}$$

This means that $e^{-tl_m} \leq e^{-t\frac{n}{m}l_n}$ and hence

$$\begin{aligned}\|e^{-t\frac{n}{m}\bar{l}_n} f - e^{-t\bar{l}_m} f\|_{\mathfrak{L}_2(H,\mu)} &\leq \sqrt{\|e^{-t\frac{n}{m}\bar{l}_n} - e^{-t\bar{l}_m}\|}\sqrt{\left(\left[e^{-t\frac{n}{m}\bar{l}_n} - e^{-t\bar{l}_m}\right] f, f\right)_{\mathfrak{L}_2(H,\mu)}} \\ &\leq \sqrt{2\left(\left[e^{-t\frac{n}{m}\bar{l}_n} - e^{-t\bar{l}_m}\right] f, f\right)_{\mathfrak{L}_2(H,\mu)}}\end{aligned}$$

(since $\|\mathcal{T}_n(t)\| \leq 1$).

By Duhamel's formula we get

$$e^{-t\frac{n}{m}\bar{l}_n} f - e^{-t\bar{l}_m} f = \int_0^t e^{-(t-s)\frac{n}{m}l_n} \left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] e^{-s\bar{l}_m} f\, ds.$$

Since $\frac{n}{m}l_n$ and l_m commute, $e^{-(t-s)\frac{n}{m}l_n}$ and e^{-sl_m} commute as well. In addition, l_n and l_m commute due to the fact that q_k and q_j, commute where

$$q_i = \frac{\partial^2}{\partial x_i^2} - \lambda_i^2 x_i \frac{\partial}{\partial x_i}.$$

Finally,

$$e^{-t\frac{n}{m}\bar{l}_n}f - e^{-t\bar{l}_m}f = \int_0^t e^{-(t-s)\frac{n}{m}\bar{l}_n - s\bar{l}_m}\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] f\,ds$$

and

$$\begin{aligned}
\|e^{-t\frac{n}{m}\bar{l}_n}f - e^{-t\bar{l}_m}f\|^2_{\mathfrak{L}_2(H,\mu)} &\le 2\left(\left[e^{-t\frac{n}{m}\bar{l}_n} - e^{-t\bar{l}_m}\right]f, f\right)_{\mathfrak{L}_2(H,\mu)} \\
&= 2\Big(\int_0^t e^{-(t-s)\frac{n}{m}\bar{l}_n - s\bar{l}_m}\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] f\,ds, f\Big)_{\mathfrak{L}_2(H,\mu)} \\
&= 2\int_0^t \left(\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] f, G_{st}f\right)_{\mathfrak{L}_2(H,\mu)} ds \\
&\le 2\sqrt{\left(\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] f, f\right)_{\mathfrak{L}_2(H,\mu)}}\int_0^t \sqrt{\left(\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] G_{st}f, G_{st}f\right)_{\mathfrak{L}_2(H,\mu)}}\,ds,
\end{aligned}$$

where $G_{st}f = e^{-(t-s)\frac{n}{m}\bar{l}_n - \bar{l}_m s}f$.

But

$$\begin{aligned}
&\left\{\int_0^t \sqrt{\left(\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] G_{st}f, G_{st}f\right)_{\mathfrak{L}_2(H,\mu)}}\,ds\right\}^2 \\
&\quad\le t\int_0^t \left(\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] G_{st}f, G_{st}f\right)_{\mathfrak{L}_2(H,\mu)} ds \\
&\quad= t\Big(\int_0^t e^{-2(t-s)\frac{n}{m}\bar{l}_n}\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] e^{-2s\bar{l}_m} f\,ds, f\Big)_{\mathfrak{L}_2(H,\mu)} \\
&\quad= t\Big(\frac{1}{2}\left[e^{-2t\frac{n}{m}\bar{l}_n} - e^{-2t\bar{l}_m}\right] f, f\Big)_{\mathfrak{L}_2(H,\mu)} \\
&\quad\le \frac{t}{2}\|e^{-2t\frac{n}{m}\bar{l}_n} - e^{-2t\bar{l}_m}\|\|f\|^2_{\mathfrak{L}_2(H,\mu)} \le t\|f\|^2_{\mathfrak{L}_2(H,\mu)}
\end{aligned}$$

(due to the inequality $\|\mathcal{T}_n(t)\| \le 1$).

We recall that by lemma 1 $\mathcal{E}_n(F,F)$ is a Cauchy sequence and hence

$$\begin{aligned}
\|e^{-t\frac{n}{m}\bar{l}_n}f - e^{-t\bar{l}_m}f\|_{\mathfrak{L}_2(H,\mu)} &\le \left\{4t\left(\left[\bar{l}_m - \frac{n}{m}\bar{l}_n\right] f, f\right)_{\mathfrak{L}_2(H,\mu)}\right\}^{1/4}\|f\|^{1/2}_{\mathfrak{L}_2(H,\mu)} \\
&= \left\{4t\Big[\mathcal{E}^C_m(f,f) - \frac{n}{m}\mathcal{E}^C_n(f,f)\Big]\right\}^{1/4}\|f\|^{1/2}_{\mathfrak{L}_2(H,\mu)} \to 0
\end{aligned}$$

as $m,n\to\infty$.

In addition $e^{-tl_n} \le e^{-t\frac{n}{m}l_n}$ for $m \ge n$, and similar by the above considerations we have

$$\begin{aligned}
\|e^{-t\frac{n}{m}\bar{l}_n}f - e^{-t\bar{l}_n}f\|_{\mathfrak{L}_2(H,\mu)} &\le \left\{4t\left(\left[\bar{l}_n - \frac{n}{m}\bar{l}_n\right] f, f\right)_{\mathfrak{L}_2(H,\mu)}\right\}^{1/4}\|f\|^{1/2}_{\mathfrak{L}_2(H,\mu)} \\
&= \left\{4t\Big[\Big(1 - \frac{n}{m}\Big)\mathcal{E}^C_n(f,f)\Big]\right\}^{1/4}\|f\|^{1/2}_{\mathfrak{L}_2(H,\mu)} \to 0
\end{aligned}$$

as $m,n\to\infty$.

Hence,

$$\lim_{m>n,n\to\infty}\|e^{-t\bar{l}_m}f - e^{-t\bar{l}_n}f\|_{\mathfrak{L}_2(H,\mu)} = 0 \quad \forall f \in D_{\mathcal{E}^C},$$

for any t within a bounded interval.

Thus we have shown that for any $t > 0$, $f \in D_{\mathcal{E}^C}$ $e^{-t\bar{l}_n}f$ is a Cauchy sequence. Since $D_{\mathcal{E}^C}$ is a dense set and the family $e^{-t\bar{l}_n}$ is uniformly bounded this statement holds also for all $f \in \mathfrak{L}_2(H,\mu)$. Hence the limit

$$\lim_{n\to\infty}\mathcal{T}_n(t)f = \mathcal{T}(t)f \quad \forall f \in \mathfrak{L}_2(H,\mu) \tag{10}$$

exists. The semigroup $\mathcal{T}(t)$ is a contraction in $C_2(H,\mu)$ since in the strong limit this property of $\mathcal{T}_n(t)$ is inherited. Moreover $\mathcal{T}1 = 1$, since $\mathcal{T}_n 1 = 1$ for all n an $\mathcal{T}$ is positivity preserving, since $\mathcal{T}_n$ are positivity preserving. Hence $\mathcal{T}$ is a Markov semigroup in $\mathfrak{L}_2(H,\mu)$. By Kolmogorov–Ionescu Tulcea construction there exists a Markov process $\xi_x(t)$ such that $E(f(\xi_x(t))) = \mathcal{T}(t)f(x)$ for μ-a.e. $x \in H$, for any bounded measurable function f defined on H.

From (9) and (10) we deduce that

$$(\mathcal{T}(t)f)x = \lim_{n\to\infty} \mathrm{E}(f(\xi_{x,n}(t))) = Ef(\xi_x(t))$$

for μ-almost all $x \in H$. □

Acknowledgments. The second and the third authors thank the Institute of Applied Mathematics, University of Bonn for the hospitality. The financial support by DFG project 436 RUS 113/593 is gratefully acknowledged.

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Received 15/03/2006