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# STRONG MATRIX MOMENT PROBLEM OF HAMBURGER

K. K. SIMONOV

Abstract. In this paper we consider the strong matrix moment problem on the real line. We obtain a necessary and sufficient condition for uniqueness and find all the solutions for the completely indeterminate case. We use M. G. Kreĭn's theory of representations for Hermitian operators and technique of boundary triplets and the corresponding Weyl functions.

## 1. Introduction

In this paper we consider the following problem. Given a bisequence of self-adjoint $N\times N$-matrices $\{S_k\}_{-\infty}^{+\infty}$, find all the self-adjoint nonnegative Borel $N\times N$-matrix measures $d\Sigma$ on $\mathbb{R}$ obeying the identities

$$\int_{-\infty}^{+\infty} t^k \, d\Sigma(t) = S_k \quad (k = 0, \pm 1, \pm 2, \ldots). \tag{1}$$

This problem is called *the strong full matrix moment problem of Hamburger*. The matrices $\{S_k\}_{-\infty}^{+\infty}$ are called *moments* and the measure $d\Sigma$ is called *a solution of the moment problem* (1).

Let us recall that for the classical moment problem one is given a sequence $\{S_k\}_0^{\infty}$ and seeks a measure $d\Sigma$ such that (1) holds only for nonnegative $k$.

Investigations of the scalar strong moment problem and orthogonal Laurent polynomials originated in papers of W. B. Jones, W. J. Thron, H. Waadeland, O. Njåstad (see [12, 9, 11]). It is worth noting that a necessary and sufficient condition for solvability of the strong moment problem was originally obtained by Yu. M. Berezanskiĭ (see [2] and Remark 3.1.1 below). A description for the solutions of the scalar strong moment problem was obtained in [19] and [21] for the Hamburger problem and in [13] for the Stieltjes problem. Note that the description of the solutions in [19] was given under an additional assumption of *regularity*, which is not used in the present paper (see Remark 4.10.1). A detailed bibliography can be found in the survey [10].

The classical matrix moment problem was investigated by M. G. Kreĭn (see [14, 15]). In [15], M. G. Kreĭn has described all the solutions of the classical matrix moment problem for the completely indeterminate case.

To solve the moment problem means to answer the following questions:

(1) Under which conditions is the moment problem solvable?
(2) If the moment problem is solvable, how to determine whether it has a unique solution?
(3) How to describe all the solutions of the moment problem?

In this paper we assume that a given bisequence $\{S_k\}_{-\infty}^{+\infty}$ is positive and normalized (see Definition 3.1). Under these conditions the moment problem (1) always has a solution. We determine a necessary and sufficient condition for the solution to be unique

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(see Theorem 4.2). When the moment problem is completely indeterminate, we describe the set of all the solutions of (1) (see Theorem 4.10).

Let us briefly outline the contents of the paper. In Section 2 we recall basic concepts of M. G. Kreĭn's theory of representations for Hermitian operators and some facts of the theory of boundary triplets.

In Section 3 we consider the space of Laurent polynomials of the form

$$\sum_{k=-m}^{m} \xi_k z^k \quad \left(\{\xi_k\}_{-m}^{m} \subset \mathbb{C}^N,\ m = 0, 1, 2, \dots\right)$$

with the inner product generated by the Hankel quadratic forms

$$\sum_{i,j=-m}^{m} \xi_j^* S_{i+j} \xi_i \quad \left(\{\xi_k\}_{-m}^{m} \subset \mathbb{C}^N,\ m = 0, 1, 2, \dots\right).$$

We introduce the multiplication operator $A$ and determine a one-to-one correspondence between the set of minimal self-adjoint extensions of $A$ and the set of all the solutions of (1) (see Theorem 3.2). We also recall some earlier results from [20] on orthogonal matrix Laurent polynomials of the first and second kind.

In Section 4 we find a necessary and sufficient condition for a solution of (1) to be unique (see Theorem 4.2). For the completely indeterminate case (see Definition 4.1), we construct a boundary triplet (see Theorem 4.7) for $A^*$ and the corresponding resolvent matrix (see Theorem 4.8). Finally, we describe the set of all the solutions of (1) in the form of a Nevanlinna type formula (see Theorem 4.10).

## 2. Representations of Hermitian operators

Let us recall basic concepts and statements of M. G. Kreĭn's theory of representations for Hermitian operators (see [14, 7]) and some facts of the theory of boundary triplets (see [8, 4, 6, 5]).

In this section, we consider a simple closed Hermitian operator $A$ with deficiency indices $(N, N)$ in a Hilbert space $\mathfrak{H}$. We assume that the domain of $A$ is dense in $\mathfrak{H}$. We will use the usual notation:

$$\mathfrak{M}_\lambda = \operatorname{ran}(A - \lambda), \quad \mathfrak{N}_\lambda = \mathfrak{H} \ominus \mathfrak{M}_{\overline{\lambda}}.$$

Let $\mathfrak{L}$ be a subspace in $\mathfrak{H}$ of dimension $N$. If there exists at least two points $\lambda_+ \in \mathbb{C}_+$ and $\lambda_- \in \mathbb{C}_-$ such that the decomposition

$$\mathfrak{H} = \mathfrak{L} \dotplus \mathfrak{M}_\lambda \tag{2}$$

holds for $\lambda = \lambda_\pm$, then $\mathfrak{L}$ is called *the module of a representation* of the operator $A$.

A point $\lambda \in \mathbb{C}$ is called *an $\mathfrak{L}$-regular point* of $A$ if $\lambda$ is a point of regular type for $A$ and the decomposition (2) holds. Denote by $\rho(A; \mathfrak{L})$ the set of all $\mathfrak{L}$-regular points of $A$ and put

$$\rho_s(A; \mathfrak{L}) = \left\{\lambda \in \mathbb{C} :\ \lambda, \overline{\lambda} \in \rho(A; \mathfrak{L})\right\}.$$

Let us define two holomorphic operator-valued functions

$$\mathcal{P}(\lambda), \mathcal{Q}(\lambda) :\ \mathfrak{H} \to \mathfrak{L} \quad (\lambda \in \rho(A; \mathfrak{L}))$$

on the set $\rho(A; \mathfrak{L})$. Let $\mathcal{P}(\lambda)$ be the skew projection onto the subspace $\mathfrak{L}$ parallel to $\mathfrak{M}_\lambda$. In other words, $\mathcal{P}(\lambda)$ obeys

$$\mathcal{P}(\lambda) f \in \mathfrak{L}, \quad (I - \mathcal{P}(\lambda)) f \in \mathfrak{M}_\lambda \quad (f \in \mathfrak{H}).$$

Define $\mathcal{Q}(\lambda)$ by the equality

$$\mathcal{Q}(\lambda) = P_{\mathfrak{L}} (A - \lambda)^{-1} (I - \mathcal{P}(\lambda)).$$

(By $P_H$ we denote the orthogonal projection onto a subspace $H$.)

The function $\mathcal{P}$ establishes an isomorphism between the Hilbert space $\mathfrak{H}$ and the space of holomorphic functions

$$\mathfrak{H}_{\mathfrak{L}} = \{f_{\mathfrak{L}}(\lambda) = \mathcal{P}(\lambda)f : \ f \in \mathfrak{H}, \ \lambda \in \rho(A;\mathfrak{L})\}.$$

By this isomorphism the operator $A$ is transformed to the multiplication operator

$$\mathcal{P}(\lambda)Af = \lambda f_{\mathfrak{L}}(\lambda) \quad (f \in \operatorname{dom} A).$$

The following assertion is useful for checking whether a given point is $\mathfrak{L}$-regular.

**Theorem 2.1** ([14, 7]). *If for a point $\lambda \in \mathbb{C}$ we can find a neighborhood $U \ni \lambda$ and a linear set $L \subset \mathfrak{H}$ such that $L$ is dense in $\mathfrak{H}$ and $f_{\mathfrak{L}}$ is holomorphic in $U$ for each $f \in L$, then $\lambda$ is an $\mathfrak{L}$-regular point of the operator $A$.*

It is easy to check the following properties of the functions $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$:

$$\begin{array}{lll} \mathcal{P}(\lambda)Af = \lambda\mathcal{P}(\lambda)f, & \mathcal{Q}(\lambda)Af = \lambda\mathcal{Q}(\lambda)f + P_{\mathfrak{L}}f & (f \in \operatorname{dom} A), \\ A^*\mathcal{P}(\lambda)^*\phi = \overline{\lambda}\mathcal{P}(\lambda)^*\phi, & A^*\mathcal{Q}(\lambda)^*\phi = \overline{\lambda}\mathcal{Q}(\lambda)^*\phi + \phi & (\phi \in \mathfrak{L}), \\ \mathcal{P}(\lambda)\phi = \phi, & \mathcal{Q}(\lambda)\phi = 0 & (\phi \in \mathfrak{L}), \\ P_{\mathfrak{L}}\mathcal{P}(\lambda)^* = I_{\mathfrak{L}}, & P_{\mathfrak{L}}\mathcal{Q}(\lambda)^* = 0_{\mathfrak{L}}, & \\ \mathcal{P}(\lambda)^*P_{\mathfrak{L}} = \mathcal{P}(\lambda)^*, & \mathcal{Q}(\lambda)^*P_{\mathfrak{L}} = \mathcal{Q}(\lambda)^*. & \end{array} \tag{3}$$

It follows from (3) that

$$\mathfrak{N}_{\lambda} = \ker(A^* - \lambda) = \mathcal{P}(\overline{\lambda})^*\mathfrak{L} \quad (\lambda \in \rho_s(A;\mathfrak{L})). \tag{4}$$

**Proposition 2.2** ([5, 6]). *The following decomposition holds.*

$$\operatorname{dom} A^* = \operatorname{dom} A \dotplus \mathcal{P}(\lambda)^*\mathfrak{L} \dotplus \mathcal{Q}(\lambda)^*\mathfrak{L} \quad (\lambda \in \rho_s(A;\mathfrak{L})).$$

**Definition 2.1.** Let $\widetilde{A}$ be a self-adjoint extension of the operator $A$, possibly in a larger Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. The extension $\widetilde{A}$ is called $\mathfrak{L}$-*minimal* if

$$\widetilde{\mathfrak{H}} = \overline{\operatorname{span}}\left\{\mathfrak{L}, \ (\widetilde{A} - \lambda)^{-1}\mathfrak{L} : \ \lambda \in \rho(\widetilde{A})\right\}.$$

**Definition 2.2.** Let $\widetilde{A}$ be an $\mathfrak{L}$-minimal self-adjoint extension of the operator $A$. Then the operator-valued function

$$P_{\mathfrak{L}}(\widetilde{A} - \lambda)^{-1}|_{\mathfrak{L}} \quad (\lambda \in \rho(\widetilde{A}))$$

is called the $\mathfrak{L}$-*resolvent* of the operator $A$ corresponding to the extension $\widetilde{A}$.

**Definition 2.3** ([8]). A triplet $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$, where $\Gamma = \{\Gamma_0, \Gamma_1\}$ is a linear operator from $\operatorname{dom} A^*$ to $\mathfrak{L} \oplus \mathfrak{L}$, is called *a boundary triplet* for the operator $A^*$ if the mapping $\Gamma$ is surjective and obeys *the abstract Green identity*

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathfrak{L}} - (\Gamma_0 f, \Gamma_1 g)_{\mathfrak{L}} \quad (f, g \in A^*). \tag{5}$$

Let us recall that a linear subspace $\widetilde{S} \subset \mathfrak{L} \oplus \mathfrak{L}$ is called *a linear relation* in $\mathfrak{L}$. Any linear operator $S$ in $\mathfrak{L}$ can be identified with its graph

$$\{\{f, Sf\} \in \mathfrak{L} \oplus \mathfrak{L} : \ f \in \operatorname{dom} S\}.$$

Therefore any linear operator can be considered as a linear relation.

A linear relation $\widetilde{S}$ in $\mathfrak{L}$ is called Hermitian (dissipative) if $(f', f) \in \mathbb{R}$ ($\Im(f', f) \geq 0$) for any pair $(f, f') \in \widetilde{S}$. A Hermitian (dissipative) relation $\widetilde{S}$ is called self-adjoint (maximal dissipative) if $\dim \widetilde{S} = N$.

Any maximal dissipative linear relation $\widetilde{S}$ in $\mathfrak{L}$ is uniquely represented in the form

$$\widetilde{S} = S \oplus \operatorname{mul} \widetilde{S},$$

where $S$ is a linear operator, which is called *the operator part of* $\widetilde{S}$, and

$$\operatorname{mul}\widetilde{S} = \left\{\{0, f\} \in \widetilde{S}\right\}$$

is called *the multivalued part of* $\widetilde{S}$.

**Proposition 2.3** ([5])**.** *A boundary triplet $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ defines a one-to-one correspondence between the set of proper extensions $\widetilde{A}$ of the operator $A$ ($A \subset \widetilde{A} \subset A^*$) and the set of linear relations $\theta \subset \mathfrak{L} \oplus \mathfrak{L}$. This correspondence is given by*

$$\widetilde{A} = \widetilde{A}_\theta \longleftrightarrow \theta = \Gamma \operatorname{dom} \widetilde{A} = \left\{\{\Gamma_0 f, \Gamma_1 f\} : \ f \in \operatorname{dom} \widetilde{A}\right\}.$$

*The extension $\widetilde{A}_\theta$ is Hermitian (self-adjoint) if and only if the relation $\theta$ has the same property.*

In particular, the operators $\Gamma_0$ and $\Gamma_1$ define two self-adjoint extensions $\widetilde{A}_0$ and $\widetilde{A}_1$ of the operator $A$ with the following domains:

$$\operatorname{dom} \widetilde{A}_0 = \ker \Gamma_0, \quad \operatorname{dom} \widetilde{A}_1 = \ker \Gamma_1. \tag{6}$$

The equality

$$\gamma(\lambda) = (\Gamma_0|_{\mathfrak{N}_\lambda})^{-1} \quad (\lambda \in \rho(\widetilde{A}_0)) \tag{7}$$

defines the operator-valued function $\gamma(\lambda) : \ \mathfrak{L} \to \mathfrak{N}_\lambda$ that is holomorphic on $\rho(\widetilde{A}_0)$.

**Definition 2.4** ([5])**.** The operator-valued function $M(\lambda)$ : $\mathfrak{L} \to \mathfrak{L}$ defined by the equality

$$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda \quad (f_\lambda \in \mathfrak{N}_\lambda, \ \lambda \in \rho(\widetilde{A}_0)) \tag{8}$$

is called *the Weyl function* of the operator $A$ corresponding to the boundary triplet $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$.

**Proposition 2.4** ([5])**.** *The functions $M(\lambda)$ and $\gamma(\lambda)$ obey the following identities:*

$$\gamma(\lambda) - \gamma(\mu) = (\lambda - \mu)(\widetilde{A}_0 - \lambda)^{-1}\gamma(\mu) \qquad (\lambda, \mu \in \rho(\widetilde{A}_0)), \tag{9}$$

$$M(\lambda) - M(\mu) = (\lambda - \mu)\gamma(\overline{\mu})^*\gamma(\lambda) \qquad (\lambda, \mu \in \rho(\widetilde{A}_0)). \tag{10}$$

**Definition 2.5** ([6])**.** It is said that a holomorphic function $\tau : \mathbb{C}_+ \to \mathfrak{L} \oplus \mathfrak{L}$ belongs to the class $\widetilde{\mathcal{N}}_{\mathfrak{L}}$ if $\tau(\lambda)$ is a maximal dissipative relation in $\mathfrak{L}$ for any $\lambda \in \mathbb{C}_+$.

It is said that $\tau$ belongs to the class $\mathcal{N}_{\mathfrak{L}}$ if $\tau(\lambda)$ is a maximal dissipative operator for any $\lambda \in \mathbb{C}_+$.

One can extend a function $\tau \in \widetilde{\mathcal{N}}_{\mathfrak{L}}$ to the domain $\mathbb{C}_-$ by the formula

$$\tau(\lambda) = \tau(\overline{\lambda})^* \quad (\lambda \in \mathbb{C}_-).$$

By identities (9) and (10), it follows that $M(\lambda)$ belongs to the class $\mathcal{N}_{\mathfrak{L}}$. Moreover, identity (10) means that $M(\lambda)$ is a $Q$-function of the operator $A$ corresponding to the extension $\widetilde{A}_0$ in the sense of [16, 17].

**Definition 2.6** ([18])**.** A $2N \times 2N$-matrix $W(\lambda) = (w_{ij}(\lambda))_1^2$ holomorphic on $\rho(A; \mathfrak{L})$ is called *an $\mathfrak{L}$-resolvent matrix* of the operator $A$ if it obeys the identity

$$W(\lambda)JW(\mu)^* = J + i(\lambda - \overline{\mu})G(\lambda)G(\mu)^* \quad (\lambda, \mu \in \rho(A; \mathfrak{L})),$$

where

$$J = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad G(\lambda) = \begin{pmatrix} -\mathcal{Q}(\lambda) \\ \mathcal{P}(\lambda) \end{pmatrix}.$$

An $\mathfrak{L}$-resolvent matrix is not unique. If $W_1(\lambda)$ and $W_2(\lambda)$ are two different $\mathfrak{L}$-resolvent matrices of $A$, then

$$W_1(\lambda) = W_2(\lambda)U \quad (\lambda \in \rho(A; \mathfrak{L})),$$

where $U$ is a $J$-unitary matrix.

There exists a natural one-to-one correspondence between the set of $\mathfrak{L}$-resolvent matrices and the set of boundary triplets. This following theorem shows how to construct the $\mathfrak{L}$-resolvent matrix corresponding to a boundary triplet.

**Theorem 2.5** ([6]). *Let $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator $A^*$. Then the matrix function*

$$W_\Pi(\lambda) = (\Gamma G(\lambda)^*)^* = \begin{pmatrix} -\Gamma_0 \mathcal{Q}(\lambda)^* & \Gamma_0 \mathcal{P}(\lambda)^* \\ -\Gamma_1 \mathcal{Q}(\lambda)^* & \Gamma_1 \mathcal{P}(\lambda)^* \end{pmatrix}^*$$

*is an $\mathfrak{L}$-resolvent matrix of $A$. $W_\Pi(\lambda)$ is called* the $\Pi\mathfrak{L}$-resolvent matrix of $A$ *corresponding to the boundary triplet* $\Pi$.

**Theorem 2.6** ([14, 7, 6]). *Let $W(\lambda) = (w_{ij}(\lambda))_1^2$ be an $\mathfrak{L}$-resolvent matrix of the operator $A$. Then the formula*

$$P_{\mathfrak{L}}(\widetilde{A} - \lambda)^{-1}|_{\mathfrak{L}} = (w_{11}(\lambda)\tau(\lambda) + w_{12}(\lambda))\,(w_{21}(\lambda)\tau(\lambda) + w_{22}(\lambda))^{-1} \quad (\lambda \in \rho(A; \mathfrak{L})) \tag{11}$$

*establishes a one-to-one correspondence between the set of all $\mathfrak{L}$-minimal self-adjoint extensions $\widetilde{A}$ of the operator $A$ and the set of all functions $\tau \in \widetilde{\mathcal{N}}_{\mathfrak{L}}$.*

*Remark* 2.6.1. Suppose that $W(\lambda)$ is the $\Pi\mathfrak{L}$-resolvent matrix of $A$ corresponding to a boundary triplet $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$. In this case, the parameter $\tau(\lambda)$ in (11) corresponds to the generalized resolvent of $A$ uniquely defined by M. G. Kreĭn's formula (see [17, 7])

$$P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}|_{\mathfrak{H}} = (\widetilde{A}_0 - \lambda)^{-1} - \gamma(\lambda)(\tau(\lambda) + M(\lambda))^{-1}\gamma(\overline{\lambda})^*,$$

where $\widetilde{A}_0$, $\gamma(\lambda)$, and $M(\lambda)$ are defined by (6)–(8).

## 3. The space of Laurent polynomials

**Theorem 3.1.** *If the moment problem* (1) *is solvable, then the quadratic forms*

$$\sum_{i,j=-m}^{m} \xi_j^* S_{i+j} \xi_i \quad \left(\{\xi_k\}_{-m}^m \subset \mathbb{C}^N\right) \tag{12}$$

*are positive definite for any $m = 0, 1, 2, \ldots$.*

*Proof.* Suppose $d\Sigma$ is a solution of the moment problem (1), $m$ is a nonnegative integer. Then

$$\sum_{i,j=-m}^{m} \xi_j^* S_{i+j} \xi_i = \int_{-\infty}^{+\infty} \left(\sum_{j=-m}^{m} \xi_j t^j\right)^* d\Sigma(t) \left(\sum_{i=-m}^{m} \xi_i t^i\right) \geq 0$$

for any $\{\xi_k\}_{-m}^m \subset \mathbb{C}^N$. □

*Remark* 3.1.1. For the scalar case, it is known (see [11]) that the positive definiteness of the quadratic forms (12) is also a sufficient condition for the solvability of the moment problem (1). Yu. M. Berezanskiĭ has pointed us that this condition for the solvability of (1) is a particular case of Theorem 5.1 from his book [2, page 722].

**Definition 3.1.** A bisequence $\{S_k\}_{-\infty}^{+\infty}$ is called *positive* if the quadratic forms (12) are strictly positive definite for each $m = 0, 1, 2, \ldots$.

A positive bisequence $\{S_k\}_{-\infty}^{+\infty}$ is called *normalized* if $S_0 = I$.

Any positive bisequence $\{\widetilde{S}_k\}_{-\infty}^{\infty}$ can be converted to a normalized bisequence $\{S_k\}_{-\infty}^{+\infty}$ by the rule

$$S_k = \widetilde{S}_0^{-\frac{1}{2}} \widetilde{S}_k \widetilde{S}_0^{-\frac{1}{2}} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

**Definition 3.2.** The moment problem (1) is called *nondegenerate* if the given bisequence of moments $\{S_k\}_{-\infty}^{+\infty}$ is positive.

In this paper, we assume that the given bisequence of moments is positive and normalized.

Consider the space of $N$-vector Laurent polynomials

$$\overset{\circ}{\mathfrak{H}} = \left\{ f(z) = \sum_{k=-m}^{m} f_k z^k : \ \{f_k\}_{-m}^{m} \subset \mathbb{C}^N, \ m = 0, 1, 2, \ldots \right\}.$$

In this space, we introduce the inner product

$$(f, g) = \sum_{i,j=-m}^{m} g_j^* S_{i+j} f_i, \tag{13}$$

where

$$f(z) = \sum_{i=-m}^{m} f_i z^i, \quad g(z) = \sum_{j=-m}^{m} g_j z^j \quad (m = 0, 1, 2, \ldots).$$

Denote by $\mathfrak{H}$ the completion of $\overset{\circ}{\mathfrak{H}}$ to a Hilbert space with respect to the inner product (13).

In the space $\mathfrak{H}$, take the $N$-dimensional subspace

$$\mathfrak{H}_0 = \left\{ f(z) \equiv f_0 : \ f_0 \in \mathbb{C}^N \right\}.$$

Since the bisequence $\{S_k\}_{-\infty}^{+\infty}$ is normalized, the subspace $\mathfrak{H}_0$ is naturally isomorphic to the space $\mathbb{C}^N$.

Consider the linear operator $\overset{\circ}{A}$ defined by

$$\operatorname{dom} \overset{\circ}{A} = \overset{\circ}{\mathfrak{H}}, \quad \overset{\circ}{A} f(z) = z f(z) \quad (f \in \overset{\circ}{\mathfrak{H}}).$$

Denote by $A$ the closure of $\overset{\circ}{A}$ in the Hilbert space $\mathfrak{H}$.

**Theorem 3.2.** *There exists a one-to-one correspondence between the set of all the solutions $d\Sigma$ of the moment problem* (1) *and the set of all the $\mathfrak{H}_0$-minimal self-adjoint extensions $\widetilde{A}$ of the operator $A$. This correspondence is given by*

$$\Sigma(t) = P_{\mathfrak{H}_0} E_t(\widetilde{A})|_{\mathfrak{H}_0},$$

*where $E_t(\widetilde{A})$ is the orthogonal spectral measure of a self-adjoint operator $\widetilde{A}$.*

*Proof.* Suppose that $\widetilde{A}$ is an $\mathfrak{H}_0$-minimal self-adjoint extension of the operator $A$ in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. Let $E_t = E_t(\widetilde{A})$ be the spectral measure of $\widetilde{A}$. We now prove that $d\Sigma(t) = d(P_{\mathfrak{H}_0} E_t|_{\mathfrak{H}_0})$ is a solution of the moment problem (1). Since $\ker \widetilde{A} = \ker A = \{0\}$, the equalities

$$z^k \phi = A^k \phi = \widetilde{A}^k \phi \quad (\phi \in \mathfrak{H}_0 \cong \mathbb{C}^N, \ k = 0, \pm 1, \pm 2, \ldots)$$

hold. Therefore

$$\int_{-\infty}^{+\infty} t^{i+j} \psi^* \, d\Sigma(t) \, \phi = \int_{-\infty}^{+\infty} t^{i+j} \, d(E_t \phi, \psi) = (\widetilde{A}^i \phi, \widetilde{A}^j \psi) = (A^i \phi, A^j \psi) = \psi^* S_{i+j} \phi$$
$$(\phi, \psi \in \mathfrak{H}_0 \cong \mathbb{C}^N, \ i, j = 0, \pm 1, \pm 2, \ldots),$$

and thus $d\Sigma(t)$ is a solution of the moment problem (1).

Let us prove the converse assertion. Suppose that $d\Sigma(t)$ is a solution of the moment problem (1). Then $\Sigma(t)$ is a linear bounded self-adjoint operator in $\mathfrak{H}_0 \cong \mathbb{C}^N$ and it obeys the conditions

$$\Sigma(-\infty) = 0_{\mathfrak{H}_0}, \quad \Sigma(+\infty) = I_{\mathfrak{H}_0}, \quad \Sigma(t-0) = \Sigma(t) \quad (t \in \mathbb{R}).$$

By the Naimark dilation theorem (see [1, 3]), there exists a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}_0$ and a resolution of identity $E_t : \widetilde{\mathfrak{H}} \to \widetilde{\mathfrak{H}}$ such that

$$\Sigma(t) = P_{\mathfrak{H}_0} E_t|_{\mathfrak{H}_0}, \quad \overline{\operatorname{span}}\,\{E_t\phi : \phi \in \mathfrak{H}_0\} = \widetilde{\mathfrak{H}}.$$

The resolution of identity $E_t$ defines the self-adjoint operator

$$\widetilde{A} = \int_{-\infty}^{+\infty} t\, dE_t$$

in the space $\widetilde{\mathfrak{H}}$. Let us show that there exists an isometric embedding $V : \mathfrak{H} \to \widetilde{\mathfrak{H}}$ such that $VAV^{-1} \subset \widetilde{A}$. Indeed,

$$\int_{-\infty}^{+\infty} t^{2k}\, d(E_t\phi, \phi) = \int_{-\infty}^{+\infty} t^{2k}\, d(\Sigma(t)\phi, \phi) = \phi^* S_{2k}\phi < \infty$$
$$(\phi \in \mathfrak{H}_0 \cong \mathbb{C}^N,\ k = 0, \pm 1, \pm 2, \ldots),$$

and therefore $\mathfrak{H}_0 \subset \operatorname{dom} \widetilde{A}^k$ for each integer $k$. Put

$$V(z^k\phi) = V(A^k\phi) = \widetilde{A}^k\phi \quad (\phi \in \mathfrak{H}_0 \cong \mathbb{C}^N.\ k = 0, \pm 1, \pm 2, \ldots).$$

The mapping $V$ is isometric since

$$\begin{aligned}(V(z^i\phi), V(z^j\psi))_{\widetilde{\mathfrak{H}}} &= (\widetilde{A}^i\phi, \widetilde{A}^j\psi)_{\widetilde{\mathfrak{H}}} = \int_{-\infty}^{+\infty} t^{i+j}\, d(E_t\phi, \psi) = \int_{-\infty}^{+\infty} t^{i+j}\, \psi^* d\Sigma(t)\phi \\ &= \psi^* S_{i+j}\phi = (z^i\phi, z^j\psi) \quad (\phi, \psi \in \mathfrak{H}_0 \cong \mathbb{C}^N,\ i, j = 0, \pm 1, \pm 2, \ldots),\end{aligned}$$

and the inclusion $VAV^{-1} \subset \widetilde{A}$ holds by construction. □

**Corollary 3.2.1.** *The moment problem* (1) *has a unique solution if and only if the operator $A$ is maximal. Otherwise the moment problem* (1) *has infinitely many solutions.*

**Definition 3.3** ([20])**.** A sequence of $N \times N$-matrix Laurent polynomials $\{P_k(z)\}_0^\infty$ of the form

$$P_{2k}(z) = \sum_{j=-k}^{k} P_{2k}^{(j)} z^j, \quad P_{2k+1}(z) = \sum_{j=-k-1}^{k} P_{2k+1}^{(j)} z^j \quad (P_k^{(j)} \in \mathbb{C}^{N\times N})$$

is called *the sequence of orthogonal Laurent polynomials of the first kind* if the following conditions hold:

(A) The coefficients $P_{2k}^{(k)}$ and $P_{2k+1}^{(-k-1)}$ are strictly positive matrices.
(B) The Laurent polynomials $\{P_k(z)\}_0^\infty$ are orthonormal, i. e.,

$$(P_i(z)\xi, P_j(z)\eta) = 0, \quad (P_k(z)\xi, P_k(z)\eta) = \eta^*\xi \quad (\xi, \eta \in \mathbb{C}^N,\ i, j, k = 0, 1, \ldots,\ i \neq j).$$

Conditions (A) and (B) uniquely determine the sequence $\{P_k(z)\}_0^\infty$.

**Definition 3.4** ([20])**.** The sequence of $N \times N$-matrix Laurent polynomials $\{Q_k(z)\}_0^\infty$ defined by

$$\eta^* Q_k(z)\xi = (R_k(\cdot, z)\xi, \eta) \quad (\xi, \eta \in \mathbb{C}^N,\ k = 0, 1, 2, \ldots),$$

where

$$R_k(\zeta, z) = \frac{P_k(\zeta) - P_k(z)}{\zeta - z} \quad (k = 0, 1, 2, \ldots),$$

is called *the sequence of Laurent polynomials of the second kind.*

Extending Definitions 3.3 and 3.4, put

$$P_{-2}(z)=0,\quad P_{-1}(z)=0,\quad Q_{-2}(z)=-I,\quad Q_{-1}(z)=0.$$

**Definition 3.5** ([11, 10])**.** The sequence $\{P_k(z)\}_0^\infty$ is called *regular* if the coefficients $P_{2k}^{(-k)}$ and $P_{2k+1}^{(k)}$ are nondegenerate matrices.

Note that in this paper we do not assume that the sequence $\{P_k(z)\}_0^\infty$ is regular (see Remark 4.10.1).

If we denote by $\{\epsilon_j\}_1^N$ the standard basis in $\mathbb{C}^N$, then the sequence

$$\{P_i(z)\epsilon_j\}=\{P_0(z)\epsilon_1,\dots,P_0(z)\epsilon_N,P_1(z)\epsilon_1,\dots,P_1(z)\epsilon_N,\dots\}$$

forms an orthonormal basis in the space $\mathfrak{H}$. Therefore any element $f\in\mathfrak{H}$ can be uniquely represented as a Fourier series

$$f(z)=\sum_{k=0}^{\infty}P_k(z)\phi_k, \tag{14}$$

where the Fourier coefficients $\phi_k\in\mathbb{C}^N$ are determined by the equalities

$$\epsilon_j^*\phi_k=(f(z),P_k(z)\epsilon_j)\quad (j=1,\dots,N).$$

Conversely, a vector $f$ of the form (14) belongs to the space $\mathfrak{H}$ if and only if it obeys the inequality

$$\|f\|^2=\sum_{k=0}^{\infty}\|\phi_k\|_{\mathbb{C}^N}^2<\infty.$$

**Theorem 3.3** ([20])**.** *The Laurent polynomials $\{P_k(z)\}_0^\infty$ and $\{Q_k(z)\}_0^\infty$ obey the recurrence relations*

$$\begin{aligned}
zP_{2k}(z)&=P_{2k-2}(z)C_{2k-2}^*+P_{2k-1}(z)B_{2k-1}^*\\
&\quad+P_{2k}(z)A_{2k}+P_{2k+1}(z)B_{2k}+P_{2k+2}(z)C_{2k},\\
zQ_{2k}(z)&=Q_{2k-2}(z)C_{2k-2}^*+Q_{2k-1}(z)B_{2k-1}^*\\
&\quad+Q_{2k}(z)A_{2k}+Q_{2k+1}(z)B_{2k}+Q_{2k+2}(z)C_{2k},\\
zP_{2k+1}(z)&=P_{2k}(z)B_{2k}^*+P_{2k+1}(z)A_{2k+1}+P_{2k+2}(z)B_{2k+1},\\
zQ_{2k+1}(z)&=Q_{2k}(z)B_{2k}^*+Q_{2k+1}(z)A_{2k+1}+Q_{2k+2}(z)B_{2k+1}\\
&\qquad (k=0,1,2,\dots),
\end{aligned} \tag{15}$$

*with the initial conditions*

$$P_{-2}(z)=0,\quad P_0(z)=I,\quad Q_{-2}(z)=-I,\quad Q_0(z)=0, \tag{16}$$

*where the coefficients $\{A_k\}_0^\infty$, $\{B_k\}_{-1}^\infty$, $\{C_k\}_{-2}^\infty$ are some $N\times N$-matrices.*

In the basis $\{P_i(z)\epsilon_j\}$, the operator $A$ has the following block-matrix form

$$\begin{pmatrix}
A_0 & B_0^* & C_0^* & & & \\
B_0 & A_1 & B_1^* & & & \\
C_0 & B_1 & A_2 & B_2^* & C_2^* & \\
 & & B_2 & A_3 & B_3^* & \\
 & & C_2 & B_3 & A_4^* & \\
 & & & & & \ddots
\end{pmatrix} \tag{17}$$

**Definition 3.6.** The matrix (17) is called *the generalized Jacobi matrix* corresponding to the matrix moment problem (1).

**Proposition 3.4** ([20])**.** *The coefficients $\{A_k\}_0^\infty$, $\{B_k\}_{-1}^\infty$, $\{C_k\}_{-2}^\infty$ of the recurrence relations* (15) *obey the following conditions.*

(i) $C_{-2}=I,\ B_{-1}=0,\ C_{2k-1}=0 \quad (k=0,1,2,\dots);$

(ii) *The following matrices are well defined:*

$$C_{2k}^{-1},\quad \widetilde{B}_0=(B_0^*-A_0C_0^{-1}B_1)^{-1},$$
$$\widetilde{C}_{2k+1}=-\left[\begin{pmatrix}B_{2k} & B_{2k+1}^*\end{pmatrix}\begin{pmatrix}C_{2k} & A_{2k+2}\\ 0 & C_{2k+2}\end{pmatrix}^{-1}\begin{pmatrix}B_{2k+2}^*\\ B_{2k+3}\end{pmatrix}\right]^{-1}$$
$$(k=0,1,2\dots);$$

(iii) *The following inequalities hold:*

$$C_{2k}C_{2k-2}\cdots C_0>0,\quad \widetilde{C}_{2k+1}\widetilde{C}_{2k-1}\cdots\widetilde{C}_1\widetilde{B}_0>0\quad (k=0,1,2,\dots);$$

(iv) *The matrices $A_k$ are self-adjoint and obey the identities*

$$A_{2k+1}=B_{2k}C_{2k}^{-1}B_{2k+1}\quad (k=0,1,2,\dots).$$

**Proposition 3.5** ([20]). *Let $\{A_k\}_0^\infty$, $\{B_k\}_{-1}^\infty$, $\{C_k\}_{-2}^\infty$ be arbitrary matrices satisfying conditions* (i)–(iv). *Then there exists a unique positive and normalized bisequence of moments $\{S_k\}_{-\infty}^{+\infty}$ such that the corresponding Laurent polynomials $\{P_k(z)\}_0^\infty$ and $\{Q_k(z)\}_0^\infty$ obey* (15) *with the given coefficients.*

**Corollary 3.5.1** ([20]). *Suppose that a sequence of vectors $\{f_k\}_0^\infty\subset\mathbb{C}^N$ obeys the following recurrence relations.*

$$\lambda f_{2k}=C_{2k-2}f_{2k-2}+B_{2k-1}f_{2k-1}+A_{2k}f_{2k}+B_{2k}^*f_{2k+1}+C_{2k}^*f_{2k+2},$$
$$\lambda f_{2k+1}=B_{2k}f_{2k}+A_{2k+1}f_{2k+1}+B_{2k+1}^*f_{2k+2}\tag{18}$$
$$(k=0,1,2,\dots),$$

*where $f_{-2}=f_{-1}=0$. Then*

$$f_j=P_j(\overline{\lambda})^*f_0\quad (j=0,1,2,\dots).\tag{19}$$

## 4. Solutions of the moment problem

**Theorem 4.1.** *The limit*

$$\mathcal{R}(\lambda)=\lim_{n\to\infty}\left(\sum_{k=0}^{n}P_k(\lambda)P_k(\lambda)^*\right)^{-1}$$

*converges for each $\lambda\in\mathbb{C}\setminus\{0\}$ and its rank satisfies the condition*

$$\operatorname{rank}\mathcal{R}(\lambda)=m_\pm\quad (\lambda\in\mathbb{C}_\pm),$$

*where $(m_+,m_-)$ are the deficiency indices of the operator $A$. The deficiency subspace $\mathfrak{N}_\lambda$ of the operator $A$ has the form*

$$\mathfrak{N}_\lambda=\left\{f_{\lambda,\phi}(z)=\sum_{k=0}^\infty P_k(z)P_k(\overline{\lambda})^*\phi:\ \phi\in\mathbb{C}^N\ominus\ker\mathcal{R}(\overline{\lambda})\right\}\quad (\lambda\in\mathbb{C}\setminus\{0\}).$$

*Remark* 4.1.1. For the classical matrix moment problem this theorem has been proved by M. G. Kreĭn [15] (see also [2]).

*Proof.* To prove this statement, we only need to show that the deficiency space $\mathfrak{N}_\lambda$ of the operator $A$ consists of the vectors

$$f_{\lambda,\phi}(z)=\sum_{k=0}^\infty P_k(z)P_k(\overline{\lambda})^*\phi\quad (\phi\in\mathbb{C}^N)$$

satisfying the condition

$$\sum_{k=0}^\infty \phi^*P_k(\overline{\lambda})P_k(\overline{\lambda})^*\phi<\infty.$$

Let us find the domain of the adjoint operator $A^*$. A vector $f \in \mathfrak{H}$ belongs to $\operatorname{dom} A^*$ if and only if there exists a vector $g \in \mathfrak{H}$ such that

$$(f(z), AP_k(z)\epsilon_j) = (g(z), P_k(z)\epsilon_j) \quad (k = 0, 1, 2, \ldots,\ j = 1, \ldots, N). \tag{20}$$

In this case $f \in \operatorname{dom} A^*$ and $g = Af$. Suppose that the vectors $f(z)$ and $g(z)$ have the form

$$f(z) = \sum_{k=0}^{\infty} P_k(z) f_k, \quad g(z) = \sum_{k=0}^{\infty} P_k(z) g_k.$$

Then the equalities (20) are equivalent to

$$g_k = C_{k-2} f_{k-2} + B_{k-1} f_{k-1} + A_k f_k + B_k^* f_{k+1} + C_k^* f_{k+2} \quad (k = 0, 1, \ldots,\ f_{-2} = f_{-1} = 0).$$

The vector $g$ belongs to $\mathfrak{H}$ if and only if

$$\|g\|^2 = \sum_{k=0}^{\infty} \|C_{k-2} f_{k-2} + B_{k-1} f_{k-1} + A_k f_k + B_k^* f_{k+1} + C_k^* f_{k+2}\|_{\mathbb{C}^N}^2 < \infty. \tag{21}$$

Thus a vector $f \in \mathfrak{H}$ belongs to $\operatorname{dom} A^*$ if and only if its Fourier coefficients $\{f_k\}_0^\infty$ satisfy condition (21).

We are now ready to find the deficiency subspace

$$\mathfrak{N}_\lambda = \mathfrak{H} \ominus \operatorname{ran}(A - \overline{\lambda}) = \ker(A^* - \lambda).$$

If a vector $f \in \mathfrak{H}$ of the form

$$f(z) = \sum_{k=0}^{\infty} P_k(z) f_k$$

belongs to $\ker(A^* - \lambda)$, then the coefficients $f_k$ obey (18). By Corollary 3.5.1, it follows that the vector $f$ has the form

$$f(z) = \sum_{k=0}^{\infty} P_k(z) P_k(\overline{\lambda})^* f_0 \quad (f_0 \in \mathbb{C}^N). \tag{22}$$

Conversely, if a vector $f$ has the form (22) and obeys the condition

$$\|f\|^2 = \sum_{k=0}^{\infty} \left\|P_k(\overline{\lambda})^* f_0\right\|_{\mathbb{C}^N}^2 < \infty, \tag{23}$$

then $f$ belongs to $\mathfrak{N}_\lambda = \ker(A^* - \lambda)$. □

**Corollary 4.1.2.** *If $A$ has deficiency indices $(N, N)$, then $A$ is a simple operator.*

By Theorem 4.1 and Corollary 3.2.1, one obtains the following necessary and sufficient condition for the uniqueness of the solutions of (1).

**Theorem 4.2.** *The moment problem* (1) *has a unique solution if and only if $\mathcal{R}(\lambda) = 0$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Otherwise the moment problem* (1) *has infinitely many solutions.*

**Definition 4.1.** The moment problem (1) is called *completely indeterminate* if the operator $A$ has deficiency indices $(N, N)$.

The moment problem (1) is completely indeterminate if and only if $\ker \mathcal{R}(\lambda) = \{0\}$ for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In the sequel, we assume that the moment problem (1) is completely indeterminate.

**Proposition 4.3.** *If the moment problem* (1) *is completely indeterminate, then the condition*

$$\mathfrak{H}_0 \cap \overline{\operatorname{ran}}(A - \lambda) = \{0\} \quad (\lambda \in \mathbb{C} \setminus \{0\})$$

*holds.*

*Proof.* Suppose that $\alpha$ is the angle between $\mathfrak{H}_0$ and $\overline{\operatorname{ran}}(A-\lambda)$. We claim that $\alpha > 0$. Indeed,

$$\sin\alpha = \inf_{\substack{\|\phi\|_{\mathbb{C}^N}=1\\ f\in\operatorname{dom} A}} \{\|\phi-(A-\lambda)f\|\} \geq \inf_{\substack{\|\phi\|_{\mathbb{C}^N}=1\\ g(\lambda)=0}} \{\|\phi-g\|\} = \inf_{\substack{\|\phi\|_{\mathbb{C}^N}=1\\ h(\lambda)=\phi}} \{\|h\|\}$$

$$= \inf_{h\in\mathfrak{H}} \left\{\frac{\|h\|}{\|h(\lambda)\|_{\mathbb{C}^N}}\right\} = \inf_{\substack{\{h_k\}_0^m\subset\mathbb{C}^N\\ m=0,1,2,\dots}} \left\{\frac{\left(\sum_{k=0}^m \|h_k\|^2_{\mathbb{C}^N}\right)^{\frac{1}{2}}}{\|\sum_{k=0}^m P_k(\lambda)h_k\|_{\mathbb{C}^N}}\right\}.$$

Using Cauchy's inequality, we obtain

$$\sin\alpha \geq \inf_{m=0,1,2,\dots} \left\{\frac{1}{\left(\sum_{k=0}^m \|P_k(\lambda)\|^2_{\mathbb{C}^N}\right)^{\frac{1}{2}}}\right\} = \frac{1}{\left(\sum_{k=0}^\infty \|P_k(\lambda)\|^2_{\mathbb{C}^N}\right)^{\frac{1}{2}}} > 0.$$

□

Let us take the subspace $\mathfrak{L} = \mathfrak{H}_0$ as the module of a representation. By Proposition 4.3, the decomposition

$$\mathfrak{H} = \overline{\mathfrak{M}}_\lambda \dotplus \mathfrak{H}_0 \quad (\lambda \in \mathbb{C}\setminus\{0\})$$

holds. Denote by $\mathcal{P}(\lambda)$ the skew projection onto $\mathfrak{H}_0$ parallel $\mathfrak{M}_\lambda$ in the space $\mathfrak{H}$. Put

$$\mathcal{Q}(\lambda) = P_{\mathfrak{H}_0}(A-\lambda)^{-1}(I-\mathcal{P}(\lambda)).$$

Then any vector

$$f(z) = \sum_{k=0}^\infty P_k(z)f_k \in \mathfrak{H}$$

obeys

$$\mathcal{P}(\lambda)f = f(\lambda) = \sum_{k=0}^\infty P_k(\lambda)f_k, \quad \mathcal{Q}(\lambda)f = \sum_{k=0}^\infty Q_k(\lambda)f_k.$$

**Proposition 4.4.** *The set of $\mathfrak{H}_0$-regular points of $A$ coincides with the domain $\mathbb{C}\setminus\{0\}$.*

*Proof.* This statement follows from Theorem 2.1 for $L = \mathring{\mathfrak{H}}$. □

**Proposition 4.5.** *The following equalities hold:*

$$(\mathcal{P}(\lambda)^*\phi)(z) = \sum_{k=0}^\infty P_k(z)P_k(\lambda)^*\phi,$$

$$(\mathcal{Q}(\lambda)^*\phi)(z) = \sum_{k=0}^\infty P_k(z)Q_k(\lambda)^*\phi$$

$$(\phi\in\mathfrak{H}_0).$$

*Proof.* Let us prove the first equality, the second equality can be proved similarly. Expand the vector $\mathcal{P}(\lambda)^*\phi$ as

$$\mathcal{P}(\lambda)^*\phi = \sum_{k=0}^\infty P_k(z)f_k.$$

Then the coefficients $f_k$ are determined from the equalities

$$\epsilon_j^* f_k = (\mathcal{P}(\lambda)^*\phi, P_k(z)\epsilon_j) = (\phi, \mathcal{P}(\lambda)P_k(z)\epsilon_j) = (\phi, P_k(\lambda)\epsilon_j)_{\mathbb{C}_N} = \epsilon_j^* P_k(\lambda)^*\phi$$

$$(k = 0,1,2,\dots,\ j = 1,\dots,N).$$

□

**Theorem 4.6.** *The adjoint operator $A^*$ has the form*

$$
\begin{gathered}
\operatorname{dom} A^* = \operatorname{dom} A \dotplus \mathcal{P}(\lambda)^*\mathfrak{H}_0 \dotplus \mathcal{Q}(\lambda)^*\mathfrak{H}_0,\\
A^* f = A f_0 + \overline{\lambda}\mathcal{P}(\lambda)^*\phi + \overline{\lambda}\mathcal{Q}(\lambda)^*\psi + \psi\\
(\lambda \in \mathbb{C}\setminus\{0\})
\end{gathered}
\tag{24}
$$

*where*

$$f = f_0 + \mathcal{P}(\lambda)^*\phi + \mathcal{Q}(\lambda)^*\psi \quad (f_0 \in \operatorname{dom} A,\ \phi,\psi \in \mathfrak{H}_0).$$

*The deficiency subspace $\mathfrak{N}_\lambda$ has the form*

$$\mathfrak{N}_\lambda = \Big\{ f_{\lambda,\phi}(z) = \sum_{k=0}^{\infty} P_k(z)P_k(\overline{\lambda})^*\phi :\ \phi \in \mathfrak{H}_0 \Big\}.$$

*Proof.* Using Proposition 2.2, we obtain the form of $A^*$. Using Theorem 4.1, we get the form of the defect subspace $\mathfrak{N}_\lambda$. □

**Theorem 4.7.** *Suppose $a \in \mathbb{R}\setminus\{0\}$. The triplet $\Pi = \{\mathfrak{H}_0, \Gamma_0, \Gamma_1\}$ given by*

$$\Gamma_0 f = \mathcal{P}(a)(A^* - a)f = -\psi, \quad \Gamma_1 f = P_{\mathfrak{H}_0} f - \mathcal{Q}(a)(A^* - a)f = \phi,$$

*and*

$$f = f_0 + \mathcal{P}(a)^*\phi + \mathcal{Q}(\lambda)^*\psi \in \operatorname{dom} A^* \quad (f_0 \in \operatorname{dom} A,\ \phi,\psi \in \mathfrak{H}_0)$$

*is a boundary triplet for the operator $A^*$. The corresponding Weyl function $M(\lambda)$ has the form*

$$M(\lambda) = \Big( I - (\lambda - a)\sum_{k=0}^{\infty} Q_k(a)P_k(\overline{\lambda})^* \Big)\Big( (\lambda - a)\sum_{k=0}^{\infty} P_k(a)P_k(\overline{\lambda})^* \Big)^{-1}. \tag{25}$$

*Proof.* Let us check that the equalities

$$-\mathcal{P}(a)(A^* - a)f = -\psi, \quad P_{\mathfrak{H}_0} f - \mathcal{Q}(a)(A^* - a)f = \phi$$

hold for a vector

$$f = f_0 + \mathcal{P}(a)^*\phi + \mathcal{Q}(a)^*\psi \in \operatorname{dom} A^* \quad (f_0 \in \operatorname{dom} A,\ \phi,\psi \in \mathfrak{H}_0).$$

Indeed,

$$
\begin{gathered}
-\mathcal{P}(a)(A^* - a)f = -\mathcal{P}(a)((A - a)f_0 + \psi) = -\psi,\\
P_{\mathfrak{H}_0} f - \mathcal{Q}(a)(A^* - a)f = P_{\mathfrak{H}_0} f_0 + \phi - \mathcal{Q}(a)((A - a)f_0 + \psi) = \phi.
\end{gathered}
$$

It is clear that $\Gamma = \{\Gamma_0, \Gamma_1\}$ is surjective and obeys the Green identity (5). Therefore $\Pi$ is a boundary triplet for $A^*$.

Now let us find the Weyl function $M(\lambda)$. Taking $f_{\lambda,\phi}(z) = \mathcal{P}(\overline{\lambda})^*\phi \in \mathfrak{N}_\lambda$, we get

$$
\begin{aligned}
\Gamma_0 f_{\lambda,\phi}(z) &= \mathcal{P}(a)(A^* - a)\mathcal{P}(\overline{\lambda})^*\phi = (\lambda - a)\mathcal{P}(a)\mathcal{P}(\overline{\lambda})^*\phi = (\lambda - a)\sum_{k=0}^{\infty} P_k(a)P_k(\overline{\lambda})^*\phi,\\
\Gamma_1 f_{\lambda,\phi}(z) &= (P_{\mathfrak{H}_0} - \mathcal{Q}(a)(A^* - a))\mathcal{P}(\overline{\lambda})^*\phi = I - (\lambda - a)\mathcal{Q}(a)\mathcal{P}(\overline{\lambda})^*\phi\\
&= I - (\lambda - a)\sum_{k=0}^{\infty} Q_k(a)P_k(\overline{\lambda})^*\phi.
\end{aligned}
$$

Thus the Weyl function $M(\lambda)$ has the form (25). □

**Theorem 4.8.** *Let $\Pi = \{\mathfrak{L}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $A^*$ defined in Theorem 4.7. Then the matrix function*

$$W(\lambda) = \begin{pmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} I + (\lambda - a)\sum_{j=0}^{\infty} Q_j(\lambda)P_j(a)^* & (\lambda - a)\sum_{j=0}^{\infty} Q_j(\lambda)Q_j(a)^* \\ -(\lambda - a)\sum_{j=0}^{\infty} P_j(\lambda)P_j(a)^* & I - (\lambda - a)\sum_{j=0}^{\infty} P_j(\lambda)Q_j(a)^* \end{pmatrix} \tag{26}$$

*is the corresponding $\Pi\mathfrak{L}$-resolvent matrix.*

*Proof.* Using Theorem 2.5, Proposition 4.5, and Theorem 4.7, we get

$$\begin{aligned}
w_{11}(\lambda)^* &= -\Gamma_0 \mathcal{Q}(\lambda)^* = \mathcal{P}(a)(A^* - a)\mathcal{Q}(\lambda)^* = I + (\overline{\lambda} - a)\mathcal{P}(a)\mathcal{P}(\lambda)^* \\
&= I + (\overline{\lambda} - a)\sum_{k=0}^{\infty} P_k(a)Q_k(\lambda)^*, \\
w_{12}(\lambda)^* &= -\Gamma_1 \mathcal{Q}(\lambda)^* = -(P_{\mathfrak{H}_0} - \mathcal{Q}(a)(A^* - a))\mathcal{Q}(\lambda)^* = (\overline{\lambda} - a)\mathcal{Q}(a)\mathcal{Q}(\lambda)^* \\
&= (\overline{\lambda} - a)\sum_{k=0}^{\infty} Q_k(a)Q_k(\lambda)^*, \\
w_{21}(\lambda)^* &= \Gamma_0 \mathcal{P}(\lambda)^* = -\mathcal{P}(a)(A^* - a)\mathcal{P}(\lambda)^* = -(\overline{\lambda} - a)\mathcal{P}(a)\mathcal{P}(\lambda)^* \\
&= -(\overline{\lambda} - a)\sum_{k=0}^{\infty} P_k(a)P_k(\lambda)^*, \\
w_{22}(\lambda)^* &= \Gamma_1 \mathcal{P}(\lambda)^* = (P_{\mathfrak{H}_0} - \mathcal{Q}(a)(A^* - a))\mathcal{P}(\lambda)^* = I - (\overline{\lambda} - a)\mathcal{Q}(a)\mathcal{P}(\lambda)^* \\
&= I - (\overline{\lambda} - a)\sum_{k=0}^{\infty} Q_k(a)P_k(\lambda)^*.
\end{aligned}$$

□

*Remark* 4.8.1. The proofs of Theorems 4.7 and 4.8 are close to those in [6] in the case of the classical Hamburger moment problem.

**Theorem 4.9** ([20])**.** *The matrix function* (26) *is holomorphic in $\mathbb{C} \setminus \{0\}$ and has the minimal exponential type at its points of singularity $\lambda = 0, \infty$, i. e.,*

$$\lim_{\lambda \to \infty} \frac{\log \|W(\lambda)\|_{\mathbb{C}^{2N}}}{|\lambda|} = 0,$$
$$\lim_{\lambda \to 0} |\lambda| \log \|W(\lambda)\|_{\mathbb{C}^{2N}} = 0.$$

Using Theorems 2.6, 3.2, and 4.8, we obtain our main result.

**Theorem 4.10.** *There exists a one-to-one correspondence between the set of all solutions $d\Sigma$ of the moment problem* (1) *and the set of all functions $\tau \in \widetilde{N}_{\mathfrak{H}_0}$. The correspondence is given by the following Nevanlinna type formula*

$$\int_{-\infty}^{+\infty} \frac{d\Sigma(t)}{t - \lambda} = (w_{11}(\lambda)\tau(\lambda) + w_{12}(\lambda))\,(w_{21}(\lambda)\tau(\lambda) + w_{22}(\lambda))^{-1},$$

*where the functions $(w_{ij}(\lambda))_1^2$ are defined by* (26).

*Remark* 4.10.1. For the scalar case, Theorem 4.10 was proved by O. Njåstad (see [19]). However the result of O. Njåstad is restricted to bisequences $\{S_k\}_{-\infty}^{+\infty}$ that give rise to regular sequences $\{P_k(z)\}_0^{\infty}$. Since we do not use the regularity condition, we strengthen the result of O. Njåstad even for the scalar case.

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Department of Mathematics, Donets'k National University, Donets'k, Ukraine
*E-mail address*: xi@gamma.dn.ua

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