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ON $*$-WILDNESS OF A FREE PRODUCT OF FINITE-DIMENSIONAL C^*-ALGEBRAS

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Abstract. In this paper, we study the complexity of representation theory of free products of finite-dimensional C^*-algebras.

0. Introduction

The presented paper is devoted to a study of representation theory for free products of finite-dimensional C^*-algebras.

The free products of algebras (C^*-algebras, groups, etc.) appear naturally in group theory, non-commutative probability, operator theory, etc., see [10] for an introduction to the subject.

These objects are typically complicated from the combinatorial point of view. For example, the free group with two generators, denoted below by $\mathcal{F}_2$, has an exponential growth. At the same time, the representation theory of $\mathcal{F}_2$ is also quite complicated. In particular, it was shown in [5] that the problem of unitary classification of irreducible representations of $\mathcal{F}_2$ contains, as a subproblem, the problem of unitary classification of irreducible representations of any finitely generated $*$-algebra. Motivated by this fact, the authors in [4, 5] have introduced a notion of $*$-wild $*$-algebra (C^*-algebra). Informally, a C^*-algebra is called $*$-wild if the problem of unitary classification of its $*$-representations contains as a subproblem the problem of classification of $*$-representations of $\mathcal{F}_2$; for a rigorous definition, see Preliminaries.

The aim of this paper is to give a criterion for $*$-wildness of free products of finite-dimensional C^*-algebras. This problem is the first step in the direction of proving a natural conjecture that "almost all" free products of C^*-algebras are $*$-wild.

1. Preliminaries

1.1. **$*$-Representation types: $*$-wild and $*$-tame C^*-algebras.** In this subsection we give some facts and definitions related to $*$-wild and $*$-tame algebras. All C^*-algebras are supposed to be unital and their representations are unital $*$-homomorphisms into $B(H)$, the algebra of all bounded linear operators on a Hilbert space H.

Let $\mathcal{A}$ be a C^*-algebra. Denote by $M_n(\mathcal{A})$ the C^*-algebra $M_n(\mathbb{C})\otimes\mathcal{A}$ with the natural $*$-structure and a unique C^*-norm. Let H be a Hilbert space and $\pi : \mathcal{A} \to B(H)$ be a $*$-homomorphism. It induces the homomorphism $\pi_n = \mathrm{id}_n \otimes \pi : M_n(\mathcal{A}) \to B(H^n)$.

In what follows we let $\operatorname{Rep}\mathcal{A}$ denote the category of all $*$-representations of $\mathcal{A}$.

Definition 1. (see [9]) *A C^*-algebra $\mathcal{B}$ majorizes a C^*-algebra $\mathcal{A}$ ($\mathcal{B} \succ \mathcal{A}$), if there exists a unital $*$-homomorphism $\psi : \mathcal{B} \to M_n(\mathcal{A})$ such that the functor $F_\psi : \operatorname{Rep}\mathcal{A} \to \operatorname{Rep}\mathcal{B}$*

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defined by

$$F_\psi(\pi) = \pi_n \circ \psi, \quad \forall \pi \in \operatorname{Rep} \mathcal{A}, \tag{1}$$

$$F_\psi(A) = I \otimes A, \text{ for any operator } A \text{ intertwining } \pi_1 \text{ and } \pi_2 \tag{2}$$

is full.

Note that in oder to verify that the functor F_ψ is full, it is sufficient to show that, for every $\pi \in \operatorname{Rep} \mathcal{A}$ in $B(H)$ and any operator $A \in B(H^n)$, the inclusion $A \in F_\psi(\pi)(\mathcal{B})'$ implies that $A = \operatorname{diag}(a, a, \ldots, a)$ and $a \in \pi(\mathcal{A})'$, see [9] for details; here $\pi(\mathcal{A})'$ denotes the commutant of $\pi(\mathcal{A})$. In particular, the representations $F_\psi(\pi_1)$ and $F_\psi(\pi_2)$ of $\mathcal{B}$ are unitary equivalent if and only if the representations π_1 and π_2 of $\mathcal{A}$ are unitary equivalent, the representation $F_\psi(\pi)$ is irreducible if and only if π is irreducible. If $\pi = \bigoplus_\lambda \pi_\lambda$, where π_λ is irreducible, then $F_\psi(\pi) = \bigoplus_\lambda F_\psi(\pi_\lambda)$. Thus the problem of unitary classification of representations of the C^*-algebra $\mathcal{B}$ contains, as a subproblem, the problem of unitary classification of representations of the C^*-algebra $\mathcal{A}$. Note also that the relation " $\succ$ " is a quasi-order, i.e., $\mathcal{A} \succ \mathcal{B}$ and $\mathcal{B} \succ \mathcal{C}$ imply $\mathcal{A} \succ \mathcal{C}$.

In the sequel, if G_1, G_2 are groups and $C^*(G_1)$, $C^*(G_2)$ are their group C^*-algebras, see Sec. 1.2, we will write $G_1 \succ G_2$ instead of $C^*(G_1) \succ C^*(G_2)$.

The next definition is based on the fact that $C^*(\mathcal{F}_2)$ majorizes any finitely generated $*$-algebra, see [4, 5].

Definition 2. *A C^*-algebra $\mathcal{B}$ is called $*$-wild if $\mathcal{B} \succ C^*(\mathcal{F}_2)$.*

For properties of $*$-wild algebras and a number of examples of $*$-wild algebras, we refer to [9, 6, 7, 2].

Similarly to the theory of finite-dimensional algebras one can introduce the notion of a C^*-algebra of finite representation type ($*$-finite algebras) and tame C^*-algebras (C^*-algebras of type **1**). For convenience of the reader, we recall formal definitions.

Definition 3. *A $*$-algebra is called $*$-finite if it has only finitely many unitary non-equivalent irreducible representations.*

For the following definition see, for example [1].

Definition 4. *A C^*-algebra is called of type* **1** *if the C^*-algebra generated by any of its irreducible representation contains the algebra of compact operators.*

Definition 5. *A C^*-algebra is called $*$-tame if it is not $*$-finite and is of type* **1**.

1.2. **Enveloping C^*-algebras.** In this subsection we recall the definitions of an enveloping C^*-algebra and the group C^*-algebra, which can be found, for example, in [1], [3].

Let $\mathcal{A}$ be a $*$-algebra. The C^*-algebra $\widetilde{\mathcal{A}}$ with a $*$-homomorphism $\varphi : \mathcal{A} \mapsto \widetilde{\mathcal{A}}$ is called an enveloping C^*-algebra of the algebra $\mathcal{A}$ if for every representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ there exists a unique representation $\widetilde{\pi} : \widetilde{\mathcal{A}} \to \mathcal{B}(\mathcal{H})$ of $\widetilde{\mathcal{A}}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "\varphi"'] \arrow[dr, "\pi"] & \\
\widetilde{\mathcal{A}} \arrow[r, "\widetilde{\pi}"'] & \mathcal{B}(\mathcal{H})
\end{tikzcd}$$

A group C^*-algebra of a group G is an enveloping C^*-algebra of the group ring $\mathbb{C}[G]$. More precisely, a group C^*-algebra, $C^*(G)$, is the completion of $\mathbb{C}[G]$ with respect to the C^*-norm

$$\|a\| = \sup\{\pi(a) : \pi \in \mathrm{Rep}(\mathbb{C}[G])\}.$$

1.3. **C^*-free products.** Finally we recall the definition of the free product of a family of C^*-algebras, see [10] for details.

Definition 6. *If $(\mathcal{A}_i)_{i\in I}$ is a family of unital C^*-algebras, then their free product is the unique unital C^*-algebra $\mathcal{A} = *_{i\in I}\mathcal{A}_i$ and the unital $*$-homomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that, given any unital C^*-algebra $\mathcal{B}$ and unital $*$-homomorphisms $\Phi_i : \mathcal{A}_i \to \mathcal{B}$, there exist a unique unital $*$-homomorphism $\Phi = *_{i\in I}\Phi_i : \mathcal{A} \to \mathcal{B}$ making the following diagram commutative:*

$$\begin{tikzcd} \mathcal{A}_i \arrow[d, "\psi_i"'] \arrow[dr, "\Phi_i"] & \\ \mathcal{A} \arrow[r, "\Phi"'] & \mathcal{B} \end{tikzcd}$$

The definition of the free product of groups (algebras) is absolutely analogous. However, in the case of groups, one can describe the free product in terms of generators and relations. Namely, let groups G_i, $i \in I$, be defined by generators and relations,

$$G_i = \langle g_{li} \mid P_{ji} = e,\ j \in J_i,\ l \in M_i\rangle,$$

where J_i, M_i are some sets and P_{ji} are words in the alphabet $g_{li},\ g_{li}^{-1}$. Then

$$*_{i\in I}G_i = \langle g_{li} \mid P_{ji} = e,\ j \in J_i,\ l \in M_i,\ i \in I\rangle.$$

2. C^*-free product of finite-dimensional C^*-algebras

In this section we show that a free product of finite-dimensional C^*-algebras is either $*$-tame or $*$-wild. Moreover, we give a criterion for $*$-wildness of such free products.

It is known, see for example [8], that every finite-dimensional C^*-algebra is $*$-isomorphic to $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ for some natural numbers n_1, n_2, $\dots$, n_k.

1. Firstly we prove $*$-wildness of the C^*-free product of two matrix-algebras.

Proposition 1. *The C^*-algebra $M_k(\mathbb{C}) * M_n(\mathbb{C})$ is $*$-wild for every $k > 1$, $n > 1$.*

Proof. Let $(e_{ij})_{i,j=1}^k$ and $(e'_{ij})_{i,j=1}^k$ be matrix units of the algebras $M_k(\mathbb{C})$ and $M_n(\mathbb{C})$, and let $\mathcal{F}_2$ be a free group with two generators u and v. Denote by $L := LCM(n,k)$. Define a $*$-homomorphism $\psi : M_k(\mathbb{C}) * M_n(\mathbb{C}) \to M_L(C^*(\mathcal{F}_2))$ as follows:

$$\psi(e_{12}) = e_{12} \otimes v \oplus \bigoplus_{1}^{\frac{1}{k}L-1} e_{12} \otimes e,$$

$$\psi(e_{ii+1}) = \bigoplus_{1}^{\frac{1}{k}L} e_{ii+1} \otimes e, \qquad \text{for } i \geq 2;$$

$$\psi(e'_{12}) = T\Bigg(e'_{12} \otimes u \oplus \bigoplus_{1}^{\frac{1}{n}L-1} e_{12} \otimes e\Bigg)T^*,$$

$$\psi(e'_{ii+1}) = T\Big(\bigoplus_{1}^{\frac{1}{n}L} e'_{ii+1}\otimes e\Big)T^*, \qquad \text{for } i\ge 2,$$

$$\text{here} \qquad T = \begin{pmatrix} 1 & -\tan x \\ \tan x & 1 \end{pmatrix}\otimes e \oplus I_{L-2}\otimes e, \quad x\in\big(0,\frac{\pi}{2}\big).$$

Consider a representation π of $\mathcal{F}_2$ and an operator $A = \{a_{ij}\}_{i,j=1}^{k}$ that commutes with each $(\mathrm{id}_L\otimes\pi)(\psi(e_{ij}))$, $i,j=1,\dots,k$ and $(\mathrm{id}_L\otimes\pi)(\psi(e'_{ij}))$, $i,j=1,\dots,n$. To verify that the functor generated by ψ, F_ψ, is full it is sufficient to show that $A = \mathrm{diag}(c,c,\dots,c)$ and $[c,\pi(u)] = [c,\pi(v)] = 0$. Commutativity of A with $(\mathrm{id}_L\otimes\pi)(\psi(e_{ij}))$, $i,j=1,\dots,k$ and $(\mathrm{id}_L\otimes\pi)(\psi(e'_{ij}))$, $i,j=3,\dots,n$ makes A diagonal. By commuting A with $(\mathrm{id}_L\otimes\pi)(\psi(e_{ii+1}))$, $i=2,\dots,k$ and $(\mathrm{id}_L\otimes\pi)(\psi(e'_{ii+1}))$, $i=1,\dots,n$, one can check that $A=\mathrm{diag}(a_{11},a_{11},\dots,a_{11})$ and $[a_{11},\pi(u)] = [a_{11},\pi(v)] = 0$ proving the fullness of the functor F_ψ. ☐

2. In the following proposition we study free products of $\mathbb{C}^k$, $k\ge 2$.

Proposition 2.

(1) *The C^*-algebra $\mathbb{C}^n * \mathbb{C}^k$ is $*$-wild iff $n\ge 2$, $k\ge 3$, $n,k\in\mathbb{N}$.*
(2) *The C^*-algebra $\mathbb{C}^2*\mathbb{C}^2*\mathbb{C}^2$ is $*$-wild.*

Proof. Let $(G_i)_{i\in I}$ be a family of groups. Then a unitary representation of the free product $*_{i\in I}G_i$ is determined by its restrictions to the groups G_i. Thus, the universal property of the C^*-free product, i.e., the property that representations of $C^*(G)$ correspond to unitary representations of G, implies that

$$C^*(*_{i\in I}G_i) = *_{i\in I}C^*(G_i).$$

Hence, we get the following isomorphisms:

$$\mathbb{C}^n*\mathbb{C}^k \simeq C^*(\mathbb{Z}_n*\mathbb{Z}_k),$$
$$\mathbb{C}^2*\mathbb{C}^2*\mathbb{C}^2 \simeq C^*(\mathbb{Z}_2*\mathbb{Z}_2*\mathbb{Z}_2).$$

As it was proved in [9], $C^*(\mathbb{Z}_2*\mathbb{Z}_2*\mathbb{Z}_2)$ is $*$-wild and $C^*(\mathbb{Z}_n*\mathbb{Z}_k)$ is $*$-wild if and only if $n\ge 2$, $k\ge 3$, which yields the statement of the proposition. ☐

3. It remains to consider the case $\mathbb{C}^k * M_n(\mathbb{C})$.

Proposition 3. *The C^*-algebra $\mathbb{C}^n * M_k(\mathbb{C})$ is $*$-wild iff $n\ge 2$, $k\ge 2$, $n,k\in\mathbb{N}$.*

Proof. Let $(e_{ij})_{i,j=1}^k$ be matrix units of the algebras $M_k(\mathbb{C})$, and let z be a generator of $C^*(\mathbb{Z}_n)\simeq\mathbb{C}^n$. It is known, see [9], that the group $\mathbb{Z}_2*\mathbb{Z}_3 =< u,v | v^2 = u^3 = e >$ is $*$-wild. To prove the statement of the proposition we are going to construct a $*$-homomorphism for sufficient large $l\in\mathbb{N}$,

$$\varphi : \mathbb{C}^n * M_k(\mathbb{C}) \to M_l(C^*(\mathbb{Z}_2*\mathbb{Z}_3)).$$

a). Let $n=k=2$. Define a $*$-homomorphism $\varphi:\mathbb{C}^2*M_2(\mathbb{C})\to M_4(C^*(\mathbb{Z}_2*\mathbb{Z}_3))$ by the following:

$$\varphi(z) = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & 0 & -e \end{pmatrix}, \quad \varphi(e_{12}) = T\begin{pmatrix} 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix}T^*,$$

$$\text{here} \qquad T = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & -e\tan x \\ 0 & 0 & e\tan x & e \end{pmatrix}, \quad x\in\big(0,\pi/2\big).$$

One can directly check that the functor F_φ generated by φ is full. Thus, $\mathbb{C}^2 * M_2(\mathbb{C})$ is $*$-wild.

b). Let $n = 2$, $k \geq 3$. Define a $*$-homomorphism $\varphi : \mathbb{C}^2 * M_k(\mathbb{C}) \to M_k(C^*(\mathbb{Z}_2 * \mathbb{Z}_3))$ as follows:

$$\varphi(z) = \begin{pmatrix} 0 & u^{-1} & 0 \\ u & 0 & 0 \\ 0 & 0 & v \end{pmatrix} \oplus (I_{k-3} \otimes e), \quad \varphi(e_{ij}) = e_{ij} \otimes e, \quad i, j = 1, \dots, k.$$

Fix a representation π of $\mathbb{Z}_2 * \mathbb{Z}_3$. Let $A = \{a_{ij}\}_{i,j=1}^2$ commute with each $(\mathrm{id}_k \otimes \pi)(\varphi(e_{ij}))$, $i, j = 1, \dots, k$, and $(\mathrm{id}_k \otimes \pi)(\varphi(z))$. Then commutativity of A with $(\mathrm{id}_k \otimes \pi)(\varphi(e_{ij}))$ implies $A = \mathrm{diag}(a_{11}, \dots, a_{11})$ and that with $(\mathrm{id}_k \otimes \pi)(\varphi(z))$ gives $[a_{11}, \pi(u)] = [a_{11}, \pi(v)] = 0$, proving the fullness of the functor F_φ.

c). Let $n \geq 3$, $k \geq 2$. Define a $*$-homomorphism $\varphi : \mathbb{C}^n * M_k(\mathbb{C}) \to M_k(C^*(\mathbb{Z}_n * \mathbb{Z}_n))$ by the following:

$$\varphi(z) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \oplus (I_{k-2} \otimes e), \quad \varphi(e_{ij}) = e_{ij} \otimes e, \quad i, j = 1, \dots, k.$$

One can directly check using similar arguments as in **b).** that the functor F_φ generated by φ is full. Since $C^*(\mathbb{Z}_n * \mathbb{Z}_n)$ is $*$-wild, see [9], we have that $\mathbb{C}^n * M_k(\mathbb{C})$ is also $*$-wild for $n \geq 3$, $k \geq 2$. □

Now we combine the results of the previous propositions to get a criterion for $*$-wildness of a C^*-free products of finite-dimensional C^*-algebras.

Theorem 1. *A C^*-free product $*_{i\in I}\mathcal{A}_i$ of a family of finite-dimensional C^*-algebras $(\mathcal{A}_i)_{i\in I}$ is $*$-wild if and only if one of the following conditions is hold:*

(1) *There exist i_1, $i_2 \in I$ such that $\mathcal{A}_{i_1}$ has $M_k(\mathbb{C})$ and $\mathcal{A}_{i_2}$ has $M_n(\mathbb{C})$ as direct summands with $k > 1$, $n > 1$.*
(2) *There exist i_1, $i_2 \in I$ such that $\mathcal{A}_{i_1}$ has $\mathbb{C}^n$ and $\mathcal{A}_{i_2}$ has $\mathbb{C}^k$ as direct summands with $n \geq 2$, $k \geq 3$.*
(3) *There exist i_1, i_2, $i_3 \in I$ such that $\mathcal{A}_{i_1}$ has $\mathbb{C}^2$, $\mathcal{A}_{i_2}$ has $\mathbb{C}^2$, and $\mathcal{A}_{i_3}$ has $\mathbb{C}^2$ as direct summands.*
(4) *There exist i_1, $i_2 \in I$ such that $\mathcal{A}_{i_1}$ has $\mathbb{C}^n$ and $\mathcal{A}_{i_2}$ has $M_k(\mathbb{C})$ as direct summands with $n \geq 2$, $k \geq 2$.*

Proof. In fact, the $*$-wildness of the algebras specified in the statements was proved in Propositions 1–3. The C^*-algebras which are left in the list of the considered C^*-free products are the following:

(1) $\mathbb{C}^2 * \mathbb{C}^2$,
(2) a C^*-free product of any number of C^*-algebras $\mathbb{C}$ and one C^*-algebra $M_n(\mathbb{C})$.

$*$-representations of C^*-algebra $\mathbb{C}^2 * \mathbb{C}^2$ are described in [9], it is a $*$-tame algebra, and the algebra from the second claim is evidently of type **1**. □

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