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UNIFORM EQUICONTINUITY FOR SEQUENCES OF HOMOMORPHISMS INTO THE RING OF MEASURABLE OPERATORS

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Abstract. We introduce a notion of uniform equicontinuity for sequences of functions with the values in the space of measurable operators. Then we show that all the implications of the classical Banach Principle on the almost everywhere convergence of sequences of linear operators remain valid in a non-commutative setting.

0. Introduction

Let (Ω, Σ, μ) be a probability space. Denote by $\mathcal{L} = \mathcal{L}(\Omega, \Sigma, \mu)$ the set of all (classes of) complex-valued measurable functions on Ω. Let τ_μ stand for the measure topology in $\mathcal{L}$. The classical Banach Principle may be stated as follows.

Let $(X, \|\cdot\|)$ be a Banach space, and let $a_n : (X, \|\cdot\|) \to (\mathcal{L}, \tau_\mu)$ be a sequence of continuous linear maps. Consider the following properties of the sequence $\{a_n\}$:

(i) $\{a_n(x)\}$ converges almost everywhere (a.e.) for every $x \in X$;

(ii) $a^\star(x)(\omega) = \sup\limits_n |a_n(x)(\omega)| < \infty$ a.e.;

(iii) $a^\star(x)(\omega) < \infty$ a.e., and the maximal operator $a^\star : (X, \|\cdot\|) \to (\mathcal{L}, \tau_\mu)$ is continuous at 0;

(iv) the set $\{x \in X : \{a_n(x)\}$ converges a.e.$\}$ is closed in X.

Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) always hold. If, in addition, there is a set $D \subset X$, $\overline{D} = X$, such that $\{a_n(x)\}$ converges a.e. for every $x \in D$, then all four conditions (i)–(iv) are equivalent.

A non-commutative Banach Principle for measurable operators affiliated with a semifinite von Neumann algebra was established in [1]. In particular, a non-commutative counterpart of condition (ii), which we call pointwise uniform boundedness, was suggested. In [2] the classical Banach Principle was extended to any topological group of second Baire category.

In the present article we propose a non-commutative version of condition (iii) which we call uniform equicontinuity of the sequence $\{a_n\}$ at 0. Then, for a complete metrizable topological group X, we show that all the implications stated above hold true in the non-commutative setting with a semifinite von Neumann algebra for both almost uniform and bilateral almost uniform convergences.

1. Preliminaries

Let M be a semifinite von Neumann algebra acting on a Hilbert space H, and let $P(M)$ be the complete lattice of all projections in M. A densely-defined closed operator x in H is said to be *affiliated* with M if $y'x \subset xy'$ for every $y' \in M'$, where M' is the

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commutant of the algebra M. Let τ be a faithful normal semifinite trace on M. If I is the identity of M and $e^{\perp} = I - e$, $e \in P(M)$, then an operator x affiliated with M is said to be *τ-measurable* if for each $\varepsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \leq \varepsilon$ such that eH lies in the domain of x. We will denote by $L = L(M, \tau)$ the set of all τ-measurable operators affiliated with M. Let $\| \cdot \|$ stand for the uniform norm in M. The *measure topology*, t_τ, in L is the one given by the system

$$\{V(\varepsilon, \delta) = \{x \in L : \| xe \| \leq \delta \text{ for some } e \in P(M) \text{ with } \tau(e^{\perp}) \leq \varepsilon\} : \ \varepsilon > 0, \delta > 0\}$$

of neighborhoods of zero.

Remark 1.1. If in definition of $V(\varepsilon, \delta)$ one replaces the condition $\| xe \| \leq \delta$ by the "two-sided" $\| exe \| \leq \delta$, then the same topology t_τ will be generated [3].

Theorem 1.2. ([4], see also [5]). *Equipped with the measure topology, L is a complete metrizable topological *-algebra.*

A sequence $\{y_n\} \subset L$ is said to converge *almost uniformly* (a.u.) (*bilaterally almost uniformly* (b.a.u.)) to $y \in L$ if for any given $\varepsilon > 0$ there exists a projection $e \in P(M)$ with $\tau(e^{\perp}) \leq \varepsilon$ satisfying $\| (y - y_n)e \| \to 0$ (respectively, $\| e(y - y_n)e \| \to 0$) as $n \to \infty$.

Clearly $y_n \to y$ a.u. implies $y_n \to y$ b.a.u. It is also known ([4], see also [1]) that $y_n \to y$ in t_τ implies that there is a subsequence $\{y_{n_k}\} \subset \{y_n\}$ converging to y a.u.

Proposition 1.3. *For a sequence $\{y_n\} \subset L$, the following are equivalent:*

(i) $\{y_n\}$ converges a.u. (b.a.u.) in L;

(ii) for every $\varepsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \leq \varepsilon$ such that $\| (y_m - y_n)e \| \to 0$ (respectively, $\| e(y_m - y_n)e \| \to 0$) as $m, n \to \infty$.

Proof. We provide a proof for the a.u. convergence; in the case of the b.a.u. convergence, the proof is similar. The implication $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (i)$. Condition (ii) implies that the sequence $\{y_n\}$ is fundamental in measure. Therefore, by Theorem 1.2, there is $y \in L$ such that $y_n \to y$ in t_τ. Fix $\varepsilon > 0$ and choose $p \in P(M)$, $\tau(p^{\perp}) \leq \varepsilon/2$, such that $\| (y_m - y_n)e \| \to 0$ as $m, n \to \infty$. Because $\{y_n\} \subset L$, it is possible to construct $q \in P(M)$ with $\tau(q^{\perp}) \leq \varepsilon/2$ satisfying $\{y_n q\} \subset M$. If $e = p \wedge q$, then $\tau(e^{\perp}) \leq \varepsilon$, $y_n e = y_n q e \in M$, and

$$\| y_m e - y_n e \| = \| (y_m - y_n)pe \| \leq \| (y_m - y_n)e \| \to 0,$$

$m, n \to \infty$. Therefore, there exists $y(e) \in M$ such that $\| y_n e - y(e) \| \to 0$. In particular, $y_n e \to y(e)$ in t_τ. On the other hand, $y_n e \to ye$ in t_τ, which implies that $y(e) = ye$. Hence, $\| (y_n - y)e \| \to 0$, i.e., $y_n \to y$ a.u. $\square$

Let (X, t) be a topological space, $x_0 \in X$, and let $a_n : X \to L$ be such that $a_n(x_0) = y_0$, $n = 1, 2, \ldots$. The family $\{a_n\}$ is *equicontinuous* at x_0 if, given $\varepsilon > 0$ and $\delta > 0$, there is a neighborhood U of x_0 in (X, t) such that $a_n U \subset y_0 + V(\varepsilon, \delta)$, $n = 1, 2, \ldots$, i.e., for every $x \in U$ and every n one can find a projection $e = e(x, n) \in P(M)$ with $\tau(e^{\perp}) \leq \varepsilon$ satisfying $\| (a_n(x) - y_0)e \| \leq \delta$.

Definition 1.4. Let (X, t) be a topological space, and let $a_n : X \to L$ and $x_0 \in X$ be such that $a_n(x_0) = y_0$, $n = 1, 2, \ldots$. Let $x_0 \in E \subset X$. The family $\{a_n\}$ will be called *uniformly equicontinuous* at x_0 on E if, given $\varepsilon > 0, \delta > 0$, there is a neighborhood U of x_0 in (X, t) such that for every $x \in E \bigcap U$ there exists a projection $e = e(x) \in P(M)$, $\tau(e^{\perp}) \leq \varepsilon$, satisfying $\sup\limits_n \| (a_n(x) - y_0)e \| \leq \delta$.

We will need the following two technical lemmas.

Lemma 1.5 [1]. *If f is the spectral projection of $b \in M$, $0 \leq b \leq I$, corresponding to the interval $[\frac{1}{2}, 1]$, then*

(i) $\tau(f^{\perp}) \leq 2\tau(I - b)$;

(ii) $f = bd$ *for some* $d \in M,\ \ 0 \le d \le 2 \cdot I$.

Let E be any set. If $a_n : E \to L$, $x \in E$, and $b \in M$ are such that $\{a_n(x)b\} \subset M$ ($\{ba_n(x)b\} \subset M$), then we denote

$$S(x, b) = S(\{a_n\}, x, b) = \sup_n \parallel a_n(x)b \parallel$$

(respectively,

$$SB(x, b) = SB(\{a_n\}, x, b) = \sup_n \parallel ba_n(x)b \parallel).$$

Lemma 1.6. *Let* $(X, +)$ *be a semigroup, and let* $a_n : X \to L$ *be a sequence of additive maps. Assume that* $\overline{x} \in X$ *is such that for every* $\varepsilon > 0$ *there exist a sequence* $\{x_k\} \subset X$ *and a projection* $p \in P(M)$ *with* $\tau(p^\perp) \le \varepsilon$ *satisfying*

(i) $\{a_n(\overline{x} + x_k)\}$ *converges a.u. (b.a.u.) as* $n \to \infty$ *for every* k*;*

(ii) $S(x_k, p) \to 0$ *(respectively,* $SB(x_k, p) \to 0$*) as* $k \to \infty$*.*

Then the sequence $\{a_n(\overline{x})\}$ *converges a.u. (respectively, b.a.u.) in* L*.*

Proof. We will prove this lemma for the a.u. convergence; proof for the b.a.u. convergence is similar. Fix $\varepsilon > 0$ and choose $\{x_k\} \subset X$ and $p \in P(M)$, $\tau(p^\perp) \le \varepsilon/2$, such that conditions (i) and (ii) hold. Pick $\delta > 0$ and let $k_0 = k_0(\delta)$ be such that $S(x_{k_0}, p) \le \delta/3$. By Proposition 1.3, there exists a projection $q \in P(M)$ with $\tau(q^\perp) \le \varepsilon/2$ and a positive integer N for which the inequality

$$\parallel (a_m(\overline{x} + x_{k_0}) - a_n(\overline{x} + x_{k_0}))q \parallel \le \delta/3$$

holds whenever $m, n \ge N$. If we define $e = p \wedge q$, then $\tau(e^\perp) \le \varepsilon$ and

$$\begin{aligned} \parallel (a_m(\overline{x}) - a_n(\overline{x}))e \parallel &\le \parallel (a_m(\overline{x} + x_{k_0}) - a_n(\overline{x} + x_{k_0}))e \parallel \\ &\quad + \parallel a_m(x_{k_0})e \parallel + \parallel a_n(x_{k_0})e \parallel \le \delta; \quad m, n \ge N. \end{aligned}$$

Therefore, by Proposition 1.3, the sequence $\{a_n(\overline{x}\}$ converges a.u. in L. □

2. The case of the almost uniform convergence

Let E be any set. A sequence $a_n : E \to L$ will be called *pointwise uniformly bounded* on E if, given $x \in E$ and $\varepsilon > 0$, there is a projection $e \in P(M)$ such that $\tau(e^\perp) \le \varepsilon$ and $S(x, e) < \infty$.

Let $(X, +, t)$ be a metrizable topological group, $0 \in E \subset X$, and let $a_n : X \to L$, $a_n(0) = a_1(0), n = 2, 3, \ldots$. In this section we will examine relationships among the following properties of the sequence $\{a_n\}$:

(CNV(E)) Almost uniform convergence on E: $\{a_n(x)\}$ converges a.u. for every $x \in E$;

(BND(E)) Pointwise uniform boundedness on E;

(CNT(E)) Uniform equicontinuity at 0 on E (see Definition 1.4);

(CLS(E)) Closedness in E of the set $C = \{x \in E : \{a_n(x)\}$ converges a.u.$\}$.

Proposition 2.1. *Any (CNV(E)) sequence* $a_n : X \to L$ *is (BND(E)).*

Proof. Pick $x \in E$ and let $\varepsilon > 0$. Since the sequence $\{a_n(x)\}$ converges a.u., there is $a_x \in L$ and $p \in P(M)$ with $\tau(p^\perp) \le \varepsilon/2$ such that $\parallel (a_n(x) - a_x)p \parallel \to 0$, $n \to \infty$. Because $a_n(x), a_x \in L$, it is possible to construct such a projection $q \in P(M)$ that $\tau(q^\perp) \le \varepsilon/2$ and $a_n(x)q,\ a_xq \in M$, $n = 1, 2, \ldots$. Defining $e = p \wedge q$, we obtain $\tau(e^\perp) \le \varepsilon$ and

$$\parallel a_n(x)e - a_xe \parallel = \parallel (a_n(x) - a_x)pe \parallel \le \parallel (a_n(x) - a_x)p \parallel \to 0.$$

Consequently, $\parallel a_n(x)e \parallel \to \parallel a_xe \parallel$ and $S(x, e) = \sup_n \parallel a_n(x)e \parallel < \infty$. □

Theorem 2.2. *Let* (X, t) *be complete,* $0 \in E \subset X$*. Assume that* $\overline{E} = E$*,* $E + E \subset E$*, and let* $a_n : (X, +, t) \to (L, +, t_\tau)$ *be a (BND(E)) sequence of continuous homomorphisms. Then the sequence* $\{a_n\}$ *is (CNT(E)).*

Proof. Fix $\varepsilon > 0$ and $\delta > 0$. For a positive integer l define the set

$$E_l = \{x \in E : S(x,b) \leq l \text{ for some } 0 \neq b \in M \text{ with } 0 \leq b \leq I, \tau(I-b) \leq \varepsilon/4\}.$$

Show that the set E_l is closed in (E,t). Take $\overline{x} \in \overline{E_l}$ and let $\{y_m\} \subset E_l$ be such that $y_m \to \overline{x}$ in t. Then we have $a_1(y_m)^* \to a_1(\overline{x})^*$ in t_τ, hence, there exists a subsequence $\{y_m^{(1)}\} \subset \{y_m\}$ for which $a_1(y_m^{(1)})^* \to a_1(\overline{x})^*$ a.u. By the same reasoning, one can find a subsequence $\{y_m^{(2)}\} \subset \{y_m^{(1)}\}$ satistying $a_2(y_m^{(2)})^* \to a_2(\overline{x})^*$ a.u. Repeating this process, for every $n \geq 3$, we choose a subsequence $\{y_m^{(n)}\} \subset \{y_m^{(n-1)}\}$ such that $a_n(y_m^{(n)})^* \to a_n(\overline{x})^*$ a.u. as $m \to \infty$. Define $x_m = y_m^{(m)}$. Since $\{x_m\}_{m \geq n}$ is a subsequence of $\{y_m^{(n)}\}$, we have

$$a_n(x_m)^* \to a_n(\overline{x})^* \ a.u., \quad m \to \infty, \quad n = 1,2,\ldots$$

By definition of E_l, one can find a sequence $\{b_m\} \subset M,\ 0 \leq b_m \leq I,\ \tau(I-b_m) \leq \varepsilon/4$, such that $S(x_m, b_m) \leq l$ for every m. Since the unit ball in M is compact in the weak operator topology, there is a subnet $\{b_\alpha\} \subset \{b_m\}$ and $b \in M$ for which $b_\alpha \to b$ weakly. Clearly $0 \leq b \leq I$. Besides, by the well-known inequality (see, for example [6]),

$$\tau(I-b) \leq \liminf_{\alpha} \tau(I-b_\alpha) \leq \varepsilon/4.$$

Show that $S(\overline{x}, b) \leq l$. Fix n. Because $a_n(x_m)^* \to a_n(\overline{x})^*$ a.u., given $\sigma > 0$, there is a projection $h \in P(M)$ satisfying $\tau(h^\perp) \leq \sigma$ and

$$\| h(a_n(x_m) - a_n(\overline{x}) \| = \| (a_n(x_m)^* - a_n(\overline{x})^* h \| \to 0, \quad m \to \infty.$$

We shall verify first that $\| ha_n(\overline{x})b \| \leq l$. For every $\xi, \eta \in H$ we have

$$\begin{aligned} &| (h(a_n(x_m)b_m - a_n(\overline{x})b)\xi, \eta) | \\ &\leq | (h(a_n(x_m) - a_n(\overline{x}))b_m\xi, \eta) | + | ((b_m - b)\xi, a_n(\overline{x})^* h\eta) | . \end{aligned} \tag{2.1}$$

Fix $\nu > 0$ and let m_0 be such that

$$\| (a_n(x_m) - a_n(\overline{x}) \| < \nu \tag{2.2}$$

whenever $m \geq m_0$. Next, since $b_\alpha \to b$ weakly, there is such an index $\alpha(\nu)$ that

$$| ((b_\alpha - b)\xi, a_n(\overline{x})^* h\eta) | < \nu \tag{2.3}$$

for all $\alpha \geq \alpha(\nu)$. Remembering that $\{b_\alpha\}$ is a subnet of $\{b_m\}$, one finds such an index $\alpha(m_0)$ that $\{b_\alpha\}_{\alpha \geq \alpha(m_0)} \subset \{b_m\}_{m \geq m_0}$. In particular, if $\alpha_0 \geq \max\ \{\alpha(\nu), \alpha(m_0)\}$, then $b_{\alpha_0} = b_{m_1}$ for some $m_1 \geq m_0$. It follows now from (2.1)–(2.3) that

$$\begin{aligned} | (ha_n(\overline{x})b\xi, \eta) | &\leq | (ha_n(x_{m_1})b_{m_1}\xi, \eta) | \\ &+ | (h(a_n(x_{m_1}) - a_n(\overline{x}))b_{m_1}\xi, \eta) | + | ((b_{m_1} - b)\xi, a_n(\overline{x})^* h\eta) | \\ &\leq l \cdot \| \xi \| \cdot \| \eta \| + \ \| h(a_n(x_{m_1}) - a_n(\overline{x}) \| \cdot \| b_{m_1} \| \cdot \| \xi \| \cdot \| \eta \| + \nu \\ &\leq l + 2\nu \end{aligned}$$

for all $\xi, \eta \in H$ with $\| \xi \| = \| \eta \| = 1$. Therefore,

$$\| ha_n(\overline{x})b \| = \sup_{\|\xi\| = \|\eta\| = 1} \| (ha_n(\overline{x})b\xi, \eta \| \leq l.$$

Now, let us pick $h_j \in P(M)$ such that $\tau(h_j^\perp) \leq \sigma_j = 1/j,\ j = 1,2,\ldots,$ and

$$\| h_j(a_n(x_m) - a_n(\overline{x}) \| \to 0 \quad \text{as} \quad m \to \infty.$$

Let ξ, $\| \xi \| \leq 1$, belong to the domain, $\mathcal{D}$, of the operator $a_n(\overline{x})b \in L$. Take $\eta \in H$, $\| \eta \| \leq 1$. Since $h_j \to I$ weakly and $\| h_j a_n(\overline{x})b \| \leq l$ for all $j = 1,2,\ldots$, we have

$$| (a_n(\overline{x})b\xi, \eta) | = \lim_{j \to \infty} \| (h_j a_n(\overline{x})b\xi, \eta) \| \leq \limsup_{j \to \infty} \| h_j a_n(\overline{x})b \| \cdot \| \xi \| \cdot \| \eta \| \leq l.$$

Therefore, $\| a_n(\overline{x})b\xi \| \leq l$ for every $\xi \in \mathcal{D}$ with $\| \xi \| \leq 1$. This means that $a_n(\overline{x})b \in M$ and $\| a_n(\overline{x})b \| \leq l$, i.e. $\overline{x} \in E_l$, hence E_l is closed in (E, t).

Note that, due to Lemma 1.5, condition (BND(E)) is equivalent to the following:

Given $x \in E$ and $\varepsilon > 0$, there is $0 \neq b \in M,\ 0 \leq b \leq I$, such that $\tau(I - b) \leq \varepsilon$ and $S(x, b) < \infty$.

Taking this and definition of E_l into account, we obtain

$$E = \bigcup_{l=1}^{\infty} E_l.$$

Because $\overline{E} = E$, the metric space (E, t) is complete. Therefore, by the Baire category theorem, it is possible to find l_0 and an open set U_0 such that $E \cap U_0 \subset E_{l_0}$. In other words, for every $z \in E \cap U_0$ there exists $b_z \in M,\ 0 \leq b_z \leq I,\ \tau(I - b_z) \leq \varepsilon/4$, satisfying

$$S(z, b_z) \leq l_0.$$

Let $z \in E \cap U_0$, and let f_z be the spectral projection of the operator b_z corresponding to the interval $[1/2, 1]$. Then, by Lemma 1.5, we have $\tau(f_z^{\perp}) \leq \varepsilon/2$ and also

$$S(z, f_z) \leq 2 \cdot S(z, b_z) \leq 2l_0.$$

Pick any $x_0 \in E \cap U_0$, and let V_0 be such an open neighborhood of zero that $x_0 + V_0 \subset U_0$. Let k_0 be a positive integer such that $k_0^{-1} \cdot 4l_0 \leq \delta$. Take W_0 to be such an open neighborhood of zero that $k_0 \cdot W_0 \subset V_0$. Pick $x \in E \cap W_0$, and let $z = x_0 + k_0 \cdot x$. Since $E + E \subset E$, we have $z \in E$; besides, $z \in U_0$, so $z \in E \cap U_0$. Thus, defining $e = f_z \wedge f_{x_0}$, we get $\tau(e^{\perp}) \leq \varepsilon$ and also

$$\begin{aligned} S(x, e) &= k_0^{-1} \cdot S(k_0 x, e) = k_0^{-1} \cdot S(z - x_0, e) \\ &\leq k_0^{-1} \cdot (S(z, e) + S(x_0, e)) \leq k_0^{-1} \cdot 4l_0 \leq \delta. \end{aligned}$$

Therefore, the sequence $\{a_n\}$ satisfies condition (CNT(E)). □

Remarks. 1. If X is a topological vector space over $\mathbb{Q}$, then clearly (CNT(E)) implies (BND(E)).

2. There is an error in [1] (bottom of p. 37). This error can be fixed by introducing necessary changes accordingly with the first part of the proof of Theorem 2.2.

Theorem 2.3. *Any (CNT(X)) sequence $a_n : X \to L$ of homomorphisms is (CLS(X)).*

Proof. Pick $\overline{x} \in \overline{C}$ and fix $\varepsilon > 0$. Since $\{a_n\}$ is a (CNT(X)) sequence, for every positive integer k, it is possible to find an open neighborhood U_k of $0 \in (X, t)$ such that, given $x \in U_k$, there exists $e_x \in P(M)$, $\tau(e_x^{\perp}) \leq \varepsilon/2^k$, satisfying $S(x, e_x) \leq 1/k$. Let $\{y_m\} \subset C$ be such that $y_m \to \overline{x}$ in t. Then we have $x_m = y_m - \overline{x} \to 0$ in t, hence, for every k there exists $x_k = x_{m_k} \in U_k$. It follows now that there is $e_k = e_{x_k} \in P(M)$ with $\tau(e_k^{\perp}) \leq \varepsilon/2^k$ for which the inequality $S(x_k, e_k) \leq 1/k$ holds. Letting $e = \wedge_{k=1}^{\infty} e_k$, we obtain $\tau(e^{\perp}) \leq \varepsilon$ and $S(x_k, e) \leq S(x_k, e_k) \to 0$ as $k \to \infty$. Taking into account that $\overline{x} + x_k = y_k \in C$, i.e. $\{a_n(\overline{x} + x_k)\}$ converges a.u., $k = 1, 2, \ldots$, by Lemma 1.6, we conclude that $\{a_n(\overline{x})\}$ converges a.u. as well. Therefore, $\overline{x} \in C$ and $= C$, meaning that the sequence $\{a_n\}$ is (CLS(X)). □

Theorem 2.4. *Let X be complete, and let $a_n : X \to L$ be a (CNV(D)) sequence of continuous homomorphisms. If $\overline{D} = X$, then all four conditions (CNV(X))–(CLS(X)) are equivalent.*

Proof. By Proposition 2.1, (CNV(X)) implies (BND(X)), while Theorems 2.2 and 2.3 entail implications (BND(X)) $\Rightarrow$ (CNT(X)) and (CNT(X)) $\Rightarrow$ (CLS(X)), respectively. Finally, (CLS(X)) together with (CNV(D)), $\overline{D} = X$, allow us to conclude that (CNV(X)) holds, which ends the proof. □

3. The case of the bilateral almost uniform convergence

Let E be a set. A sequence $a_n : E \to L$ is called *pointwise bilaterally uniformly bounded* on E if for every $x \in E$ and $\varepsilon > 0$, there is a projection $e \in P(M)$, $\tau(e^{\perp}) \leq \varepsilon$, such that $SB(x, e) < \infty$.

Let $(X, +, t)$ be a metrizable topological group. Let $0 \in E \subset X$. In this section we will discuss relationships among the following properties of a sequence $a_n : X \to L$, $a_n(0) = a_1(0),\ n = 2, 3, \ldots$:

(B.CNV(E)) Bilateral almost uniform convergence on E:
$\{a_n(x)\}$ converges b.a.u. for every $x \in E$;

(B.BND(E)) Pointwise bilateral uniform boundedness on E;

(B.CNT(E)) Bilateral uniform equicontinuity at 0 on E:
given $\varepsilon > 0$, $\delta > 0$, there exist a neighborhood U of $0 \in (X, t)$ such that for every $x \in E \cap U$ there is $e = e(x) \in P(M)$ with $\tau(e^{\perp}) \leq \varepsilon$ satisfying $SB(x, e) \leq \delta$;

(B.CLS(E)) Closedness in E of the set $C_B = \{x \in E : \{a_n(x)\}$ *converges b.a.u.*$\}$.

Remark. Due to Remark 1.1, conditions (B.CNT(E)) and (CNT(E)) are equivalent.

Proof of the next statement is similar to that of Proposition 1.3.

Proposition 3.1. *Every (B.CNV(E)) sequence $a_n : X \to L$ is (B.BND(E)).*

Let $(X, +, \leq)$ be an *ordered group*, i.e. $(X, +)$ is a group with a partial order " $\leq$ " such that $x + z \leq y + z$ for all $x, y, z \in X$ with $x \leq y$. A homomorphism $a : X \to L$ is called *positive* if $a(x) \geq 0$ for every $x \in X_+ = \{y \in X : y \geq 0\}$.

Theorem 3.2. *Let $(X, +, \leq, t)$ be a ordered complete metrizable topological group. Let $\overline{E} = E \subset X_+$ and $E + E \subset E$. Then every (B.BND(E)) sequence $a_n : X \to L$ of positive continuous homomorphisms is (B.CNT(E)).*

Proof. Fix $\varepsilon > 0, \delta > 0$. For a positive integer l define

$$E_l = \Big\{x \in E : \sup_n \| a_n(x)^{1/2} b \| \leq l \text{ for some } 0 \neq b \in M \text{ with } 0 \leq b \leq I, \tau(I - b) \leq \frac{\varepsilon}{4}\Big\}.$$

Let $\{y_m\} \subset E_l$ be such that $y_m \to x$ in t. Since $\overline{E} = E$, we have $x \in E \subset X_+$. Repeating verbatim the argument of the proof of Theorem 2.7 in [3], one can verify that $x \in E_l$, i.e. the set E_l is closed. Next, by Lemma 1.5, condition (B.BND(E)) is equivalent to the following:

Given $x \in E$ and $\varepsilon > 0$, there is $0 \neq b \in M$, $0 \leq b \leq I$, such that $\tau(I - b) \leq \varepsilon$ and $SB(x, b) \leq \infty$.

Consequently, exactly as it is done in [3], we get

$$E = \bigcup_{l=1}^{\infty} E_l.$$

Because X is complete and $\overline{E} = E \subset X$, (E, t) is a complete metric space, which allows us to apply the Baire category theorem. It follows that there exist such l_0 and an open set $U_0 \subset (X, t)$ that

$$E \cap U_0 \subset E_{l_0},$$

i.e., given $z \in E \cap U_0$, there exists $b_z \in M$, $0 \leq b_z \leq I$, satisfying $\tau(I - b_z) \leq \varepsilon/4$ and

$$\sup_n \| a_n(z)^{1/2} b_z \| \leq l_0.$$

Therefore, if $z \in E \cap U_0$ and f_z is the spectral projection of b_z corresponding to the interval $[1/2, 1]$, by Lemma 1.5, $\tau(f_z^{\perp}) \leq \varepsilon/2$ and also

$$\begin{aligned} SB(z, f_z) &= \sup_n \| f_z a_n(z) f_z \| \\ &= \sup_n \| a_n(z)^{1/2} f_z \|^2 \leq \sup_n (2 \| a_n(z)^{1/2} b_z \|)^2 \leq 4 l_0^2. \end{aligned}$$

Now, repeating the ending of the proof of Theorem 2.2, we conclude that the sequence $\{a_n\}$ is (B.CNT(E)). □

Proposition 3.3. *Let $(X, +, \leq, t)$ be a ordered complete metrizable topological group with $\overline{X_+} = X_+$. Assume that for every neighborhood $0 \in U \subset (X, t)$ the set $U \cap X_+ - U \cap X_+$ is also an neighborhood of 0. Then every (B.BND(X_+)) sequence $a_n : X \to L$ of positive continuous homomorphisms is (B.CNT(X)).*

Proof. Given $\varepsilon > 0, \delta > 0$, setting $E = X_+$ in Theorem 3.2, one can find a neighborhood $0 \in U \subset (X, t)$ such that for every $x \in X_+ \cap U$ there is $e \in P(M)$, $\tau(e^{\perp}) \leq \varepsilon$, with $SB(x, e) \leq \delta/2$. Therefore, for every z from the neighborhood $U \cap X_+ - U \cap X_+$ of zero we have $SB(z, e) \leq \delta$, which means that $\{a_n\}$ is a (B.CNT(X)) sequence. □

With a slight modification of the proof of Theorem 2.3 utilizing Lemma 1.6 we obtain the following.

Theorem 3.4. *If $(X, +, t)$ is a metrizable topological group, then every (B.CNT(X)) sequence $a_n : X \to L$ of homomorphisms is (B.CLS(X)).*

Theorem 3.5. *Let $(X, +, \leq, t)$ be as in Proposition 3.3. Assume that a sequence $a_n : X \to L$ of positive continuous homomorphisms is (B.CNV(D)) with $\overline{D} = X$. Then all four conditions (B.CNV(X))–(B.CLS(X)) are equivalent.*

Proof. (B.CNV(X)) implies (B.CNV(X_+)), hence, by Proposition 3.1, properties (B.BND(X)) and (B.BND(X_+)) hold. Due to Proposition 3.3, we arrive at (B.CNT(X)), which, by Theorem 3.4, implies (B.CLS(X)). Finally, (B.CLS(X)) together with (B.CNV(D)), $\overline{D} = X$, yield (B.CNV(X)), and the proof is complete. □

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